THE DYNAMO

THE DYNAMO

ITS THEORY, DESIGN AND MANUFACTURE

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VOLUME I

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PREFACE TO THE SIXTH EDITION

In the interval since the publication of the fifth edition of the present book but few striking changes in the design or manufacture of dynamos and alternators have been made. The more economical use of materials and higher speeds may perhaps have doubled the power obtained from a machine of given weight or occupying a given floor space, but every advance has been won by close attention to minor improvements in details, unless the wider adoption of artificial ventilation in modern high-speed machines be claimed as a new departure in principle. For central-station work the artificially-ventilated turbo alternator has continued to oust its engine-driven rival, and the sizes in which it is successfully manufactured have very greatly increased to meet the growing demands for power. The progress of electric generating machinery has indeed been mainly towards units of larger and larger capacity. In 1910 Prof. Miles Walker summarized the then position in the following words :

"If we look at the growth in the kilowatt capacity of machines during the last thirty years, we are driven to the conclusion that the kilowatt capacity of large units in the immediate future will be as great as 15,000 or 20,000 kW. In the year 1880 a 10-kW machine was considered large; in 1885 a 100-kW; in 1890 a 300-kW; in 1895 a 500-kW; in 1900 a 1,000-kW; in 1905 a 5,000-kW; in 1910 we have 10,000-kW steam turbine-driven generators and 17,000-kVA water turbine-driven generators."[1] In spite of the interposition of the great war the progress has been maintained, and now in 1921 we have 40,000 kVA single-unit steam turbine-driven alternators[2] and 32,500 kVA water turbine-driven alternators. It would, however, appear that a halt may shortly be anticipated, even if it be only temporarily.

But as pointed out by Dr. Hans Behn-Eschenburg[3] so long ago as 1911, the real "modern" development in electrical generators

[1] *Journ. I.E.E.*, Vol. 45, p. 319.

[2] And in rare cases even higher up to 50,000 kVA (Connors Creek station of the Detroit Edison Co.) and 60,000 kVA (Rheinisch Westfälische Kraftwerke, Cologne).

[3] "Charakterische und mechanische Eigenschaften moderner Generatoren insbesondere solcher höherer Tourenzahl," Turin International Congress, Sept. 1911.

is not so much the output and size of machines as the disappearance of the limitations which were formerly regarded as restricting the power that could be developed *per pole*, and this has been entirely due to the demands of the steam- and the water-turbine. Two-pole alternators for 12,500 kW or more at 3,000 revolutions per min. and a frequency of 50 are now in use, and machines of 16,000 kW (20,000 KVA, at 0·8 power-factor) at the same speed are proposed in 1921. On the other hand, low-speed alternators and continuous-current generators have again to a limited extent been called for in connection with Diesel and semi-Diesel oil-engines.

In spite of continuous and costly experimenting the directly-coupled continuous-current turbo-dynamo has succumbed to the medium-speed generator driven by the steam turbine through mechanical reduction gearing, or to the competition from the rotary converter fed from an alternating-current transmission system. The output of the continuous-current machine remains, therefore, far below that of the turbo-alternator, and seldom exceeds from 2,000 to 3,000 kW. Neither the homopolar dynamo nor the asynchronous induction generator can be said to have found more than a limited use in special cases up to the present.

In conclusion, it may safely be predicted that in the future as in the past, the course of electrical design and the type of machine will, in the main, be determined by the economical and technical considerations which govern the power of the most favoured prime movers, unless some new form of using and distributing electrical energy, as *e.g.* at telephone or even higher frequencies, assumes commercial importance.

Turning to the present revision, greater space has been devoted to the treatment of the E.M.F. of the dynamo by vectorial methods, the flux-curve of the field being assumed to be resolved into its fundamental and harmonics. The theory of armature winding has also been re-considered and expanded—in both the above cases the new treatment being largely influenced by the papers of Dr. S. P. Smith, published in the *Journal of the Institution of Electrical Engineers*. In the last edition, the drum armature having almost entirely superseded the ring armature, but little reference was made to the latter; in the present edition the drum armature in its toothed form is given still greater predominance to the practical exclusion of the older smooth-core armature. Yet not to the extent that could have been wished. It must be confessed that a theory of the E.M.F. of the toothed drum which will be complete and take into account ripples in the E.M.F. wave and all minor phenomena with rigid accuracy has not yet been presented, or, so far as the writer is aware, worked out in a simple form. In

consequence the student has still to approach the fundamentals of design through a theory of an ideal machine which presupposes a steady and unvarying field, so that many of the statements deduced from it are only strictly true of a smooth-core armature. The writer has, therefore, confined himself to an endeavour, in Chapter V at least, to indicate the difficulties of the problem by a fuller discussion of the actual physical causes of an induced E.M.F. Even if the views there put forward prove to be mistaken, it is hoped that they will re-direct the attention of engineering readers to a fundamental and most interesting portion of the subject and suggest new lines of thought more in consonance with the theories underlying the modern study of electrons and of the electric waves with which radio-telegraphy deals.

As the starting-point for the elementary theory of the electric machine, whether as dynamo or motor, is now placed the mechanical-force equation; the "watts per rev. per min." being the decisive guide to the size, cost and value of the rotating machine, it is believed that the torque—resisting or driving—should be treated as the primary and most fundamental property, and that the equation expressing it should on this account take precedence of the E.M.F. equation, which can be derived from it by means of the principle of the conservation of energy. The logical deduction of the usual equation for the E.M.F. of the continuous-current heteropolar dynamo has also been given more attention.

Other new matters that have been added in the present volume are a section on the oscillation of a mechanical system as affording an analogy to the electrical effects of capacity and as bearing on the critical whirling speed of shafts and the running of alternators in parallel—a discussion of the compressive stress on the mica plates in high-speed commutators—an analysis of the unbalanced magnetic pull when a rotor is displaced eccentrically to the bore of the stator—commutating poles and their leakage flux—and the winding of shunt coils with two sizes of wire.

The symbols have been altered throughout to agree with those recommended by the International Electrotechnical Commission and with the list published and adopted in *Papers on the Design of Alternate Current Machinery* by Messrs. Smith, Neville, and the present writer. It is hoped that by this change readers will be enabled to pass readily and without hindrance from the one book to the other.

Mr. F. Wallis having been unable, owing to other calls on his time, to take part in preparing either this or the last edition, the name of the present writer alone appears on the title-page. In so extensive a revision, undertaken single-handed in the intervals of other work, many slips have doubtless passed undetected,

and mistakes must have crept in; for all such the reader's indulgence must again be asked.

My thanks are due to the several firms and companies who have been good enough to furnish photographs for reproduction or other information as acknowledged in the text; to Mr. S. Neville for valuable criticisms and suggestions, and lastly in especial to Dr. S. P. Smith for kindly undertaking the onerous work of reading the greater part of the proofs.

C. C. H.

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LIST OF CHIEF SYMBOLS EMPLOYED IN VOLUME I.

A . . . = amperes or area.
AT . . . = ampere-turns.
AT_b . . . = half of demagnetizing or back ampere-turns of continuous current armature (Chap. xvi, § 4).
AT_c . . . = ampere-turns of excitation over half length of path in armature core.
AT_f . . . = ampere-turns of excitation on a *half* magnetic circuit or per pole (Chap. xvi, § 2).
AT_g . . . = ampere-turns of excitation over single air-gap.
AT_m . . . = ampere-turns of excitation over one field magnet core.
AT_t . . . = ampere-turns of excitation over length of one tooth.
AT_y . . . = ampere-turns of excitation over half length of path in yoke.
AT_p . . . = half of ampere-turns of excitation acting between pole-pieces (Chap. xvi, § 5).
AT_{pc} . . . = magnetic potential of commutating pole face.
AT_{se} . . . = ampere-turns of series winding per pole.
AT_{sh} . . . = ampere-turns of shunt winding per pole.
a . . . = number of pairs of armature paths = $q/2$ (Chap. x, § 7).
 = projection at one end of armature beyond pole-face.
 = sectional area.
a_c . . . = twice sectional area of armature core below slots in square centimetres.
a_g . . . = sectional area of air-gap in square centimetres.
a_m . . . = sectional area of magnet-core in square centimetres.
a_y . . . = twice sectional area of yoke-ring in sq. cm.
a'' . . . = projection of straight armature bar from slot at one end (eq. 83 and 84).
ac . . . = ampere-conductors per unit length of armature circumference.
at . . . = specific ampere-turns per cm. length of path.
at_c, at_m, at_t, at_y = specific ampere-turns per cm. length of path in armature core, magnet core, tooth and yoke respectively.

B . . . . = magnetic induction or flux-density in C.G.S. lines per sq. cm.
B_c . . . . = flux-density in armature core below teeth.
B_g . . . . = flux-density in air-gap.
B_{gx} . . . . = flux-density at point x in air-gap.
$B_{g\,max}$ . . maximum instantaneous flux-density in air-gap at a point (Chap. xvi, § 7).
$B_{g\,max}$ . . maximum flux-density in air-gap, averaged over a tooth-cycle or tooth-pitch (Chap. xvi, § 7).
$B_{g\,av}$ . . . average flux-density in air-gap.
B_{g1}, B_{g3}, B_{g5}, etc. fundamental and harmonics of flux-density curve.
B_s . . . . flux density in slot.
B_t . . . . flux-density in teeth.
B_t' . . . uncorrected flux-density in teeth (Chap. xvi, § 8).
B_{t1} . . . flux-density in teeth at top.
B_{t2} . . flux-density in teeth at bottom.
B_y . . flux-density in yoke.
B_m . flux density in field magnet.
bending moment.
B_e . . equivalent bending moment.
b . . damping force per unit velocity (Chap. vi, § 23).
b_a . . breadth of arm of hub.
β . . . ratio of pole-arc to pole-pitch.

C . capacity in farads.
no. of coils or of sectors in continuous-current machine.
C' . - ditto in parent machine (Chap. xii).
c . - half interpolar gap measured on the armature circumference (Fig. 252).
- no. of coils or sectors per slot.
ratio of excitation of air-gap to excitation over half magnetic circuit, $= AT_g/AT_f$.
controlling force per unit displacement (Chap. vi, § 23).

D . . = diameter of armature.
d . . = distance between adjacent end-connectors in barrel winding (eq. 83 and 84).
d' . . = diameter of journal in inches.
d_1 . . = diameter of insulated wire.
δ . = deflection of shaft or displacement of armature core from centre.
= total thickness of insulation on wire.
Δt = difference of temperature.

E . . . = modulus of elasticity.
E_c . . . = modulus of elasticity of copper.
E_m . . . = modulus of elasticity of mica.
e, E, $\mathbf{E}$ . . = instantaneous, virtual and maximum instantaneous value of alternating E.M.F. in volts.
e_a, E_a . . = instantaneous and virtual value of E.M.F. generated per phase in alternator armature.
e_i, E_i, $\mathbf{E}_i$. = instantaneous, virtual, and maximum value of impressed alternating E.M.F.
e_r, E_r, $\mathbf{E}_r$. = instantaneous, virtual, and maximum value of resultant or active alternating E.M.F.
e_s, E_s, $\mathbf{E}_s$. = instantaneous, virtual, and maximum value of E.M.F. of self-induction.
e_s', E_s', $\mathbf{E}_s'$. = instantaneous, virtual, and maximum value of E.M.F. consumed by self induction.
ε . . . = base of natural logarithms.
= angular difference between coil-pitch and pole-pitch (Chap. ix, § 12).
η . . . = hysteretic coefficient (Chap. xiv, § 8).
= efficiency.
= angle of lag of displacement behind applied force (Chap. vi, § 23).
= compression or expansion of commutator copper and rings (Chap. xiii, §§ 26, 27).

F . . . = mechanical force.
F_c . . . = total centrifugal force summed up round periphery of cylinder (Chap. xiii, § 5).
F_{sc} . . . = centrifugal force of whole commutator.
f . . . = frequency in complete periods per second (Chap. viii, § 5, and Chap. ix, § 6).
= H.C.F. of S and p, *i.e.* no. of repetitions.
f_c . . . = centrifugal force per unit arc of cylinder.
f_{sc} . . . = centrifugal force per sector of commutator.
f_s . . . = safe permissible shearing stress.
f_t . . . = safe permissible tensile stress.

g . . . = acceleration due to gravity.
γ_e . . . = electrical angle of displacement between two coils.

H . . . = magnetic difference of potential in C.G.S. units, or magnetizing intensity.
H_w . . . = loss by hysteresis in watts (eq. 97).
h . . . = specific loss by hysteresis in joules per cycle per c.cm. (eq. 96).
= height of copper commutator sector.

h	= H.C.F. between y_q and C, *i.e.* the no. of independent windings (Chap. xi, § 9).
h_a	= thickness of arm of armature hub (eq. 69).
h_c	radial depth of armature core below slots.
h_s	= depth of slot in toothed armature.
h_w	= winding depth between wedge and bottom of slot.
h_3	depth of wooden wedge in slot.
I	= moment of inertia.
i, I, I	instantaneous, virtual, and maximum values of alternating current in amperes.
I_a	total current of continuous-current armature.
I_e	current in external circuit (R.M.S. value for alternator).
I_s	shunt current of continuous-current machine.
J	virtual amperes in any one conductor on armature.
	intensity of magnetization.
K	voltage factor in E.M.F. equation (38*b*) of alternator (Chap. ix, § 15), $k_f . k_d$.
	= extension coefficient for air-gap length (Chap. xvi, § 7).
K_s	ratio of iron + air to net iron (eq. 117).
K_1	coefficient for interpolar fringe (Chap. xvi, § 6*a*).
K_2	coefficient for pole-flank fringe (Chap. xvi, § 6*b*).
K_3	coefficient for air ducts (Chap. xvi, § 6*c*).
kW	kilowatts.
kVA	kilovolt-amperes.
k	= heating coefficient $\frac{t^\circ . S_c}{W}$ (Chap. xvi, § 16), = $1/\xi$.
	= ratio of average to maximum flux-density in air-gap = $2/\pi$ for sinusoidally distributed field.
	= ratio of applied to natural undamped frequency (Chap. vii, § 23).
k_f	= form factor, *i.e.* ratio of R.M.S. to average value of a varying quantity = $\pi/2\sqrt{2}$ for sinusoidally varying quantity.
k_d	= differential factor.
k_{dc}	= differential factor of continuous-current armature.
k_{ds}	= differential factor with sinusoidally distributed field (Chap. ix, § 11).
k''_{s1}, k''_{s3}, k'' etc.	pitch differential factor of coil for fundamental and harmonics of flux curve (eq. 23).

L . . . . = gross length of armature core.
= axial length of magnet bobbin.
$\mathcal{L}$ . . . . = inductance in henrys.
L_f . . . = width of pole-face along axis of armature core.
L_i . . . . net axial length of iron in armature core.
l_o . . . . length of turn on outside of magnet-coil.
l_c . . . . length of magnetic path in armature core in cm.
l_g . . . . length of magnetic path through single air-gap.
l_t . . . . length of magnetic path through one armature tooth.
l_m . . . . length of magnetic path through one magnet core.
l_y . . . . length of magnetic path in yoke.
l_x . . . . mean length of exciting turn.
l_{mx} . . . mean length of series exciting turn.
l_{sx} . . . mean length of shunt exciting turn.
l'. . . . . length of journal in inches.
= length of end-connector at one end of drum coil (eq. 84).
l_p . . . . axial projection of winding at one end of barrel-wound drum (eq. 83).
λ . . . . angle of lead of brushes.
λ_c . . . . ratio of pitch of coil to pole-pitch (Chap. ix, § 12).

M . . . . mass.
m . . . . "creep" per coil traversed in terms of sectors, a/p (Chap. xi, § 11).
half pitch of armature winding measured on circumference (Chap. xiii, § 22).
μ . . . . permeability.
coefficient of friction.
electrical angle of displacement between coils in continuous-current winding (Chap. x, § 8).

N . . . . number of revolutions per minute.
N_c . . . . total number of field-magnet coils.
N_o . . . . crucial speed of self-excitation (Chap. xvii, § 11).
N_{ph} . . . number of phases.
N_s . . . . number of linkages of self-induced lines with circuit.
n . . . . number of revolutions per second.
n_a . . . . = number of arms in armature hub.
= number of conductors abreast in one layer in slot.
n_d . . . . = number of air-ducts in armature core.
n_l . . . . = number of layers of conductors in a slot.
ν . . . . = leakage coefficient $= \Phi_m/\Phi_a$.

P . = number of poles.
w = load on bearing in pounds.
P_m . = total uniform magnetic pull summed up all round armature (eq. 98*a*).
[illegible] = resultant magnetic pull on one-half of armature when its value is non-uniform (Chap. xiii, § 5).
P_m'' . . = magnetic pull all round armature if supposed to be uniformly at its maximum value.
P_t . . = pull due to transmitted torque (Chap. xiii, § 6).
P_u . . = unbalanced magnetic pull (eq. 101).
$\mathcal{P}$ . . = permeance.
$\mathcal{P}_l$ . . = leakage permeance.
p . . . = number of pole-pairs in heteropolar dynamo, or number of polar projections in homopolar alternator (Chap. viii, § 5).
intensity of pressure in pounds in square inch.
= $2\pi/T_e$ (Chap. vi, § 23).
p' . . = number of pole pairs in parent machine.
p_m'' . . = maximum abnormal pull due to deflection of yoke-ring per radian (Chap. xv, § 16).

q, Q, $\mathbf{Q}$. = instantaneous, virtual, and maximum value of charge on condenser in coulombs.
q . = number of parallel paths through armature winding, per phase in an alternator, and = $2a$ in a continuous-current machine.
q_1 . . = number of vectors to be added together (Chap. ix, § 12).

R . . = radius.
[illegible] = resistance in ohms.
R_a . . = resistance of armature in ohms.
R_b . . = resistance of brushes in ohms.
R_c . . = resistance of a connection between brushes of same sign (Chap. xii, Note).
R_e . . = resistance of external circuit in ohms.
R_m . . = resistance of series winding in ohms.
R_r . . = resistance of rheostat in ohms.
R_s . . = resistance of shunt winding in ohms.
R_o, R_m, R_i = outer, mean, and inner radius of hollow cylinder.
R_p . . = radius to pole-face.
$\mathcal{R}$ . . = magnetic reluctance.
$\mathcal{R}_a$ . . = magnetic reluctance of armature.
$\mathcal{R}_g$ . . = magnetic reluctance of air-gap.
$\mathcal{R}_l$ . . = magnetic reluctance of leakage paths.
$\mathcal{R}_m$ . . = magnetic reluctance of field-magnet.

r_g . . . . = radius to centre of gravity.

r_a . . . . = radius of arm of hub (eq. 69).

r_n . . . . = radius to nave of hub (eq. 69).

ρ . . . . = specific electrical resistivity, *i.e.* resistance between opposite faces of a cm. cube in ohms.

S . . . . = number of slots or teeth in toothed armature.

S' . . . . = ditto in parent machine (Chap. xii, § 5).

S_c . . . . = cooling surface of field-magnet bobbins or armature (Chap. xvi, § 16).

s_a . . . . = stress on material of armature core due to centrifugal force and magnetic pull (eq. 66)

s_a' . . . . = stress on material of armature core at junction with arm (eq. 67)

s_c . . . . = stress on material due to centrifugal force.
= compressive stress on commutator copper and mica (Chap. xiv, § 29)

s_b . . . . = bending stress on arm of hub (eq. 69).

σ . . . . = space factor, ratio of copper volume to total volume.
= ratio of final to initial displacement in oscillating system (Chap. xiii, § 10).
= ratio of wound arc to pole-pitch (Chap. xviii, § 4).

T . . . . = number of turns.
= torque.
= driving tension in belt.

T_c . . . . = turns of coil.

T_f . . . . = turns of field-magnet coil.

T_p . . . . = periodic time in seconds.

T_r' . . . . = periodic time of forced oscillation in seconds (Chap. vi, § 23).

T_m . . . . = twisting moment of shaft (eq. 68).
= number of series turns.

T_s . . . . = number of shunt turns.

T° . . . . = temperature.

t . . . . = number of turns in coil, or of active conductors in one belt corresponding to one pole and phase on alternator armature (Chap. ix, § 15).
= time in seconds.
= depth of winding of bobbin.

t° . . . . = rise of temperature.

t_1 . . . . = tooth-pitch.

U . number of coil-sides or elements in drum winding (Chap. xi, § 5).
u . number of coil-sides per slot in toothed armature.

V . velocity in cm. per second.
V_b . voltage at brushes.
V_c . volume of iron in c.cm.
V_e . voltage at terminals of external circuit (R.M.S. value if alternating).
V_x . exciting voltage.
v . peripheral speed of journal in feet per minute (Chap. xiii, § 12).
v_g . linear velocity of centre of gravity.

W, w watts.
W . weight.
W_a . weight of armature.
W_c . weight concentrated at one point.
W_d . weight distributed along shaft.
W_p . weight of pole.
W_y . weight of complete yoke-ring.
w . weight of unit volume.
w_d . width of ventilating air duct in armature core.
w_3 . width of slot-opening.
w_s . width of slot.
w_t . width of tooth.
w_{t1} . width of tooth at top.
w_{t2} . width of tooth at bottom.

X . reactance.
X_f . ampere turns of excitation on *one* magnet circuit or per pair of poles, $= 2.4T_f$.
X_p . ampere turns acting between pole pieces $= 2.4T_p$.

Y . pole-pitch.
y . total or resultant pitch (Chap. x, § 6), $= y_b + y_f$ algebraically.
y_{eq} . equipotential pitch (Chap. xii, §§ 7 ff).
y_f . front pitch of armature winding at commutator end in elements (Chap. x, § 6).
y_b . back pitch of armature winding in elements (Chap. x, § 6).
$y_b{}^1$. back pitch of armature winding in slots (Chap. xi, § 12).
y_c . average pitch, or pitch in commutator sectors (Chap. x, § 8), $= y/2 = \frac{1}{2}(y_b + y_f)$.

y_{ph} . . . = phase-pitch in armature winding tapped for N phases.

Z . . . = modulus of section.
= impedance.
= total number of conductors on armature.

z . . . = number of conductors per slot.

Φ . . . = magnetic flux, number of C.G.S. lines.

Φ_a . . . = total number of useful lines passing through one pole-pitch into armature core.

Φ_a' . . . = total number of lines entering armature of homopolar alternator within the pole-pitch (Chap. viii, § 6).

Φ_l . . . = flux linked with loop or circuit.

Φ_1, Φ_3, Φ_5, etc. = fundamental and harmonics of flux.

Φ_m . . . = total number of lines passing through a magnet-core.

Φ_r . . . = total flux of one commutating pole.

ϕ . . . = angle subtended by pole face.
= angle of lag or lead of alternating current or E.M.F.

ϕ_a . . . = angle of lag of armature current behind E.M.F.

ϕ_e . . . = angle of lag of external current vector behind terminal voltage.

ϕ_a' . . . = useless lines entering armature of homopolar alternator beyond the pole-pitch (Chap. viii, § 6).

ϕ_l . . . = number of leakage lines in one magnet circuit.

ϕ_r . . . = useful flux of one commutating pole.

ξ . . . = reluctivity (Chap. iii, § 11).
= multiplier in approximate expression for length of path in air-fringe (Chap. xvi, § 6).
= loss in watts per unit area per 1° rise.

χ . . . = angular width in radians of belt of distributed winding (Chaps. ix, § 12, and x, § 12).

ψ . . . = angle of displacement of vectors (Chap. ix, § 12).

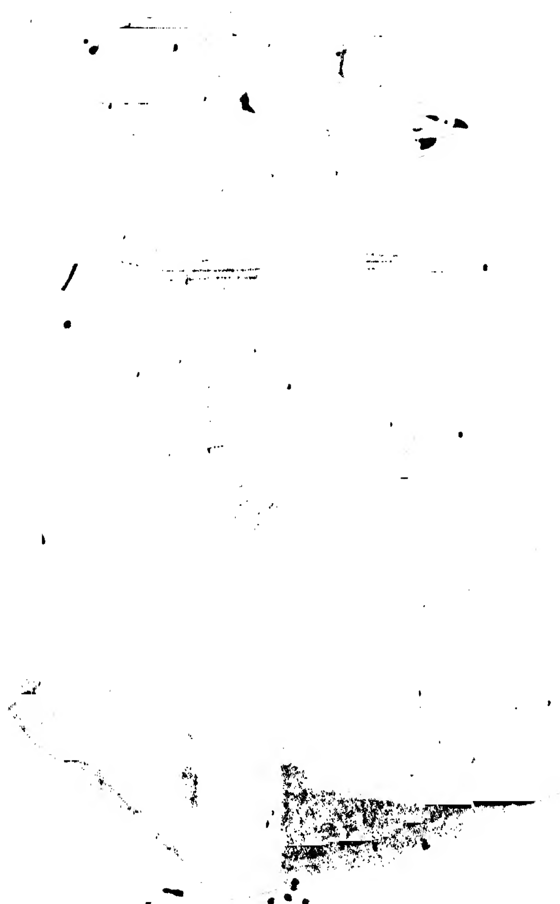
ω . . . = mechanical angular velocity in radians per sec.
= resistance of unit length of copper of given section.

ω' . . . = resistance of 1,000 yards of copper of given section at 68° F. (20° C.)

ω_{crit} . . . = critical angular velocity for whirling of shaft (Chap. xiii, § 10.)

ω, or ω . . = electrical angular velocity, $2\pi f$, in radians per sec.

NOTE.—Clarendon (that is, heavy) letters indicate the maximum value of quantities varying in time.



FARADAY'S ORIGINAL DISC DYNAMO.

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THE DYNAMO

CHAPTER I

INTRODUCTORY

§ 1. Definition of the dynamo.—The theory of the dynamo is one among the many meeting-points of electricity and magnetism, and in its use the engineer finds what is perhaps the chief industrial application of the two sister sciences. From their union the dynamo derives its characteristic dual nature, which is reproduced in its structure as a machine; for it may broadly be regarded as built up of a copper or electric portion and an iron or magnetic portion. The mechanical is, however, of no less importance than the electro-magnetic aspect as shown by the word "*dynamo*-electric" in its full form. Its fundamental principles are therefore three in number—the mechanical, the electrical, and the magnetic; and its design is correspondingly based on three fundamental equations. The first two, dealing with the mechanical and electrical sides, both involve a magnetic element, but require to be supplemented by a third equation dealing solely with the facts and laws of magnetism.

The dynamo may be defined as *a machine in which a system of conductors forming part of an electric circuit is given continuous motion relatively to a magnetic field or fields, and so is caused to cut across the magnetic flux, or to be linked with a varying number of lines; an electromotive force is thereby induced in the conductors, so that when the circuit is closed a current flows, and mechanical energy is converted into electrical energy.*

§ 2. The dynamo as a generator of electric pressure.—The function of the dynamo is, primarily and essentially to generate an electric pressure or electromotive force, and to maintain it when the circuit is closed and a current flows. The entire circuit is divisible at two points, say, A and D, into two portions, the points A and D forming the *terminals* at which the *internal* unites with the *external* portion of the circuit. Within the former, which comprises the dynamo, an electromotive force is set up, which results in a difference of potential between the terminals A and D, so that the one is positive or at a higher potential than the other which is negative. When A and D are joined by the external portion of the circuit, or, as it is called for shortness, the external circuit, a current flows from the positive to the negative terminal

in the external circuit and from the negative to the positive terminal within the dynamo itself. The potential would thus become equalized throughout the entire circuit, were it not for the fact that the dynamo maintains the difference of potential between A and D, which constitutes the *voltage* of the machine. Thus the dynamo does not generate electricity, any more than an hydraulic pump driving a motor through a closed pipe circuit generates the water. Neither is it the electricity that is consumed or lost in passage round the circuit, but the pressure under which it flows, just as the "head" of the water is lost in its passage through the pipe and motor, only to be renewed by the action of the pump.

§ 3. The output of the dynamo. The *output* of the dynamo is the rate at which it develops electric energy in the external circuit, *i.e.* it is the product of the voltage V, at its terminals, and the current I, in amperes in the external circuit. Since the watt is but a small unit of power, the output is usually expressed in units of 1,000 watts, *i.e.* in kilowatts (1 H.P. = 746 watts, so that 1 kilowatt nearly = $1\frac{1}{3}$ H.P.).

As the product of the factors, volts and amperes, the same output may be due to many different combinations of the two. Thus for transmission of electric energy over considerable distances machines of reasonably high voltage are required, for traction and direct lighting machines of moderate voltage, while, at the opposite extreme, for electro-deposition and other chemical or metallurgical processes large currents at low voltages are required. Although their power may be the same, the construction of the machines is very widely different, owing to the different natures of the work for which they are respectively suited.

§ 4. The conversion of mechanical into electrical energy, and the efficiency of the process. By the principle of conservation of energy it is impossible that any form of energy can be created; in any machine it can only be transmuted from one form to another, and in the case of the dynamo in which electrical energy is developed, our definition states that the energy is supplied to it in a mechanical form, one portion of the machine being continuously moved relatively to another portion against a resisting force opposing the motion. The movable portion of the dynamo may be driven by means of a belt and pulley, or by rope gearing; or it may be coupled directly to the main shaft of the prime mover, as to the crank shaft of a steam-engine or the shaft of a steam or water turbine; but whatever be the method of driving, the *input* is mechanical power which reappears mainly as electrical power. The dynamo is thus distinguished from the transformer; in this there is no relative motion, and the input is electrical power which reappears at a different pressure but still in an electrical form. The extensive use of electrical energy for commercial purposes has, in fact, been

rendered possible by reason of the ease and cheapness with which mechanical energy can be applied to drive the dynamo. The cost of the chemical materials which voltaic batteries require prohibits their use on a large scale, while frictional machines only yield very small currents at inconveniently high pressures, so that it was the invention of the dynamo by Faraday in 1831 which first led the way to a ready means for obtaining electrical energy economically and in a convenient form.

But in no machine can the conversion of energy from one form to another be carried on without some loss, by which is meant its re-appearance in a useless form or in useless places. Apart from the loss by friction in the bearings of the dynamo and from windage, there is a necessary loss from its total rate of development of electrical energy, owing to the absorption of power by the current in passing through the electrical resistance of the dynamo itself and to other secondary causes. In consequence of these mechanical and electrical losses, if the rate of supply of mechanical energy to the shaft of the dynamo be one horse-power, its output must be something less than 746 watts. Nevertheless, the *efficiency* of the dynamo or the ratio between the useful power obtained from it and the power supplied to it mechanically is very high, since in all but very small machines it is practically and commercially possible to obtain at least as much as 90 per cent. of the mechanical input, returned at the terminals of the dynamo in its new form of electrical energy, and in large machines even higher percentages are usually attained. Much improvement cannot therefore be expected on the score of efficiency, although the dynamos of the future may be cheaper to construct. Indeed, of all machines yet invented, the dynamo may rank as one of the most perfect converters of energy, only surpassed by the transformer, in which, however, there is no true conversion of the nature of the energy, but only a transformation of electrical energy from one pressure to another.

CHAPTER II

THE MAGNETIC CIRCUIT AND ITS LINES OF FLUX

§ 1. The inductive property of the magnetic field and its investigation by an exploring coil.—With the exploration of the field of force surrounding a magnet or solenoid by means of an ideal unit N. pole, and with its mapping by lines of force, the reader will be familiar. It is, however, with the *inductive* property of the magnetic field that the student of the dynamo is more directly concerned. Like the dynamic property in virtue of which a magnetic field exerts force on a unit pole, this second property is a directed quantity, expressible at each point by a vector having direction, sense and magnitude. In place of the unit pole, the appropriate instrument in order to investigate it is the *unit exploring coil, i.e.* a single loop having an area of one square centimetre connected by twisted leads to a ballistic galvanometer, the total resistance of the circuit so formed being one absolute electromagnetic unit of resistance on the C.G.S. system, or 10^{-9} ohm.

Let the loop be placed in the magnetic field due to a solenoid or electromagnet or other electric circuit of which the exciting current can be made or broken or reversed, and let it be turned in every direction about its centre, so that its axis or the perpendicular passing centrally through its plane occupies different positions in space. In each position let the exciting current be broken or reversed; it will then be found that except when the perpendicular falls in one particular plane, a current is induced and a certain quantity of electricity passes which can be measured by the ballistic galvanometer; further that this quantity varies according to the direction of the perpendicular and reaches a definite maximum for one particular direction. This maximum occurs when the loop is itself in the plane which has been described as containing all positions of the axis that gave no inductive effect, and when in consequence the perpendicular is at right angles to this plane.

§ 2. Lines of magnetic induction—their direction, sense and number.—If the unit exploring loop is placed in any part of a magnetic field with its perpendicular in the position for greatest inductive effect, and the quantity of electricity that passes through the secondary circuit of the loop and ballistic galvanometer when the primary or exciting current is broken is one absolute unit or one deca-coulomb, the inductive property now under investigation is defined to have *unit magnitude*. Expressed in terms of this unit, its value at any spot is termed the "*induction*," and is symbolized by *B*.

The *direction* of the magnetic induction at any spot is that of the perpendicular to the loop when in the position that gives the greatest inductive effect on making or breaking the exciting current. It is further defined by the convention that the *sense* of the induction is related to that of the current in the secondary in the same way as the forward or backward movement of a right-handed screw is related to its rotation (compare Fig. 1).[1] Thus, on breaking the primary, if the secondary current is clockwise from the observer's end, the induction passes through the loop of the exploring coil away from the observer or in the same direction as from the face of the dial to the works of the clock. If a line be drawn so that the tangent to any point in it gives a continuous record of the direction of the induction as we pass from point to point along its length, such a line is called *a line of induction*, and if the process of drawing it be repeated for all parts of the field an infinite number of such lines can be obtained.

But as in the case of magnetic force, a certain number of lines may now be marked out or labelled such that their density per unit area on a plane normal to their direction is equal to the magnitude of the induction at that place; they then become *C.G.S. lines.*[2] In a field of uniform induction if any surface be taken on the normal plane the density of such lines multiplied by the area of the surface will give the total induction passing through it. Thus by means of the unit exploring coil the direction and magnitude of the induction can be surveyed, and a complete quantitative system of lines be derived by a process exactly analogous to the mapping of the lines of force by a free unit N. pole.

§ 3. Distinction between lines of induction and lines of force.—Whenever the space under investigation is a vacuum or is filled with air or other non-magnetic medium, identically the same system of lines is obtained whether the starting-point be the force on a unit pole or the inductive property of the magnetic field. The lines of induction are linked with a current-carrying wire, circular loop, or solenoid when these are immersed in air, just as the lines of force; they follow the same paths, and are the same in number. The two systems are not, however, identical, and their difference comes out when we pass to the interior of any iron mass. For the exploring loop may be applied to investigate the internal state of an electro-magnet. Thus it may be wound on the outside of the iron-cored electromagnet, of which the primary current is made or broken as before. Or, if need be, the coil can actually be

[1] Cp. Clerk Maxwell, *Treatise on Electricity and Magnetism*, Vol. II, Pt. IV, p. 138 *et seq.* and Jeans, *Electricity and Magnetism*, p. 427.

[2] At the Paris Conference of 1900 the term "Maxwell" was recommended for adoption to express a unit of induction, or one C.G.S. line.

buried in the substance of the iron so as to measure the induction at any spot when the primary current is altered in value. It may even with certain modifications be applied to throw light on the internal condition of the permanent magnet. We are, in fact, able to trace the passage of the lines of induction through the mass of an iron magnet by direct experiment, whereas the determination of the lines of force *within the iron* is only possible by theoretical and indirect methods.

§ 4. Lines of induction always closed curves. If now the exploring coil be used to map out the induction within an iron-cored solenoid or permanent bar magnet, it will be found that as many lines of induction pass through a section of the iron taken at the centre of its length as there are in its external field, and that each line within the iron finds its appropriate continuation in one of the lines of the external field (Fig. 3). Every line of induction is, in fact, a closed curve, which either loops round an electric current

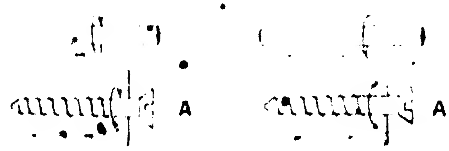


FIG. 1.

or currents (and in this case may be entirely in air or entirely in iron, or partly in one and partly in the other), or at some portion of its endless path passes through a piece of permanently magnetized steel or iron. Wherever lines of induction issue out of iron into air or other non-magnetic material, there a north pole is developed; and wherever they enter from air into iron, a south pole results; within the iron their direction is from south to north; without it, from north to south.

It will only here be necessary to remind the reader that round a straight wire carrying an electric current, the direction of the lines is related to the direction of the current, just as the direction of rotation of a right-handed screw is associated with the direction of its forward or backward movement as viewed from its head *A* to which the screw-driver is applied (Fig. 1). When bent up into a loop, the same relation holds, so that if the current in the loop be as shown in the upper part of Fig. 1, the lines will pass through the loop in the direction of the horizontal arrows (cp. Fig. 2). This leads to the following simple and convenient rule—

Curve the right hand round the outside of the loop, keeping the palm towards its axis, so that the direction of the flow of current is from the wrist to the tips of the fingers; then the outstretched thumb will

point along the positive direction of the lines within the loop, the N. face from which they issue being therefore on the same side of the hand as the thumb.

Similarly, when several loops are combined in a straight solenoid, if the *right* hand be curved round the outside of the helix with the palm towards its axis and with the current flowing from wrist to finger-tips, the outstretched thumb will point towards the N. pole or end from which the lines issue (Fig. 3).

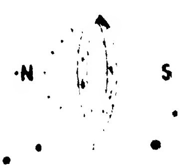


FIG. 2. Field of loop

If the solenoid be bent round until its ends meet, or if a number of insulated turns are wound round an endless ring-core of circular cross-section, a *toroid* is formed (Fig. 4). When closely and uniformly over-wound, the circular lines are confined to the section within the turns, and there are no poles.

§ 5. The total flux of a magnetic circuit.

In every case the lines of induction may be regarded as a circuital stream flowing round a closed path either in air or iron, or partly in one and partly in the other. They may therefore be regarded as a *magnetic flux* passing round a circuit. Their total number within any area may be called the *total flux*, and the number passing through unit area, *i.e.* on the C.G.S. system through one square centimetre, on a

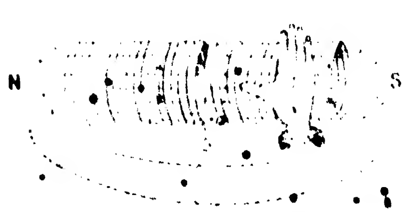


FIG. 3. Field of solenoid.

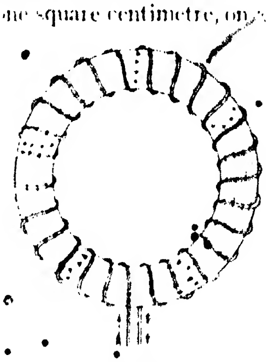


FIG. 4. Field of toroid.

plane normal to their direction is the *flux-density* at the spot, the latter expression being synonymous with "the induction" B. The lines of flux in the case of the uniformly wound closed toroid, such as Fig. 4, are endless curves within the interior of its ring-core, whether this be made of iron or wood or simply consists of air. All the lines of the straight solenoid or electromagnet are not, however, linked with the entire number of the exciting turns; some lines escape from the surface of the bobbin without traversing its whole length.

Yet every line passes through the coils for a greater or less distance in its path. And in the same way, though some of the lines of the permanent bar magnet issue from or enter into the iron short of the ends, yet all pass through its cross-section at the centre. We thus reach the conception of a *magnetic circuit* through which a certain flux passes under a magnetomotive force, just as an electric current flows round an electric circuit under an electromotive force. The expression "magnetomotive force" requires to be further defined, and will be found to form a connecting link between the two systems of magnetic force and induction.

§ 6. Magnetomotive force of a circuit. If any two points are taken in a magnetic field, and a free unit N. pole be moved from one point to the other along any line joining them, a certain amount of work will be done if the direction of movement of the pole has any component along the lines of force of the field. If the motion be against the sense of the lines the work will be done on the pole, or if it be with their sense it will be done by the pole. If the field be *in vacuo* or in air or other magnetically indifferent medium, this work, measured in ergs, is the *difference of magnetic potential* existing between the two ends of the length under consideration. Its value is most simply calculated if the path chosen coincides throughout its whole length with the direction of the lines of force; thus if a length of one centimetre be taken anywhere along a line of force, and the intensity of the field is measured by one dyne throughout the centimetre length, the work done by a unit N. pole in moving from one end to the other against the force will be one erg, and unit difference of potential exists between the two ends. Or in general, if the intensity of the field *in vacuo* or in air has the uniform value H, and a centimetre length be taken along its direction, the difference of magnetic potential over that length is also measured by H. If an endless line looped round an electric current be subdivided into very small elements, and the successive differences of potential are calculated for each small element until the whole length of the path has been traversed once, the sum of their values or the *line-integral of the magnetic force* is called the *magnetomotive force* along the line in question. When the line round which the differences of potential are integrated follows the direction of the field, and the field has a uniform intensity throughout its length, the magnetomotive force is simply the product of the intensity of the field and the length of the line, or if we use the convenient abbreviation of M.M.F. for magnetomotive force, on the analogy of E.M.F. for electromotive force, we then have

$$\text{M.M.F.} = H \times l$$

A straight wire carrying an electric current of which the return is a long way off gives the fundamental case connecting magnetomotive

force and current. If any one of the circular concentric lines be taken of which the distance from the axis of the wire is r centimetres, and the wire be carrying A amperes ($= A/10$ absolute units of current on the C.G.S. electromagnetic system), the intensity of the field *in vacuo* or in air at radius r has the uniform value of $2A/10r$, and its direction is throughout that of the chosen line, of which the length is $2\pi r$; the line-integral of the force is thus simply the product of the two, or $\frac{2A}{10r} \times 2\pi r = \frac{4\pi A}{10}$. In the case of the straight solenoid of Fig. 3, of which the length l is great as compared with the radius of its cross-section, the intensity of the field at a point in its interior well away from its ends is $\frac{4\pi}{10} \times \frac{AT}{l}$ where T is the total number of turns and A is the current in amperes flowing through them. Since, however, the intensity decreases as we approach the ends of the coil or pass outside it, the magnetomotive force of the coil is not easily calculated in this particular form, and the case must be simplified by bending the solenoid round until its two ends meet and a toroid is formed. Since the coil now has no ends, the intensity in its interior is throughout the same as at the centre of the straight coil, or $\frac{4\pi}{10} \times \frac{AT}{2\pi R}$ where $2\pi R$ is the length of the circle corresponding to the mean radius of the ring; this length may be immediately multiplied by the intensity to obtain the magnetomotive force, or

$$\text{M.M.F.} = \frac{4\pi}{10} \cdot \frac{AT}{2\pi R} \times 2\pi R = \frac{4\pi}{10} \cdot AT$$

Thus the single current round which the lines of the straight wire were looped is replaced in the coil by T turns each carrying A amperes.

If now an iron core be inserted in either the toroid or solenoid, and its M.M.F. is to be determined by the imaginary operation of taking a unit pole round a circuit linked with the coil, theory shows that certain precautions are necessary in carrying out the process. If a very small hole be supposed to be drilled along the direction of magnetization at any place, and the free unit pole be brought to the place in question, it will be acted upon by a certain force in dynes; if on the other hand the incision into the iron takes the form of a transverse cut at right angles to the direction of magnetization, and the free unit pole be placed within the infinitely thin crevasse formed by the cut, it will be acted on by a different force in dynes. It is the former force, or the intensity of the field as measured by the unit pole within the thin tube or hole, which determines the difference of potential for each small element of length; and if the pole be moved round the circuit through the thin tube once, the

work done will be the line-integral of the force, or the magnetomotive force. For the exact nature and reason of the above necessary precautions in determining H in iron the reader must be referred to larger works[1] dealing more fully with the subject; it will only here be added that the value of H as found by the first-described theoretical process is identical with the original intensity of the field, H_o, in the same toroid in air before the introduction of the iron core. Hence the effect of the increase in the number of the lines of induction as compared with the original lines of force in air has been eliminated, and the line-integral of H, determined in the above manner as indicated by theory, for any closed line passing through the iron-cored coil is $\frac{4\pi}{10}$ AT, or the same as for the hollow toroid. No difference in this respect has been introduced by the presence of the iron core, and in general the magnetomotive force of any coil, whether iron-cored or not, is equal to $4\pi/10$ or 1·257 times the ampere-turns.

§ 7. Magnetic flux as related to the magnetomotive force of the circuit.—The magnetic flux is related to the magnetomotive force in such a definite way that the latter may be regarded as the cause of the former. The relative proportion of the amperes to the turns is immaterial, but for the same number of ampere-turns the flux may be very different, according to the nature of the material of which a given circuit is composed. The quantitative relation between the two, or the fundamental equation of the magnetic circuit, is most easily established by again returning to the circular closed toroid uniformly wound over its whole periphery with turns fitting close to the core. Since the lines of flux are in this simple case entirely confined to the inside of the turns, there can be no question as to the area of cross-section of the magnetic circuit; it is $a = \pi r^2$, where r is the radius of the circular core. There is further no doubt as to the exact path of the lines; since each is circular, their mean length is $l = 2\pi R$, where R is the radius of the circular axis running through the centre of the core. Such a toroid therefore supplies us with the type of a perfect magnetic circuit. From its symmetrical shape, the flux-density across any radial section through the coil with which it is overwound must be the same, and if r be small as compared with R, the flux-density may be taken as uniform over the whole area a of the cross-section. The total number of lines is thus $\Phi = B \times a$, when a is measured in square centimetres. Now when H is measured as directed in the preceding section,

[1] For a full treatment of this and other points in magnetic theory, see Prof. Ewing's *Magnetic Induction in Iron and other Metals*, and Dr. H. Du Bois' *The Magnetic Circuit*; and for a clear résumé of the subject within short compass, Prof. Fleming, *The Alternate Current Transformer*, vol. 1, chap. 2 (2nd edit.).

$B = \mu H$, where μ is a definite coefficient for a given material under given conditions, and $Hl = \frac{4\pi}{10} AT$. The complete equation for the magnetic circuit in its simplest form is thus,—

$$\Phi = B \times a = \frac{4\pi}{10} AT \times \mu \frac{a}{l}$$

or if $\xi = 1/\mu = H/B$,

$$\Phi = \frac{1 \cdot 257 AT}{\xi\, l/a} \qquad (1)$$

§ 8. Equation of the simple magnetic circuit. Reluctivity and reluctance.—The denominator of the above fraction bears a close resemblance to the expression for the electrical resistance of a conductor, namely, $R = \rho l/a$, where ρ is the specific electrical resistivity of the material of which the conductor is composed, l and a being its length and sectional area; and considered from this point of view, the fundamental equation is seen to be of the same form as the well-known expression for a continuous current in terms of electromotive force and resistance, namely, $I = E/R$. The comparison of the flow of lines of induction to the flow of a current through an electrical circuit is more, therefore, than a mere verbal illustration; there is a genuine analogy between the two of sufficient accuracy to guide us in the solution of many magnetic problems. The entire path of the lines forms in all cases a magnetic circuit closed upon itself, and having a certain length and area. Through this circuit under the action of a magnetomotive force there flows a stream of lines, and this total number is the quotient of the magnetomotive force divided by the magnetic resistance or (to give this property its distinctive name) the *reluctance* of the circuit. Thus analogous to Ohm's law for continuous currents,

$$\text{electric current} = \frac{\text{E.M.F.}}{\text{resistance}}$$

we have

$$\text{magnetic flux or total number of lines} = \frac{\text{M.M.F.}}{\text{reluctance}}$$

The magnetomotive force is proportional to the number of ampere-turns wound round the circuit, and is the equivalent of the internal electromotive force of, *e.g.*, an electric battery. Both produce a difference of potential, in the one case electric and in the other magnetic, which is gradually expended over the resistance or reluctance of their circuits. The current or the flow of lines is, in fact, the rate of change of the potential over the resistance or the reluctance. The magnetic reluctance of an entire circuit, or of any portion of it, is proportional directly to its length and inversely

to its area. Further, just as different substances under similar conditions have a different specific resistance or resistivity to the passage of an electric current, so the specific magnetic reluctance for unit length and unit area of cross-section, or the *reluctivity*, ξ, varies with different materials.

§ 9. Permeability and permeance.—The different resistivity of various substances may also be expressed in converse terms by saying that all substances do not conduct electricity equally well. In the same way we may say that all substances are not equally permeable to the passage of magnetic flux, and this specific property of *magnetic permeability* is related to reluctivity as its reciprocal, just as electric conductivity is related to resistivity. In many cases of electric circuits it is convenient to deal with conductance instead of with its reciprocal, resistance, and in magnetic circuits it is still more common to speak of the *permeance* rather than of its reciprocal, the reluctance. The fundamental equation may then also be expressed as

$$\text{magnetic flux} = \text{M.M.F.} \times \text{permeance},$$

the expression for the permeance being of the form $\mu \cdot a/l$. Thus $\mu = 1/\xi$ represents the permeability of a substance through which magnetic flux of density B is passing, and when in the particular condition implied by the fact of such a flow of lines.

On the C.G.S. system the permeability of vacuum is unity, and for all non-magnetic substances, such as air, wood, copper or brass, it is also sensibly equal to unity. The same, too, may be said of any of the metals, with the single exceptions of iron, cobalt, and nickel, and certain of their alloys and compounds. With these three exceptions, therefore, we are justified in ranking all substances as on a level, and in classing them as non-magnetic or magnetically indifferent. Even in the case of bismuth, which shows the greatest divergence as compared with a vacuum, the difference is only in the fourth or fifth place of decimals, or as 0·99982 : 1.

But within iron or steel the importance of μ may be judged from the fact that it may amount to as much as 2,500, and the lines of force and of induction are not necessarily identical in either number or direction.

§ 10. Consequences from the relative permeabilities of various substances.—The difference between the high permeability of iron or steel under certain conditions and the permeability of air is of the same order as that between the conductivity of silver or copper and carbon. It is therefore evident that the great contrast between the conductivities of metals and bad conductors or insulators, such as indiarubber, cannot be matched by any equal difference between the permeabilities of different substances. There are in fact no magnetic insulators, bismuth itself, which is the least permeable,

being only very slightly inferior to air. When air, gun-metal, brass, or wood are spoken of and often used as magnetic insulators, it must be remembered that they are so only to a comparatively small degree. At the best their permeability to magnetic flux is only some two or three thousand times less than that of iron, and at the worst this difference may sink to practically nothing. Yet it is noteworthy that in spite of this the distinction between the magnetic and non-magnetic classes is more sharply defined than between good electric conductors and insulators. Out of the three magnetic substances the magnetic quality of cobalt and nickel, though sensible, is not to be compared with that of iron or steel, so that practically the contrast lies between iron with its alloys on the one side and all remaining substances on the other side.

Next, it may be noted that the flux-density obtainable from an electromagnet with an iron core is much greater than that due to any permanent magnet, and hence the total flux from an electromagnet is much greater than can be obtained from any combination of permanent magnets, when both are of reasonable dimensions. Owing to this fact the magnetic field which must be present in any dynamo is now always produced by means of electromagnets, strongly excited only when the dynamo is in use; so long as the electrical engineer was dependent on permanent magnets for supplying him with a magnetic field, the powerful dynamos of the present day were impossible.

§ 11. Analogy of magnetic reluctance and electric resistance not exact.—On one point a warning must at once be given as to a vital distinction between the magnetic and electric circuits. The passage of an electric current through a conductor does not alter its specific resistivity; it may raise its temperature and thereby alter its length and its resistivity, but when the secondary effects of temperature have been eliminated or corrected for, the resistance of the conductor has been proved to be the same whether a current of a milliampere or of many thousands of amperes is passing. On the other hand, the permeability of a magnetic substance is dependent on the value of the induction, or, in other words, on the density of the flux which it is at the moment passing. It is not, therefore, a constant quantity, and our definition of $\mu = B/H$ was so worded as to imply that the value of μ could not be given unless at the same time B was specified. Further, the value of μ for the same induction depends upon how that value has been reached, and lastly, is affected by the temperature of the iron. On these and other points more will be said in Chapter XIV.

§ 12. Magnetic reluctances: (a) in series and (b) in parallel.—The example on which was based the fundamental equation of the magnetic circuit was a closed ring of iron, of the same nature and of the same cross-section throughout its length. But a magnetic

circuit may also be made up of different materials having different permeabilities; these again may have different areas of cross-section and different lengths, while further they may be arranged either so that each line of induction in its closed path traverses them in succession (*i.e.* they are "in series") or so that the path followed by some lines is different from that of others, either entirely or at some portion of its length (*i.e.* they are "in parallel"). In such cases the analogy of the electric and magnetic circuits, although not perfect, yet is useful as affording a ready clue to a mode of treatment which is very closely correct in most practical instances. Just as the total resistance of a number of electric conductors in series is the sum of their separate resistances, so if a circuit or a portion of a circuit is divisible into sections having different lengths, areas, and permeabilities, and these are traversed by the same group of lines in succession, the total magnetic reluctance which must be overcome by the magnetomotive force is the sum of the separate reluctances of the different sections; *e.g.* if $l_1 a_1 \mu_1$, $l_2 a_2 \mu_2$, $l_3 a_3 \mu_3$ . . . . . are the respective lengths, cross-sections, and permeabilities of the different portions, their total magnetic reluctance is

$$\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \ldots = \frac{l_1}{a_1\mu_1} + \frac{l_2}{a_2\mu_2} + \frac{l_3}{a_3\mu_3} + \ldots$$

Similarly the total permeance is the reciprocal of the sum of the reluctances, or

$$\mathcal{P} = \frac{1}{\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \ldots}$$

Next, if at any point in their path the lines of a group separate and follow different parts or sections of the total area of the magnetic circuit, the law which they obey is analogous to the law of electric circuits in parallel, namely, that the number of lines flowing through any section of the magnetic circuit is the quotient of the difference of magnetic potential between opposite ends of the section, and the reluctance of the section; whence it also follows that when there are several paths for the lines and the same difference of magnetic potential exists between their ends, the total flux divides between the several paths directly as their permeances, as determined by the respective lengths, cross-sections, and permeabilities of the different portions; and the joint permeance of the parallel paths is simply the sum of their several permeances, or

$$\mathcal{P} = \frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2} + \frac{1}{\mathcal{R}_3} + \ldots$$

§ 13. Example of a magnetic circuit with leakage.—As an illustration of these laws, let ampere-turns be grouped entirely on

one side of a closed iron ring composed of a curved electromagnet and an armature or keeper A (Fig. 5, I): the arrangement may be likened to a battery, shown at its side, of which each cell represents a current loop, and the ends of which are joined by a thick piece of copper wire R_e: there is a rise of magnetic potential as we pass through the loops from south to north, analogous to the rise of electric potential through the cells of the battery; and since the ends between which there is a difference of potential are joined so

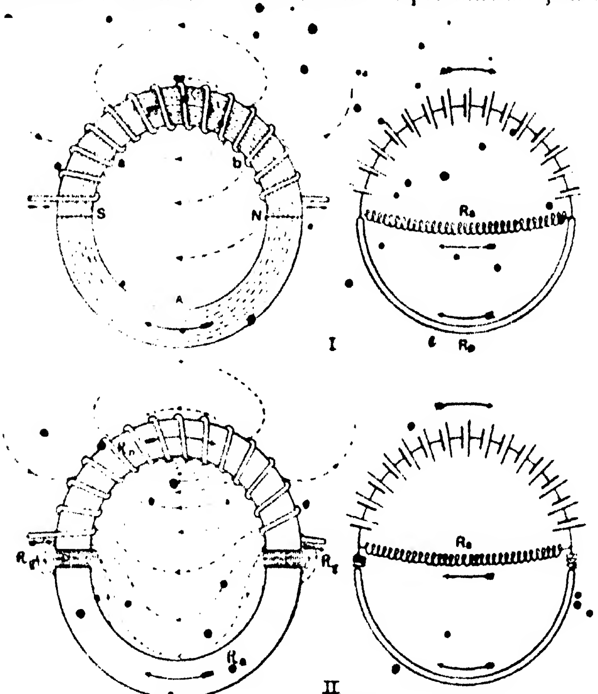


FIG. 5 Analogy of magnetic and electric circuits.

as to form a closed circuit, there is a flow of lines or of current round the circuit which within the magnetic or electric battery is from the – or S. end to the + or N. end, but without is from the + or N. to the – or S. end. But at this point there enters a very important difference: as there are no insulators for magnetism, it follows that, if between *any* two points there exists a difference two of magnetic potential, there will be a flow of lines between those points equal to the difference of potential divided by the magnetic reluctance of the path, whatever be the substance of which it is composed. The flux will therefore not be confined entirely to the

iron, as it was in Fig. 4, but will pass across from the N. end of the magnet to the S. end through the air as well as through the iron of the keeper. Thus there will be a flow from, *e.g.* *b* to *a*, since *b* is at a higher magnetic potential than *a*, *i.e.* from one portion of the magnet to the other; such flow will increase as we pass away from the centre of magnet towards its ends, and will again decrease as we near the centre of the keeper. If, therefore, the analogy between the electromagnet and the battery is to be retained, we might regard the latter as immersed in a conducting liquid or electrolyte, through which currents may flow across even from one cell of the battery to another in addition to the external current through the wire joining the terminals. Such currents would be a leakage as far as useful current in R_e is concerned, and in the same way the flux in the air of Fig. 5, I, and in similar cases, is frequently spoken of as *magnetic leakage*, stray lines, or waste field from the point of view that only lines passing through the keeper or armature are useful and desirable. The whole illustrates the case of two or more magnetic paths joined together "in parallel" to form a composite magnetic circuit. And this leads to a second point which may again be emphasized: it is that, owing to the non-existence of magnetic insulators, it is impossible to define the exact limits of the magnetic circuit or the various paths followed by the lines, except in a few simple cases such as a closed toroid. Consequently the equation of the magnetic circuit, which demands a knowledge of the exact length, area, and permeability of the different portions of the magnetic circuit, only admits of an approximately accurate application. In the case of the toroid uniformly wound over its whole periphery there was no doubt about the dimensions of the magnetic circuit and its permeance, all the lines were confined entirely to the iron, and the number flowing through any section across it was identical; there was, in fact, no polarity and no "leakage" of lines out of the loops and across the air. The reason is that all points of the ring-core are at the same magnetic potential, since the magnetomotive force supplied by the loops over any length just suffices to pass the total number of lines through that length. It may be likened to a closed loop of wire in which an E.M.F. is generated, the amount generated per unit length of the wire being uniform throughout the whole circuit; a current flows under the E.M.F., but all points of the loop are at the same potential, since the E.M.F. developed in any length just suffices to pass the current through the resistance of that length.

But now in the new case of an iron ring, in which the winding is grouped on one side only, the total magnetic reluctance of the circuit is a complex combination of the internal reluctance of the magnet, the reluctance of the armature, and the reluctance of the external air-paths. How, then, is the reluctance of the air to be

calculated, since the paths of the lines through it are so manifold and differ so much in their length? They flow not only from the extreme ends, but across from all points of the magnet, so that the number of lines carried by the magnet, and therefore their density within the iron, is continually varying; at its centre they are a maximum, but thence they gradually leak out on all sides, especially towards the ends, so that a smaller number passes through the iron of the keeper than through the magnet. The calculation can most readily be performed by certain assumptions which are only approximately true. We must represent the whole by an electric battery, between the terminals of which there are two paths, one of a low resistance and one of high resistance R_s (Fig. 5, I), placed as a shunt to the other. The second represents the reluctance of the air, which is very high as compared with that of the iron.

For the sake of simplicity, therefore, the reluctance of the air is regarded as being in parallel with the reluctance of the keeper, as if all the lines passed right through the iron of the magnet from one end to the other, and then at its ends divided into two groups, some going through the keeper and some through the air, these latter being the leakage lines, which are thus supposed to issue forth from and enter into the magnet only at its ends. On this assumption, then, if $\mathcal{R}_a$ and $\mathcal{R}_l$ are the reluctances of the two paths, afforded respectively by the keeper and the air, the external reluctance, being their joint reluctance in parallel, is $\dfrac{\mathcal{R}_a \times \mathcal{R}_l}{\mathcal{R}_a + \mathcal{R}_l}$ and this is in series with the internal reluctance of the magnet $\mathcal{R}_m$. The total reluctance, therefore, of the entire magnetic circuit is

$$\mathcal{R}_t = \mathcal{R}_m + \frac{\mathcal{R}_a \mathcal{R}_l}{\mathcal{R}_a + \mathcal{R}_l}$$

and the total number of lines produced by a given number of ampere-turns encircling the circuit is

$$\Phi = \frac{0{\cdot}4\pi AT}{\mathcal{R}_m + \dfrac{\mathcal{R}_a \mathcal{R}_l}{\mathcal{R}_a + \mathcal{R}_l}} = 0{\cdot}4\pi AT \times \mathcal{P}_t$$

where $\mathcal{P}_t$ is the reciprocal of the total reluctance.

$\mathcal{R}_a$ is equal to $l_a/\mu_a a_a$, where l_a, a_a and μ_a are the mean length, area, and permeability of the iron of the keeper, and $\mathcal{R}_m$ similarly $= l_m/\mu_m a_m$. The reluctance $\mathcal{R}_l$ of the air-paths, or their permeance $\mathcal{P}_l$, can be calculated by assigning a certain mean length of path and a certain area of cross-section to them; or we may determine experimentally the number of lines flowing through the middle of the magnet Φ_m, and the number flowing through the keeper, Φ_a; the difference between the two, $\Phi_m - \Phi_a$, gives us ϕ_l, or the number of lines which leak through the air.

On the same assumptions the total number of ampere-turns required in order to produce a given number of useful lines, Φ_a, through the iron of the keeper can also be calculated; just as the product of a current and a resistance through which it flows gives the number of volts which must be applied to the ends of the resistance in order to produce the current, so the product of a number of lines of flux and a reluctance through which they flow gives the difference of magnetic potential which must exist between the ends of the reluctance in order that the given number of lines may flow through it, i.e. M.M.F. $= \Phi \mathcal{R}$. Hence the difference of magnetic potential which must exist between the ends of the armature in order that Φ_a lines may pass through it is $\Phi_a \times \mathcal{R}_a = 0{\cdot}4\pi AT_1$. But this is also the difference of magnetic potential under which the leakage lines are assumed to flow; consequently, if we multiply $0{\cdot}4\pi AT_1$ by $\mathcal{P}_l$, the permeance of the air, the product is the number of leakage lines, which we call ϕ_l; the sum $\Phi_a + \phi_l = \Phi_m$ is the number of lines flowing through the magnet, and the difference of magnetic potential required to drive this larger number through the magnet is $\Phi_m \times \mathcal{R}_m = 0{\cdot}4\pi AT_2$. The total magnetomotive force required is therefore $0{\cdot}4\pi(AT_1 + AT_2) = 0{\cdot}4\pi AT$.

§ 14. Example of an imperfect magnetic circuit with double air-gap. Next, let the keeper be held at a little distance away from the magnet, so as to interpose in the magnetic circuit two equal air-gaps, one at either end of the keeper, with parallel faces and having a certain definite length (Fig. 5, II).

At once the total flux produced by a given number of ampere-turns wound on the magnet is enormously decreased, owing to the high reluctance of the two air-gaps. The effect may be repeated in the case of the battery if we add in series to each end of the external portion of the circuit a short length of wire having a very high resistivity, but of the same diameter as our original external resistance: these will reduce the total current, and the current through R_i will bear a much larger proportion to the current through R_e than formerly, since the two resistances R_e and R_i have become more nearly comparable. In the same way the number of lines through the armature is greatly decreased by the interposition of the two air-gaps, but the magnetic reluctance of the rest of the surrounding air remains the same as before, and the number of lines therein will be the same if the difference of magnetic potential at the poles of the magnet remains the same; the proportion, therefore, which they bear relatively to the number in the armature or to the number in the magnet is very much increased. Before, almost all the lines passed round the magnetic circuit entirely through iron; now, a much larger proportion go partly through the surrounding air. The two air-gaps being in series with the iron of the keeper, the total reluctance of the three

is the sum of their separate reluctances. Hence, if the reluctance of one air-gap be $\mathcal{R}_g$, and they are both equal, the total reluctance $2\mathcal{R}_g + \mathcal{R}_i$ must be substituted for $\mathcal{R}_i$ in the equations for the number of ampere-turns or lines of flux. It might be thought that it would be easy to calculate $\mathcal{R}_g$, since the exact length of each air-gap can be measured, as also the exact area of the parallel faces of the iron where the circuit is divided. But here again a caution is required; although most of the lines pass straight across from face to face of the gap in the iron, yet at the edges they spread outwards, and pass across by curved paths, which form a kind of "fringe," gradually shading off in density. Consequently the exact area and length of their paths is *not* known, and an allowance for the fringe has to be made by the aid of theory and experiment.

The imperfect magnetic circuit illustrated above, in which the iron is completely divided by two air-gaps, is typical of the magnetic circuit in most dynamos. For the production of the E.M.F. there is required as large a total flux as possible, within a small space, and produced with a reasonable expenditure of electrical energy in the coils through which the magnetizing current flows. An electromagnet with an iron core must, therefore, be employed in order to produce the field, and in order to obtain as many lines as possible round a whole magnetic circuit or through any portion of it with a given excitation in ampere-turns, the length of the path under consideration must be as short as possible, its area large, and its permeability great. But we cannot have a closed magnetic circuit entirely of iron, since there must be at least one and usually there are two air-gaps in each circuit, to permit of the continuous motion of one portion of the magnet system or of the electric conductors alone or of a portion of the magnetic circuit carrying with it the electric conductors, relatively to the rest of the system.

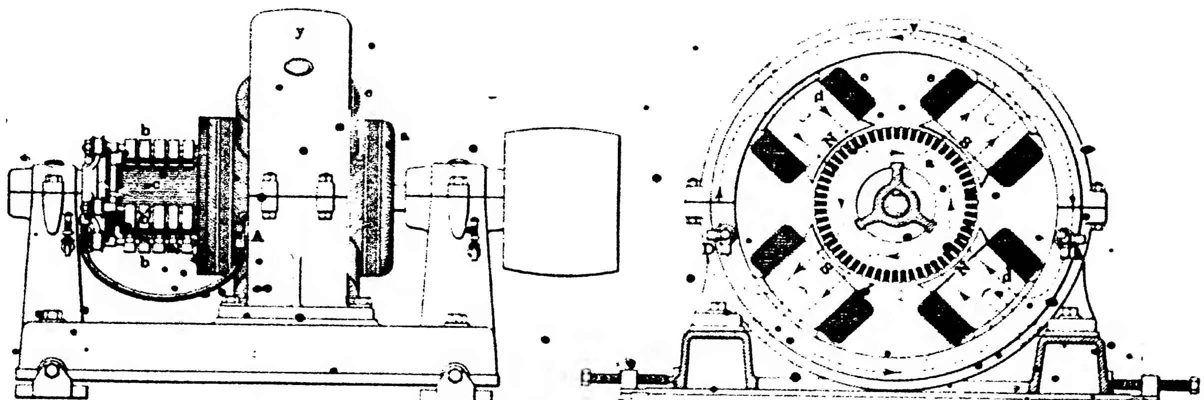
CHAPTER III

THE CLASSIFICATION OF DYNAMOS

§ 1. The two structural portions of the dynamo.—The variety of different forms which the dynamo may take is very great, yet in all there may be traced the two structural portions which the dual nature of the machine requires, and between which there is to be relative motion.

There is, first, the *field-magnet*, the function of which is to serve as the path for the lines of flux constituting the field, and which is therefore composed entirely of iron. It may usually be divided into three parts, and these can be distinguished in Fig. 6, which shows a typical four-pole belt-driven machine of small size. They are: (1) the magnet poles or cores (*dd*), on which the exciting coils (*ee*) are wound or placed, and through which the lines pass up to or away from the pole-pieces; (2) the "yoke" (*y*), which joins together the poles of the magnet; and (3) the pole-pieces themselves (*N*, *S*). These latter by reason of their particular shape, cause the lines to issue forth into and pass through the air-gaps in such directions and with such density as will best adapt them to be cut by the active conductors.

Secondly, there is the *armature* (*a*), which consists of a number of conductors—wires or bars—almost invariably of copper, which are systematically arranged and connected in a definite and particular order; in the dynamo of Fig. 6 they form portions of a large number of loops or coils of wire, each of which is entirely insulated from the structure upon which they are wound and also from every other coil, save where the end of one coil is electrically joined to the beginning of the next. In the active conductors E.M.F.'s are generated, and by their grouping are added together in series or placed in parallel, or in general brought into practical use at certain definite points or "terminals" (*A*, *D*) to which the external circuit is applied. When we leave the sphere of theoretical diagrams, it is evident that the active conductors must be supported in some way, and in most cases they are in practice wound upon, or embedded in slots in the surface of, a cylindrical iron structure, which itself forms part of the magnetic circuit and is traversed by the lines of flux; it is this structure of iron to which, in the first instance, was given the name of the armature or "keeper" of the magnet or magnets employed to produce the field. Thus in the ordinary two-pole dynamo the cylindrical armature of iron (Fig. 15) may be likened to the "keeper" placed between the



Side Elevation.

End Elevation in Part Section.

FIG. 6.—Four-pole continuous-current belt-driven dynamo.

two poles of a simple horseshoe magnet; but from its intimate connection with the electrical conductors which it supports, the term has now been extended to cover the system of conductors, the iron portion itself being called the "core" of the armature.

§ 2. Rotary motion of the active conductors or field-magnet.—The student will doubtless be familiar with the idea of an E.M.F. as being generated by movement of a conductor through a magnetic field so as to cut the lines of flux. Although more will be said on this subject in Chapter V with especial reference to the toothed armature, it will for the present introductory purpose be sufficient to assume such "line-cutting" as the source of the E.M.F. of the dynamo, since from it may correctly be obtained the necessary criteria upon which to classify the several types of machine.

Evidently simple rectilinear motion of a conductor in one and the same direction across a magnetic field between two pole-pieces (*cp.* Fig. 30) cannot practically be maintained for any length of time, since the length of the pole-pieces would have to be infinite in order to provide the infinitely long magnetic field. But the definition of the dynamo demands "continuous relative motion," and hence, when the conductor has been moved to one end of the magnetic field, its direction of movement must be reversed. We thus pass to *oscillatory or reciprocating motion*, first in one direction and then back again in the opposite direction. Such a motion would be obtained by attaching the active conductor directly to the piston of a steam engine; it would give an E.M.F. alternating in direction, and if the field were of uniform density the value of the E.M.F. would vary as the speed of movement, reaching a maximum during the middle of the stroke, and reversing at either end. But the arrangement is not convenient,[1] since it is not easy to secure a speed high enough to give an appreciable number of volts with any practicable length of wire and density of field in the air-gap. No such difficulty meets us when we have recourse to *rotary motion*, and hence in all practical dynamos, without exception, it is by rotation that the "continuous relative motion" of field and conductor is secured, and an E.M.F. continuously generated.

Thus in Fig. 6 the field-magnet is stationary but its four pole-pieces, alternately N. and S., are bored out so as to embrace the circumference of the rotating armature *a*. In other cases the armature is itself stationary and forms the external "stator," as it is called, within the bore of which revolves the magnet system or "rotor."

§ 3. Bases for classification of dynamos.—In the classification of dynamos several bases of division present themselves, each of

[1] It was used in the "oscillator" devised by Tesla (*Electr. Eng.*, vol. 13, p. 83) for the production of small currents at high frequencies.

which has its advantages in throwing light upon the affinity of one machine to another. It might be thought that the distinction between dynamos yielding an E.M.F. and current always in the same direction round the external circuit, and those yielding an alternating E.M.F. and current in the external circuit, would be the most fundamental; yet it will be found that of two so-called "continuous-current" dynamos giving a current unidirected in the external circuit, the one may be entirely distinct from the other in its whole nature and structure. As a matter of fact, dynamos which primarily and in themselves give an E.M.F. and current continuously in the same direction are few and comparatively unimportant—at least up to the present time.

The fundamental questions to be asked and answered when considering any form of dynamo are as follows—

(1) Does the E.M.F. produced in each active conductor always have the same direction along its length, or does it alternate by reason of the same lines being cut twice over in each revolution in alternate directions?

(2) Is the active length of the conductors (*a*) parallel to, or (*b*) at right angles to, the axis of rotation?

The first deals with the question whether each active conductor is the seat of an alternating E.M.F. The second, dealing with the position of the length of the active conductor and the direction of the field relatively to the axis of rotation, determines the form of field-magnet required and the structure of the armature core; from (*a*) arises the "axial" or cylindrical type of dynamo, from (*b*) the "radial" or disc type.

Thirdly and lastly comes the question whether the external voltage and current are alternating or uni-directed. Although of the greatest importance so far as the use of the dynamo is concerned, it is not so fundamental in the theory of its action as shown *e.g.* by the rotary converter, which under suitable conditions can supply from the same armature either direct or alternating current to an external circuit.

§ 4. Dynamos divided into (I) homopolar and (II) heteropolar.—By the first criterion, dynamos are divided into Class I, *homopolar* dynamos, or those in which each active conductor, when cutting lines, is always cutting them in the same direction, and therefore the E.M.F. induced in it is always in the same direction along its length, and Class II, *heteropolar* dynamos or those in which each active conductor in each revolution cuts the same lines twice over in opposite directions—once as they issue from a N. pole and once as they enter into a S. pole—so that the E.M.F. induced in it alternates in direction.

In the *homopolar dynamo* (Class I) there is a single field, and from the nature of the flux in an imperfect magnetic circuit divided

by an air-gap, it exists between a pair of poles; but this single field is only cut once in each revolution, and the cutting always takes place in the same direction, so that there is no reversal of the direction of the E.M.F. along the length of a conductor. Since the lines are always cut in the same direction, it may be said that they are cut as *e.g.* they emerge from a N. pole; the necessary presence of the second or S. pole may therefore in imagination be left out of sight, since it is not wanted to concentrate and collect the lines for them to be cut a second time in the same revolution, and hence the type of dynamo of Class I has also been termed "unipolar." The fundamental principle of the homopolar dynamo is illustrated by Fig. 7, which shows a single conductor *ab* rotated between two

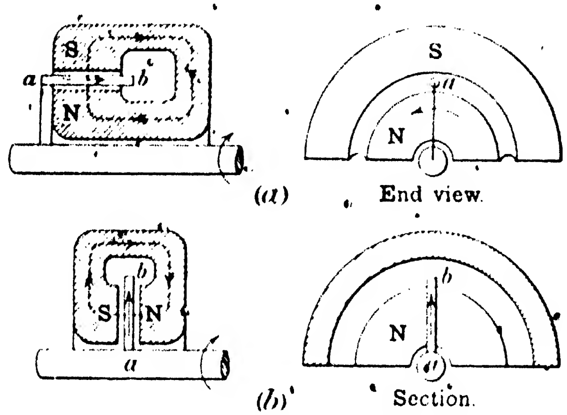


FIG. 7. Homopolar dynamos of (*a*) and (*b*) types with single active conductor.

curved pole-pieces, its length being either (*a*) parallel to the axis of rotation, or (*b*) at right angles thereto. The E.M.F. generated in both cases will always be in the same direction along the conductor's length, and it will last so long as it is moving past the pole faces, quickly dying away practically to zero in the neutral gap. There is, however, nothing to forbid the single field of the homopolar or unipolar field-magnet from being divided into two or more groups of dense lines or "fields," separated by as many gaps of very weak density, each such field corresponding to a polar projection. It may thus become in a sense multipolar, but the fields are always of the *same* sign, *i.e.* have the same direction, and an active conductor never cuts the same lines twice in a revolution, so that the dynamo never loses its distinctive "homopolarity."

In *heteropolar dynamos* (Class II) successive fields are of *different* sign, corresponding to a N. and S. pole in a bipolar dynamo or to

N. and S. poles occurring in alternate succession in the multipola dynamo. Each group of lines belonging to a distinct magneti circuit is therefore cut twice by each active conductor in eac revolution, opposite to a N. and S. pole-face respectively, and th E.M.F. in each conductor is alternating. Fig. 6 illustrates heteropolar dynamo of the (*a*) or axial type.

§ 5. Homopolar dynamos with field uniform in the path of th movement (Class I. i).—Let the pole-faces in either of the two type of Fig. 7 be extended to cover the whole circle of the path traversed by the rotating conductor, which will thus become continuously active, *i.e.* let the pole-faces be made cylindrical in case (*a*), or annular in case (*b*). We thus reach homopolar dynamos in which the field is of uniform density in the path of movement as required by

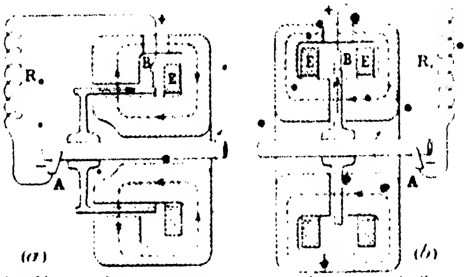


FIG. 8. Homopolar continuous current dynamos of Class I. i. with single active element

the heading of the present section. The flux-density and the rate of line-cutting may vary along the length of the conductor, but it will be uniform and unvarying along any circle concentric with the axis of rotation. The E.M.F. of the conductor will therefore with constant speed be maintained continuously at an absolutely constant value without break or fluctuation.

A number of conductors can be similarly arranged in a circle round the axis of rotation in the (*a*) type parallel to the axis, and in the (*b*) type radially thereto. When connected in parallel or touching one another, a larger current can be carried by them without increase of the E.M.F. Finally, when the whole circle is filled with contiguous strips in parallel with each other, we arrive in case (*a*) at a tube or hollow cylinder of copper concentric with the shaft, and in case (*b*) at a solid circular disc of metal. If connection to a stationary external resistance R_e is then made by brushes which press on and so make contact with the back and front ends of the cylinder or outer and inner peripheries of the disc, homopolar dynamos are obtained which give a continuous unidirected and constant E.M.F. and current, both externally and

within the armature itself, without the need for any "commutator" c, such as forms part of the continuous-current heteropolar dynamo of Fig. 6.

Fig. 8 shows the complete cylindrical or annular poles with their exciting coils E, E, combined with a complete cylinder and disc, as the single active element forming the two fundamental types of the homopolar continuous-current dynamo. As shown, the stationary brushes A and B press on the shaft and on the end of the cylinder in type (a) or the outer circumference of the disc in type (b). To the latter type belongs the original disc dynamo of Faraday shown in the frontispiece—a type which still survives in essentials in certain ampere-hour meters. Stationary brushes can be placed on the periphery of the disc or on either end of the cylinder at as many points of collection as are desired.

The present type of dynamo with uniform field is not, however, confined to the case of the complete cylinder or disc. It also covers separate conductors, forming parts of a cylinder or disc but insulated from one another. In this case each conductor, in order to maintain contact with the stationary brushes, must be connected to slip-rings on which the brushes may continuously press as the active conductor rotates. By the use of such slip-rings, it becomes possible to connect two or more active conductors in series so as to add up their E.M.F.'s, as will be further explained in Chapter VII.

The reason for treating the above homopolar dynamos in which the field is uniform in the path of movement as a separate group is that the fact of the pole-faces forming a complete circle, cylindrical and concentric with the shaft or annular in a radial plane, carries with it an important consequence, namely, that it is only by the use of sliding contacts and slip-rings that the E.M.F. of two or more separate conductors can be added up (*see* § 4 Chapter VII). There is therefore no method of *winding* the armature of such a type of machine in the true sense of the word.

§ 6. Homopolar dynamos in which the field is non-uniform in the path of movement (Class I. ii).—Reverting to Fig. 7, if, as there shown, the pole-faces are *not* complete circles, let a second similar conductor be added in the gap on the opposite side of the shaft, and let a connecting wire be taken peripherally round to join the two b ends. From the a end of the second conductor a connection can be taken to the a end of a third conductor. The E.M.F. of the third conductor can then be added to that of the first, the counter E.M.F. of the second conductor which acts as a connecting piece being very small owing to its position in the very weak field of the region beyond the pole-face. Thus a loop can be formed, and by such loops without the use of slip-rings, it becomes possible to connect two or more active conductors in series by a true winding. The loop in the homopolar machine does not itself join two active

conductors in series; it is only a means by which this may subsequently be done. It will now be noticed that as rotation proceeds, the two sides of the loop change their rôle; the conductor, which first serves as a simple connector, itself enters the field within the air-gap and generates E.M.F., while the former active conductor becomes a simple connector. Thus there is never more than one active conductor at any one time in each loop, and as the E.M.F. generated in each conductor is always in the same direction along its length, it *alternates in direction round any complete loop*. The characteristic feature of the preceding Class I. i. is thereby entirely lost, since the homopolar dynamo is now made to yield an alternating E.M.F. Further, it now becomes immaterial whether it be the field-magnet or the armature that is rotated. The same E.M.F. is in either case generated, and the former is the arrangement

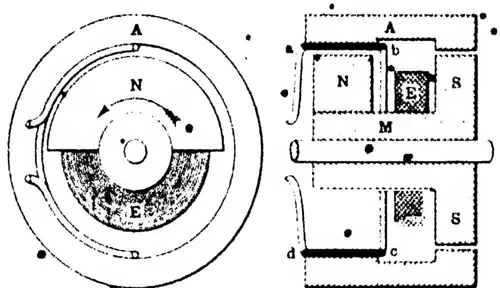


FIG. 9.—Homopolar drum alternator of Class I. ii.

that has usually been adopted in practice. In Fig. 9 is therefore shown a homopolar machine of type (*a*) with a single rotating polar projection N, and a stationary external armature wound with one *drum* loop. The magnet is excited by the coil *E* which may either rotate or be stationary, and at one side has a polar projection with semicircular face between which and the stationary armature core there exists a strong magnetic field. The remaining half of the internal circle of the armature core which at any time is not covered by the pole forms a neutral space through which few or no lines pass. At the other end the second break or moving joint in the magnetic circuit at S. has as short a length of air-gap as is found practicable, and the necessity for it is only due to the fact that it is mechanically convenient to attach the loops to a stationary portion *A* of the iron circuit. The lines of the flux are, as it were taken out of the end of the magnet, and so do not cut the conductors a second time. While the rotating field is sweeping across one side of the loop at *ab*, and generates an E.M.F. in it, the other side *cd* is a simple connector. But immediately afterwards the inactive

connecting piece will itself become active when in its turn it is cut by the flux, while *ab* will cease to be active. As the field rotates, one side of the loop is active for one-half of the revolution, and then during the other half of the revolution becomes a simple connector, or in general with a number of loops the same conductors serve either purpose alternately. A homopolar drum alternator is thus obtained when the ends *ad* of the complete loop are brought to a pair of terminals.

The loops must have a span at least equal to the angular width of the pole; otherwise there would be no distinction always present between the wires which are active for the time being, and those which are mere connectors. If this distinction be not maintained, and both sides of the loop are allowed to be cut by the flux simultaneously, the E.M.F.'s generated in the two sides act in opposition

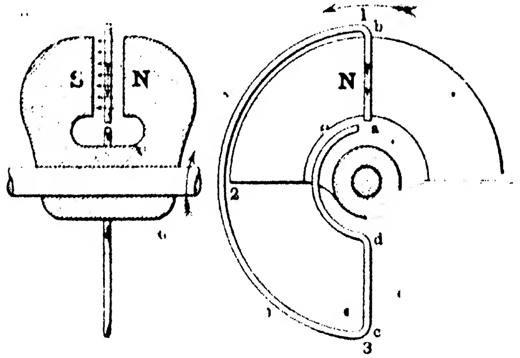


FIG. 10. Homopolar disc alternator of Class I. ii.

to each other, or, as it is termed, act "differentially," the net resultant E.M.F. being simply the difference between the two. Any such differential action is therefore detrimental to the full use of the machine. For the same reason it is not electrically permissible for the width of the field, *e.g.* with a single polar projection, to extend over more than half the circumference of the path traversed by it; *i.e.* the neutral space must be at least equal in peripheral width to the polar projection.

Fig. 10 represents similarly a homopolar disc alternator of (*b*) type with stationary armature. While the leading edge of the rotating field-magnet moves from 1 to 3, no lines traverse *cd*, and therefore no E.M.F. is induced in it; but when by further rotation the leading edge moves from 3 to 1, *ab* does not in turn cut any lines itself, and consequently the E.M.F. is induced only in *cd*.

§ 7. Heteropolar dynamos (Class II) of (a) type.—Passing to heteropolar dynamos of Class II, in the first place those in which

the length of the active conductors is *parallel to the axis of rotation*, *i.e.* the (*a*) or axial type, will be considered. The fundamental active element of an armature of this type takes the form of a straight conductor mounted on an iron core with its length parallel to the axis of rotation (Fig. 11). In order to reduce the interferric or air-gap space as much as possible, the inner surface of the pole-pieces is bored out so as to follow the path traversed by the rotating conductor, and the latter is mounted on a cylindrical iron armature, only such "clearance" being allowed as is required from considerations of mechanical safety and satisfactory working. There are thus only two short air-gaps, and the density under the pole-pieces will far exceed that of any straying lines between the pole-tips. Besides

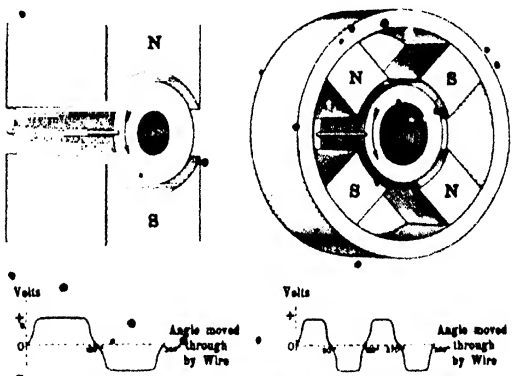


FIG. 11. Simple active element of heteropolar dynamos of (*a*) type.

its function of reducing the exciting ampere-turns to a minimum for a given number of lines by shortening the length of the air-path, the cylinder also serves as a supporting structure on which the active conductors can be secured. When the armature is the part that is to be rotated, and the iron core is used as a support for the wires, it must itself be rotated with them, although on any other ground this is entirely unnecessary. The problem of rotating the winding only round a stationary iron core has possessed great attraction for inventors; the difficulties inherent to it have, however, proved too great, and the mechanical simplicity of rotating both core and winding has led to its universal adoption. Apart from certain secondary and minor effects which need not here be considered, the rotating of the iron has no effect on the field of flux. But owing to the insertion of the iron core the lines enter the core nearly radially, and are distributed over the arc embraced by the pole-piece

almost uniformly (Fig. 12). Upon the exact distribution depends the shape of the curve of instantaneous E.M.F. In order to determine this, the value of the field-density in the air-gap or B_g for each point round the circumference of the armature core must be known; at the foot of Fig. 12 this is plotted from a starting-point in the centre of an interpolar gap, and the same curve to some other scale will also represent the curve of the instantaneous induced E.M.F. for one revolution at a constant speed of rotation (Fig. 11). If the density of the lines were strictly uniform over the polar arc and confined thereto, their directions at all points being exactly radial, the curves of flux-density and of E.M.F. would consist of a series of detached rectangles; but in reality the corners are rounded off by the existence of a weak field or "fringe" of lines at each pole-tip which is not truly radial, and the curve of induced E.M.F. for each half-period is flat-topped with sloping sides, or intermediate between a sine curve and a rectangle (Fig. 11).

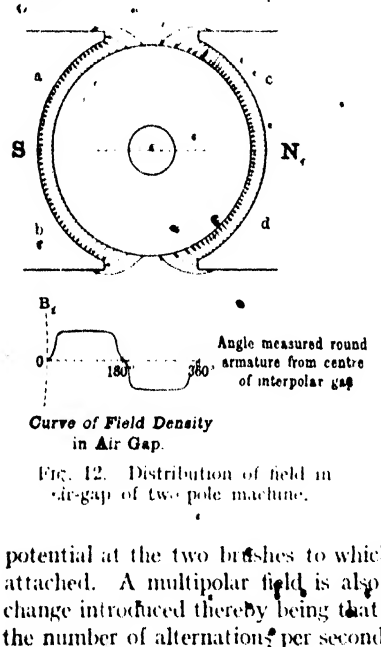


FIG. 12. Distribution of field in air-gap of two pole machine.

If the ends of the active conductor are electrically connected with two collecting rings fixed on the shaft and insulated from it, against which stationary brushes are pressed, the elementary *heteropolar alternator* is obtained, yielding an alternating difference of potential at the two brushes to which an external circuit can be attached. A multipolar field is also shown in Fig. 11, the only change introduced thereby being that for a given speed of rotation the number of alternations per second is increased in proportion to the number of pairs of poles.

But the E.M.F. generated by one single conductor of reasonable length thus rotating in a field of high density, and at as high a speed as is practicable, is very small, not amounting to more than a few volts at the most; hence in almost all commercial dynamos there is a number of active conductors, and they must be connected in series so as to add up their separate small E.M.F.'s. If arranged symmetrically round the armature core (Fig. 13) parallel to our original conductor, at any moment each of the conductors moving under the N. pole has an E.M.F. induced along its length, in the

opposite direction (as viewed by an observer at either end) from the E.M.F. induced in each of the wires moving under the S. pole, as shown by the crosses and dots in Fig. 13.[1] This fundamental fact must be the guide to any method by which the conductors can be connected together electrically in a useful manner.

Two methods of adding together the inductive action of a pair or more of elements at once present themselves; by the first, conductor 1 of Fig. 13 is connected in series with another conductor such as 2′, which in the bipolar machine is situated nearly diametrically opposite on the other side of the core, by the second it is connected with another active conductor, 2, which is next to it on the surface of the core or lies immediately above it, and under the same pole-piece.

§ 8. (1) Drum winding.—The first or *drum* method is identified with the name of "Siemens." In its complete form it was first introduced in 1871 by von Hefner Alteneck as a modification of the original Siemens shuttle-wound armature. It is the simplest method, and consists in joining the farther end of one active conductor by a connecting piece of wire passing across the end of the core to the farther end of another conductor situated nearly diametrically opposite to the first and under the opposite pole-piece (Fig. 14).

FIG. 13.—Active conductors parallel to axis of rotation.

The E.M.F.'s induced in the two active conductors will assist each other round the loop thus formed, being in opposite directions along their length as viewed from either end. The core now becomes, as it were, a cylindrical "drum," which may, if necessary, be solid right down to the shaft. By bringing the wire across the near end of the drum, *i.e.* by the addition of a second connector, the loops may be multiplied, the end of one loop forming the starting-point of the next. It will be seen that the winding of Fig. 9 is also an instance of a drum winding on an external stationary armature, but as applied to a homopolar machine.

§ 9. (2) Ring winding.—The second or *ring* method was employed by Pacinotti in 1860, and described by him in 1865, but is also frequently called by the name of the French electrician Gramme, who reintroduced it in 1870. If the iron core, which serves to guide the lines through the space between the pole-pieces,

[1] The marking of the direction of currents or E.M.F.'s by crosses and dots, according as they are away from or towards one's point of view, is a useful convention; it is easy to recall if we consider the cross to represent an end view of the feathers of a receding arrow, and the dot to represent its advancing point.

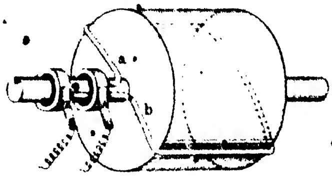
be supported at some distance radially from the shaft by means of an open hub or spider of some non-magnetic metal, such as gun-metal, a connecting piece of wire can be brought through the inside space between the core and the shaft, passing between the arms of the hub; by it the far end of wire 1 (Fig. 13) can be connected to the near end of 2, the wire next to it on the surface of the core, and the whole arrangement is shown in the ring-wound armature of Fig. 14. Without the iron core the flux would go straight across the gap from pole to pole; the inner connector would therefore be cutting lines as well as, and in the same direction as, the outer active wire, and an E.M.F. would be produced in each. These two E.M.F.'s thus acting round the loop would oppose and tend to neutralize each other, so that the net E.M.F. would be merely the difference between the two. But with the iron core interposed the lines take a more or less curved path through the mass of the iron, as shown by dotted lines in Fig. 6; and if the core be supported by a non-magnetic hub very few lines will leak across the internal centre space; consequently there is little or no E.M.F. induced in the inner wire, which thus serves purely as a conducting connector to sum up the E.M.F.'s produced in the two external active wires, and does not itself cut the lines of the field. If any E.M.F. is induced in the inner wire it is simply harmful, but with proper methods of construction and right proportioning of the iron in the armature core it becomes so very small that it may be neglected.

It will be seen that this method really amounts to threading the same wire through the central hole, so as to form a "loop," and then winding a second loop of the same wire continuous and side by side with the first. A coil of two loops is thus formed, containing two active wires on the outside of the core, whose E.M.F.'s are added together in series, and the core becomes a "ring" over-wound with a coil, whence the term "ring-winding" arises. As before, two active conductors have been placed in series, and precisely the same amount of the surface of the armature core has been covered up and rendered useful; and, therefore, two loops of the ring winding are the exact equivalent of one loop of the drum. To both the ring and the drum the iron core is essential, but in the ring there are internal conductors to be shielded from any inductive action, while in the drum all the winding is external. It is evident that in the ring the process of winding the wire round the core may also be continued so as to form a coil of three or any larger number of loops that may be desired.

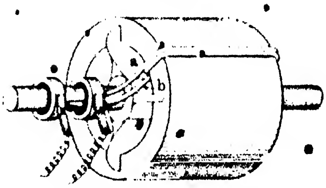
§ 10. Simple alternator and unidirected-current armatures.—If the curve of E.M.F. induced by either the drum or ring arrangement of Fig. 14 be plotted, then with the same field and speed of rotation the ordinates will have twice the height that they have in Fig. 11; and if the free ends of the loops *ab* be taken along the shaft

and connected to two insulated collecting rings mounted thereon; we have the simple bipolar "drum" and "ring" *alternator* armatures of Fig. 14. This figure also shows the curve of E.M.F. acting during one revolution at the brushes A and B, which lead to the external circuit, the angle through which the loops have rotated being reckoned as before from the time when they are situated midway between the pole-pieces.

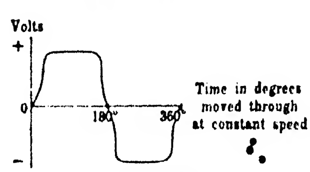
For a variety of purposes an alternating E.M.F. is inconvenient or positively useless; yet, from the nature of the dynamos under Class II, the E.M.F. induced in each loop is necessarily alternating, since it is produced by first cutting lines in one direction, and then in the opposite, by continuous rotation, and therefore any current flowing under that E.M.F. must necessarily be alternating in the armature conductors themselves, whatever it may be in the external circuit. It remains to be inquired how the alternating E.M.F. of the armature loops can be "commuted" or changed to an E.M.F. always acting in one direction in the external circuit. When in the course of rotation the loop or loops arrive at a central position under the two pole-pieces, the E.M.F. is a maximum and collecting ring a is, say, positive; later, when the loops have turned through 90°, or have reached a position in the gap between the poles, no E.M.F. is being generated, since no lines are being cut, and they have reached the position of reversal. Immediately after passing this point, ring a will be negative, ring b positive. What is required, therefore, in order that the E.M.F. at the brushes may act always in the same direction round the external circuit, is that the rings in contact with the springs or brushes should be automatically reversed at the instant when the direction of the E.M.F. in the loop is reversed, so that a should now touch B, and b touch A. This is easily effected by making a and b each half of one and the same



Drum-Wound Armature
One Loop



Ring-Wound Armature
Two Loops.



Curve of E.M.F.

FIG. 14. Alternator armatures with two active conductors in series.

split ring, the two halves being separated from each other by air or other insulating material, and further, by so setting the brushes that they pass over respectively from *a* to *b*, and from *b* to *a* at the instant of reversal (Fig. 15). This device is the simplest form of *commutator*, by which each brush always remains either positive or negative, as the case may be, and therefore the current flowing in the external circuit is *unidirected*, although in the loops of the armature itself it is alternating in direction.

A ring and a drum unidirected-current machine have now been obtained, and the curve of E.M.F. at the terminals or brushes *A*, *B* will be entirely above the horizontal line (Fig. 15). Given, therefore,

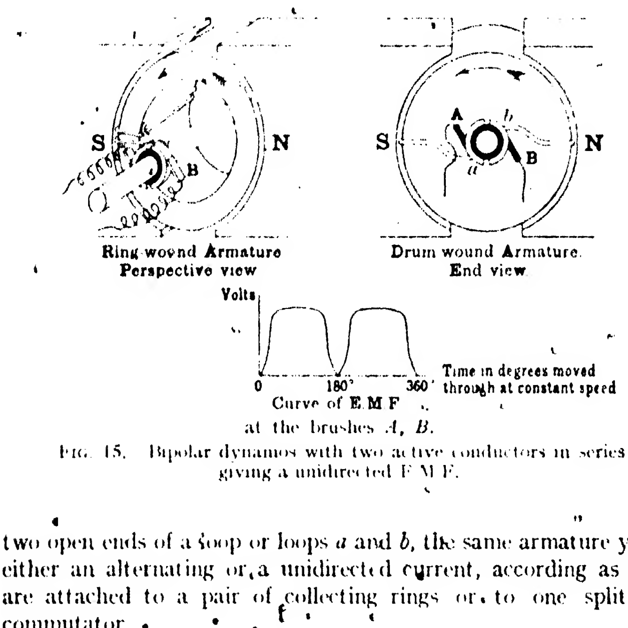


FIG. 15. Bipolar dynamos with two active conductors in series giving a unidirected E.M.F.

two open ends of a loop or loops *a* and *b*, the same armature yields either an alternating or a unidirected current, according as they are attached to a pair of collecting rings or to one split-ring commutator.

§ 11. Heteropolar dynamos (Class II) of the radial or (b) type.—Heteropolar dynamos of the second or (b) group of Class II, in which the length of the active wires is *at right angles to the axis of rotation*, seldom are used in practice, and therefore may be briefly dismissed as of less importance. For a dynamo to belong to Class II it is necessary that the conductors should cut the same lines twice in different directions. A second air-gap must, therefore, be brought into the circular path of the active conductor by breaking the magnetic circuit at a second place.

(3) *Disc winding.* In the true disc heteropolar machine, opposite poles are of opposite sign, and the lines pass straight across each air-gap from one pole-piece to the other. The method of connecting two radial wires in series is then to join the outer end of one active wire by a connecting piece passing round the periphery to the outer end of another active wire, situated nearly diametrically opposite, under the second pair of poles (Fig. 16), the loop

being completed, if required, by a second connector, joining the inner ends of two active wires, so as to render a series of continuous loops possible. It will be seen that the radial wires are actually in the central plane of rotation, and the peculiarity of the true disc machine is that it is indifferent whether the loop be wound on an iron core or not. A supporting core may be necessary, but it need not be of iron so as to guide the lines in the right direction

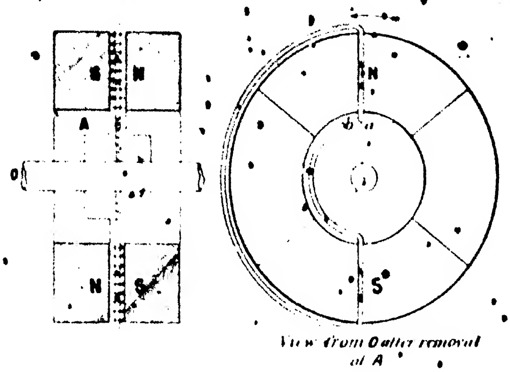


FIG. 16.—Disc armature with two active conductors in series.

through the air-gap, since the two opposite poles can be brought close together, only allowing sufficient room for the coils to pass between them. It will be recognized that Fig. 10 shows the same type of winding as applied to a homopolar machine.

(4) *Discoidal or flat-ring winding.* The winding of the discoidal machine is exactly analogous to the ring armature of Fig. 14, if it is imagined to be flattened out until its radial depth is greater than its length parallel to the

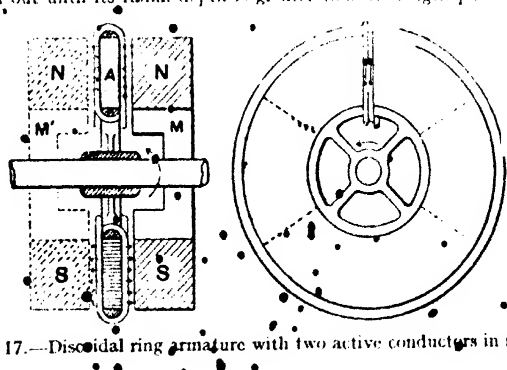


FIG. 17.—Discoidal ring armature with two active conductors in series.

axis of rotation, and it assumes a discoidal form; the polar surfaces are then presented to the ends of the cylinder instead of to its periphery. But now a second magnet, M' (shown dotted, Fig. 17) can be presented to the other face of the flat-ring core, the poles of this second magnet being exactly opposite to those of the first and of the same sign. The lines will thus enter the core from both sides alike, and will pass in opposite directions round the armature, to leave it where the second set of poles is presented to its surface. By this

means both sides of the winding become active, and the inactive connectors are reduced to the short lengths of wire at the top and bottom of the core. The entire loop of the flat ring machine, although containing two inductive portions, should therefore be regarded as equivalent to a single loop of the ring, the lines of one field being simply divided into two portions, one on either side of the core.

§ 12. Superiority in practical importance of the drum method of winding. While the above brief description of the four methods of winding, drum, ring, disc and discoidal, is necessary in order to elucidate their characteristic differences, it remains to be added that except in special cases the drum winding possesses such marked superiority that it is almost universally used in preference to the three other methods. Attention may therefore be henceforth almost exclusively directed to the drum loop and coil as the important constituent of the modern dynamo. The *formation of the coil*

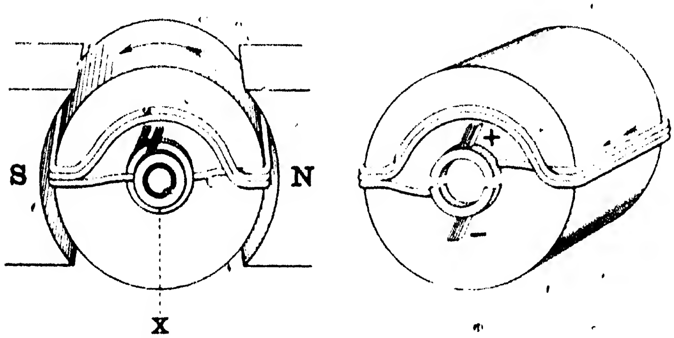


FIG. 18. Drum armature with coil of three loops.

of many loops by winding on more wire is illustrated in the case of the drum by Fig. 18, which shows a bipolar armature with a coil of three turns, by which six active wires are joined together in series with a consequent increase in the E.M.F. As before, the two free ends of the coil may be taken either to a pair of collecting rings to form an alternator (as in the diagram on the left where one of the rings is shown of larger diameter in order to be visible behind the other), or to a split-ring commutator (as in the right-hand diagram) and by means of the brushes a current, alternating or unidirected, in the two cases is led into the external circuit. The ring and discoidal-ring windings can be used without alteration in a multipolar field, but in the multipolar drum or disc machines the nature of the windings requires that the angular span of the loops, which is approximately 180° or across the diameter in the two-pole machine, should be reduced to approximately 90° in the four-pole (*cp.* Fig. 19) or 60° in the six-pole field, and so on. As the number of poles

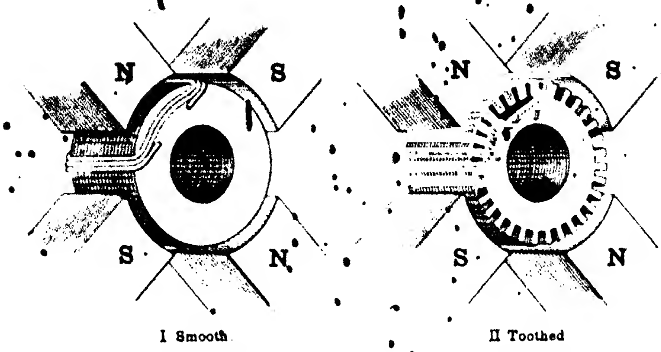


FIG. 19. Rotating multipolar armatures with single drum coil.

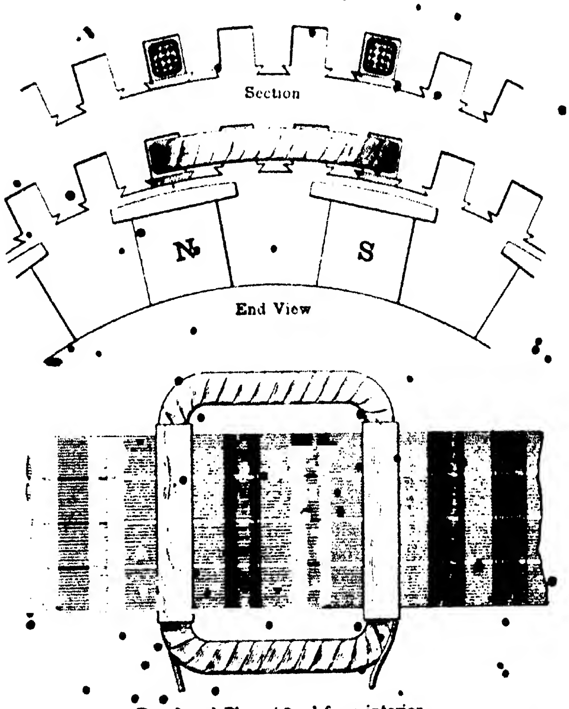


FIG. 20.—Stationary alternator armature with single drum coil.

is increased the chord becomes shorter, which has the practical effect of reducing the curvature of the coil.

Thus Figs. 19 and 20 show the drum coil of many turns as applied either on the outer surface of an armature revolving within an external multipolar field, or to the inner surface of an external stationary armature; so that the resemblance to a drum is to a great extent lost. In either case, again, the active surface of the armature may be *smooth* as in Fig. 19 (i), with the wires fastened along the outside of the core, or it may be *toothed* with the winding embedded in slots sunk radially into the periphery of the core as in Fig. 19 (ii) or 20, where the coils shown are adapted respectively for a direct-current machine and for an alternator. From the fact that the slotted core allows of a shorter air-gap and better mechanical protection for the winding, combined with certain practical advantages in manufacture, the toothed or slotted armature is much the most usual construction in both alternators and continuous-current dynamos.

The further consideration of the several modes by which coils, each of one or more loops or turns, may be connected together in order to increase still further the total E.M.F. of different types of machines, will be resumed in Chapters VIII and IX.

CHAPTER IV

THE MAGNETIC "PULL"

§ 1. The electrodynamic action between current and magnet.— In the year 1819 a Danish physicist, H. C. Oerstedt,[1] discovered that when a magnetic needle was brought near a wire carrying an electric current it was deflected into a definite position relatively to the wire; in other words, between the conductor conveying an electric current and the magnet he found that there existed a certain mechanical force which, if the conducting wire were stationary and the needle movable, would cause the latter to set itself in a definite direction. For example, if the conductor conveying the current is a long straight wire held horizontally over a compass needle, the latter tends to set itself at right angles to the length of the wire; and further, the direction in which its north pole points when it has so set itself depends upon the direction in which the current is flowing in the wire.

In the above great fact of the *electrodynamic action of an electric current on a magnet* is involved the whole principle of dynamo-electric machinery. The simple experiment of Oerstedt was, in fact, the first instance that had been observed of the conversion of electrical energy into mechanical work, and as such illustrated the principle of the electric motor; for the needle as it moves could be made to do mechanical work.

§ 2. The mechanical torque of dynamo or motor. But it is equally the fundamental principle upon which the dynamo rests. It cannot be too strongly insisted upon that the measure of the usefulness of the dynamo-electric machine, whether as dynamo or motor, depends on the mechanical force which has to be exerted upon or is exerted by the moving member, *i.e.* on the mechanical torque that the rotating member can absorb or give out; for upon this in conjunction with the speed depends the electrical power or mechanical horse-power that it can develop. Just as the purchaser of a steam-engine buys a machine capable of exerting a certain torque on its crankshaft when supplied with steam at a given pressure, so the purchaser of a dynamo or motor buys a machine capable of absorbing or exerting a certain torque when its field is fully excited and it is supplied with mechanical or electrical

[1] For an appreciation of the work of Oerstedt and his contemporaries Ampere and Arago, one hundred years ago, see *Journ. Amer. I.E.E.*, vol. 39, Dec. 1920, pp. 1021–33.

energy. The speed at which it is driven or at which it runs, although fundamental to its design and use in so far as it determines the power and also the voltage, yet in comparison is a secondary matter; for upon the torque depends primarily the *size* and *cost* of the machine, and it may even in practice be adapted to run at any speed within a fairly wide range. So important is the torque that electric machines are comparable as to their relative usefulness, size and cost in terms of the *watts per revolution per minute* that they can give, a "watt per revolution per minute" being a special unit of torque which owes its origin to its appropriateness in electrical work, and which has the same dimensions as a dyne-centimetre or a pound-foot. One watt being 10^7 ergs per second, the particular value of this unit $= \frac{60}{2\pi} \times 10^7$ dyne-centimetres, or

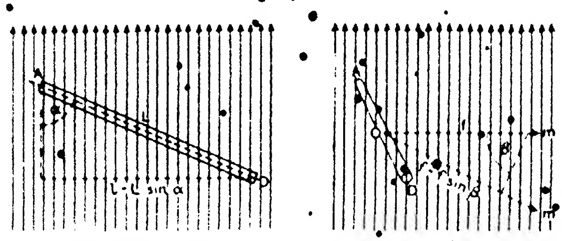
$$1 \text{ watt per rev. per min.} = 9{\cdot}55 \times 10^7 \text{ dyne-centimetres}$$
$$= \frac{9{\cdot}55 \times 10^7}{9{\cdot}81 \times 10^7} = 0{\cdot}974 \text{ kilogrammetres}$$
$$= 7{\cdot}05 \text{ lb. ft.}$$

§ 3. The mechanical force on a current-carrying conductor in a magnetic field. In order then to develop the first fundamental equation of the dynamo dealing with the connection between mechanical force and electric current, the elementary case of a straight conductor carrying an electric current and immersed in a magnetic field will first be considered.

Let a straight conductor AD, in which a current is flowing along its length from D to A, be placed in any position in a uniform magnetic field (Fig. 21); the rest of the circuit which must necessarily exist is for the present disregarded.

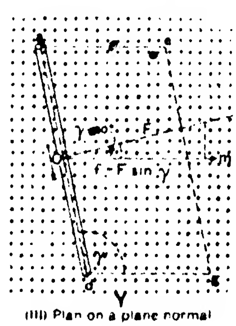
Whatever the relative positions of the lines which represent the directions of field and length of conductor, they fall under one or other of two cases: either the length of the conductor is parallel to the direction of the field, or it is not. Now, in the first case, the conductor conveying the current is entirely unacted on by any force due to the external magnetic field in which it is placed, and it has no tendency to move in any direction. But in the second case it becomes the seat of a mechanical force which, unless resisted by an equal and opposite force, will cause it to move in a definite direction. This second case is equally well expressed by saying that, if there is to be a mechanical force acting on the conductor, the projection of its length on a plane normal to the direction of the field must be a line, and not a point. In Fig. 21 let AD be a conductor whose length does not coincide with the direction of field in which it is placed, and let ad be the projection of its length

on a plane normal to the field. At right angles to *ad* and also in the normal plane draw a line *OM*. Then the conductor *AD* is acted upon by a force equally distributed throughout its entire length (since it is entirely immersed in a uniform field), and the direction of this force at each portion of its length is in the plane normal to the field and parallel to *OM*; in other words, if it is free to move, the conductor will move parallel to itself through the



(I) End Elevation from M

(II) Side Elevation from Y



(III) Plan on a plane normal to the Field.

FIG. 21.—Force on current-carrying conductor in magnetic field.

field owing to the action of forces, which may be summed up and represented by one force along the line *OM*.

The "sense" of this magnetic force or pull is determined as follows. In Fig. 21, if the direction of the lines of the field be vertically upwards, and the current flows in the conductor from *D* to *A*, the direction of the pull on the conductor will be from left to right towards *M*. Hence if the *left* hand is placed outstretched along the length of the conductor, so that the direction of current is from wrist to the finger-tips, while the thumb points down the direction of the lines of the field, the direction of the mechanical pull on the conductor will be across the hand from the back to the palm; in other words, the palm will face the direction towards which the conductor tends to move. The force is a mutual action between the conductor and the field-magnet, so that if the conductor is held fast, and the pole-pieces between which the field exists are free to move, the direction in which they will move is of course opposite, *i.e.* from the palm of the hand across to the back.

§ 1. The mechanical force equation.—Next, as regards the magnitude of the force, if l be the length of the conductor *AD* as

projected on to the plane normal to the field of uniform density, then the force acting on AD at right angles to its length, and tending to move it parallel to itself along OM, is proportional to the product of the density of the flux, the projected length of the conductor, and the current, or $F \propto B_g il$.

On the C.G.S. system, the force measured in dynes is equal to the above product when l is in centimetres, B_g in C.G.S. lines per square centimetre, and i is in absolute electromagnetic units of current. But if i is in amperes, since one ampere $= \frac{1}{10}$th of an absolute electro-magnetic unit,

$$F = B_g il \times 10^{-1} \text{ dynes}$$

But $l = L \sin a$, where a is the angle which the length of the conductor makes with the direction of the field (Fig. 21, i), so that

$$F = B_g iL \sin a \times 10^{-1} \text{ dynes}$$

or $F = B_g \cdot i/10$	dynes	per cm. of projected length				
$= B_g \frac{i}{10} \times \frac{1}{981}$	grammes	,,	,,	,,	,,	,,
$= 1{\cdot}02\ B_g i \times 10^{-7}$	kilogrammes	,,	,,	,,	,,	,,
$= 5{\cdot}7\ B_g i \times 10^{-7}$	lb.	per inch	,,	,,	,,	

There is no tendency for the conductor to rotate so as to alter the angle a, but for a given length of conductor and given current, the force is a maximum when the length of the conductor is itself at right angles to the direction of the lines, *i.e.* when it lies wholly in the plane normal to the field, as shown in Fig. 22 and $a = 90°$. The maximum force is then

$$F = B_g iL \times 10^{-1} \text{ dynes} \quad . \quad . \quad . \quad . \quad (2)$$

$$= 5{\cdot}7\ B_g iL'' \times 10^{-7} \text{ lb.} \quad . \quad . \quad . \quad . \quad (2a)$$

when i is in amperes, and in the second alternative L'' is in inches although B_g is retained in C.G.S. lines per sq. cm. Since in dynamo and motor alike, the length of the active conductors is arranged as far as possible to lie in the plane normal to the field, the above expression may be regarded as the first fundamental equation of the dynamo or motor.

When the conductor is inclined to the direction of the field, the actual force acting on it due to the magnetic pull is always

$$F = B_g iL \sin a \times 10^{-1} \text{ dynes}$$

along the line OM. But the whole or part of this may be balanced by other external forces. The component force along *any* line of action, which may or may not be unbalanced, is then simply obtained by resolving the force F along the given line. Thus in Fig. 21 let Om' be the line of action under consideration, its

projection on the normal plane (Fig. 21, iii) being along Om; then the component of the force F acting along the line Om is

$$f = F \sin \gamma$$

where γ is the angle which the projection of the line of action makes with the projected length of the conductor. Further, the component of f along the line Om' is

$$f' = f \sin \beta$$

where β is the angle which the line of action makes with the direction of the field (Fig. 21, ii). Hence the force acting on the conductor along the given line of action Om' is

$$f' = B_g iL \sin \alpha \sin \beta \sin \gamma \times 10^{-1} \text{ dynes} \quad . \quad . \quad . \quad . \quad (3)$$

One feature still remains to be mentioned. When the conductor of Fig. 21 is unconstrained and moves parallel to itself in the direction of the line OM, it passes broadside through the field, and for a given distance moved through, the number of lines cut is a maximum. Further, since the force F has no component in a vertical direction or along the length of the conductor, it is evident that there is no tendency for the conductor to move up and down the lines or end-on through them. It is essential, therefore, for the existence of force along any line of action that the conductor, by movement in that direction, should cut lines of the field along its length.

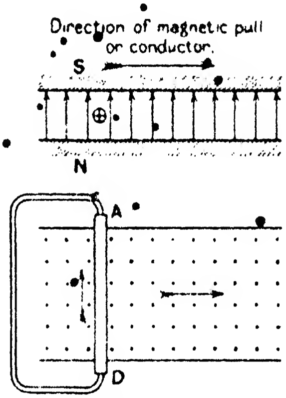


FIG. 22.—Position for maximum pull on current-carrying conductor in magnetic field.

§ 5. The magnetic pull as arising from the interaction of two magnetic fields.—The mechanism from which arises the mechanical pull on a current-carrying conductor is to be found in the interaction of its own magnetic field with the external field in which it is situated. Thus the composition of the circular lines surrounding the straight conductor of Fig. 22 with the original field would lead to the resultant lines being bent round the conductor so that the density becomes greater on its left and less on its right side (*cp.* Fig. 23 for curved pole-pieces). Again, take the case of a rectangular loop of wire round which a current flows, and placed so that its plane is parallel to the direction of the field between two external pole-pieces (Fig. 24). Then it follows from the electrodynamic law that if the direction of the current round the loop be as shown by the arrow, the upper side will be subjected to a force tending to move it towards

the right, as seen from L, while the lower side will be subjected to an equal force tending to move it towards the left; the ends will while parallel to the field have no force exerted on them, and when in a plane normal to the field, or in any intermediate position, the force tending to move the one in one direction will be counterbalanced by the force tending to move the other in the opposite

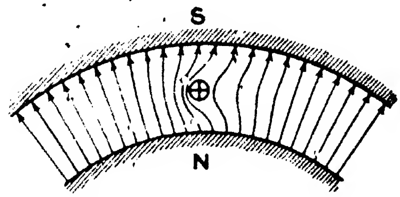


FIG. 23. Resultant field

direction. The whole loop will therefore be acted upon by a couple tending to rotate it in a clockwise direction, and the position which it will take up if free to move is one in which its plane is normal to the direction of the field, as shown by dotted lines. When the loop is vertical and is subjected to a magnetic pull, the lines that would be due to the current alone would pass through it from right to left, as seen from D. Such a supposed field must, however,

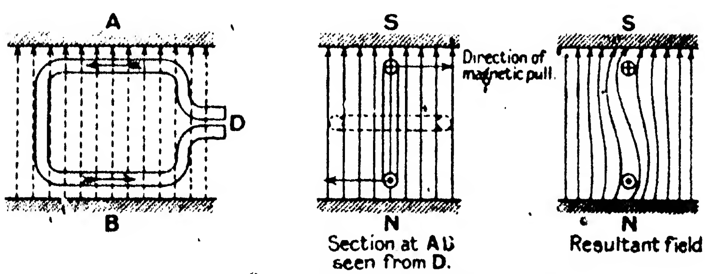


FIG. 24. Torque and resultant field for loop.

be compounded with the original vertical field and the result of this composition is that the lines of the actual field in nature are twisted out of their direct path. When the loop has rotated into its position of rest, the lines that would be due to the current alone coincide in direction with the lines of the original field, *i.e.* pass vertically upwards through the loop. Hence, as the loop rotates, the lines of the resultant field shorten their path, and further become a maximum.

If a magnetic field be divided up into tubes of flux, each of which, though possibly of varying area, conserves the same amount of induction throughout its whole circuital length, Clerk Maxwell has

shown that the state of stress in a medium which is magnetically indifferent [1] may be specified if it be supposed that the tubes of flux are subjected to

(i) a hydrostatic *pressure*, the same in all directions, and of magnitude $H^2/8\pi$ dynes per sq. cm., and

(ii) a longitudinal *tension* $BH/4\pi$ dynes per sq. cm. in the direction of the lines of induction. When, therefore, the lines of an external field are distorted or deflected out of their original normal path by the presence of the current-carrying loop and are thereby caused to traverse a longer path, the interaction of loop and field tends to cause relative movement through which the lines are enabled again to straighten themselves. The following general laws, illustrated by the above-described case, may therefore be stated—

(1) To the lines of a compound field if distorted from their initial path by current-carrying conductors may be attributed a tendency to shorten themselves, and if relative movement between coil and magnet be possible, this tendency will cause movement in such direction that the lines become progressively straighter, and shorter.

(2) Every closed circuit carrying a current tends to set itself so that the lines embraced by it are a maximum, and the direction of the movement will be such as to cause the linked flux to increase.

In simple cases the forces are easily calculable, [2] and even approximate diagrams, such as Fig. 23 or 24, of the flux round current-carrying conductors immersed in a magnetic field are always useful as pictures suggestive of the direction of the mechanical thrust or torque on them or on the surrounding iron, [3] although in complicated cases they cannot be used as a means for quantitative calculation of the magnetic pull.

§ 6. Magnetic pull on the conductors of a smooth armature.—Let us now consider the case of a smooth iron cylinder along the outside of which and parallel to its length are arranged a number of conductors conveying currents, the whole being placed in a magnetic field between two pole-pieces (Fig. 25). The direction of the lines in the air-space between the iron poles and the iron cylinder will be nearly radial, except for a small fringe of lines near the interpolar gaps, and even here the immediate entrance of the flux into the armature core or its emergence therefrom is practically radial.

[1] *Electricity and Magnetism*, vol. 2, § 642. In a ferromagnetic medium if B and H diverge in direction, the case is more complicated.

[2] See especially G. F. C. Searle "On the Magnetic Field due to a Current in a Wire placed parallel to the Axis of a Cylinder of Iron," *Electrician*, vol. 40, p. 453 (28th Jan., 1898).

[3] Even in the case of a dynamo with compensating field-winding; cp. chap. 19, § 19.

As will be seen later, when current flows through the conductors the flux-density does not remain uniform even under the actual pole-faces, but if B_{gx} is the flux-density at any part where a conductor is situated at any moment, the force acting on it is $B_{gx}\, iL \times 10^{-1}$ dynes, where L is the length in cm. of the conductor within the influence of the field. If the conductor is situated at a radius of $D/2$ cm., its torque is $B_{gx} iL \frac{D}{2} \times 10^{-1}$ dyne-centimetres. Let $B_{g\ max}$ be the maximum density, and let k be a coefficient reducing its value to the average density over a whole pole-pitch so that $kB_{g\ max} = B_{g\ av}$. If then by rotation the conductor passes through all values of B_{gx}, its average torque during passage through the field of a pole-pitch will for a given constant current be proportional to the average density, $kB_{g\ max}$, i.e. it will be $= kB_{g\ max}\, iL \frac{D}{2} \times 10^{-1}$ dyne-centimetres.

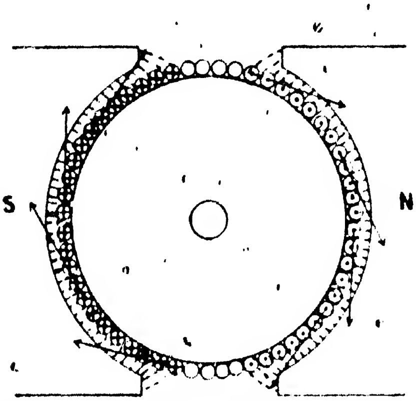


FIG. 25. Magnetic pull on the active conductors of a smooth armature.

(a) Now as shown in Fig. 25 let the directions of the currents in the one half of the conductors (marked with a dot) under one pole-piece be towards an observer, looking at the cylinder from its end, and in the other half (crossed) away from the observer. By application of the left hand it will be found that each conductor is subjected to a pull tangential to a circle passing through the centre of the conductors. The directions of a few of these forces are shown on the diagram. The whole may be considered as a number of loops all tending to set themselves vertical, so as to embrace all the lines of the field, while the ends of the loops will exert neither an upward nor a downward pull. The total result of the action will be that, unless otherwise constrained, the whole system of conductors will rotate in a clockwise direction, and if the above represents an electric motor, the conductors will so rotate. The tangential pull on each conductor is removed and put on again twice in each revolution as it passes over from under one pole to the other, but is always in the same direction relatively to the rotation.

Further, let the current in each conductor, although in different

directions in the two halves of the armature, have the same steady value of J amperes. The case of the armature of a 2-pole continuous-current dynamo or motor is then reproduced. The average magnetic drag or pull on one conductor passing through a double pole-pitch (the directions of current and flux changing simultaneously) is $kB_{g\ max} JL \times 10^{-1}$ dynes and its average torque is—

$$\tfrac{1}{2} kB_{g\ max} JDL \times 10^{-1} \text{ dyne-centimetres.}$$

The same also holds if the machine is multipolar; hence if there are Z conductors in all on the armature, the total torque will be Z times the average torque of a single conductor, *i.e.* it will be—

$$T = \tfrac{1}{2} kB_{g\ max} JZDL \times 10^{-1} \text{ dyne-centimetres} \quad . \quad . \quad (4)$$

or in the special unit of torque of § 2.

$$T = \frac{\tfrac{1}{2} kB_{g\ max} JZDL \times 10^{-1}}{\dfrac{60 \times 10^{7}}{2\pi}}$$

$$= kB_{g\ max} JZ \frac{\pi}{60} DL \times 10^{-8} \text{ watts per rev. per min.} \quad . \quad (4a)$$

Let Φ_a = the total number of C.G.S. lines which issue from any one pole and pass into the armature, or which emerge from the armature and enter into a pole of opposite sign, in either case forming a single field in the air-gap extending over a pole-pitch. Then $kB_{g\ max}\ \pi DL$ is equal to the total flux of all fields, *i.e.* $= 2p\Phi_a$, where p is the number of pairs of poles. Therefore, alternative expressions for the total torque are—

$$T = \frac{p\Phi_a JZ}{\pi} \times 10^{-1} \text{ dyne-centimetres} \quad . \quad . \quad . \quad . \quad (4b)$$

$$= \frac{p\Phi_a JZ}{4{\cdot}25} \times 10^{-8} \text{ lb.-feet} \quad . \quad . \quad . \quad . \quad . \quad (4c)$$

$$= \frac{2p\Phi_a JZ}{60} \times 10^{-8} \text{ watts per rev. per min.} \quad . \quad . \quad (4d)$$

Multiplying (4) or (4*b*) or (4*c*) by ω the angular velocity in radians per sec. ($=2\pi N/60$ where N is the number of revs. per min.), we obtain the rate of absorption or development of mechanical energy in the machine either in ergs per second in the first case, or in foot-pounds per second in the second case, and multiplying (4*a*) or (4*d*) by N, it is given directly in watts.

In the older types of dynamos or motors with smooth-core armature, such as that shown in Fig. 25, the density of field in the air-space through which the conductors moved was usually at least 5,500 C.G.S. lines per square centimetre. The force acting on each conductor per foot of active length is then 0·0376 lb. for every

ampere of current flowing through it, or nearly 3¼ lbs. for 100 amperes. The active wires had therefore to be securely fastened to the armature of the dynamo, and prevented from slipping under the action of the magnetic drag. This was effected by thin hard-wood driving strips (preferably of hornbeam) let into shallow grooves milled longitudinally across the armature core.

(*b*) Next, in reference to alternating-current machines let it be assumed that the density of the field of each pole-pair is constant in time but is distributed after a sine law over the double pole-pitch; that the currents in the conductors follow a sinusoidal law in time, and, therefore, also spacially at any one moment over the double pole-pitch; and lastly, that the maximum value of the current **I** ($= \sqrt{2}$ times the virtual current J) coincides with a position of the conductor carrying it in the maximum flux-density which we will symbolize by $B_{gs\ max}$ to indicate that the field is assumed to be sinusoidally distributed. Reckoning the angular position of a conductor from a point where current and field change direction, the instantaneous value of the current is $\mathbf{I} \sin a = \sqrt{2}J \sin a$ and of the flux-density in which it is situated is $B_{gs\ max} \sin a$. Since one pole-pitch corresponds to π electrical radians, the magnetic drag or pull from one conductor averaged over a pole-pitch is—

$$\frac{1}{\pi}\int_0^{\pi} B_{gs\ max} \sin a\, \sqrt{2}J \sin a\, da \times 10^{-1} \text{ per cm. length}$$

$$= \frac{\sqrt{2}}{\pi} B_{gs\ max} J \times \int_0^{\pi} \sin^2 a\, da \times 10^{-1} \text{ per cm. length.}$$

$$= \frac{1}{\sqrt{2}} B_{gs\ max} J \times 10^{-1} \text{ dynes per cm. length.}$$

If L be the length in cm. of the conductor within the influence of the field, the average total pull of its whole length during movement over the pole-pitch is—

$$\frac{1}{\sqrt{2}} \cdot B_{gs\ max} JL \times 10^{-1} \text{ dynes}$$

and at radius $D/2$ in cm., its average torque is—

$$\frac{1}{2\sqrt{2}} \cdot B_{gs\ max} JLD \times 10^{-1} \text{ dyne-centimetres.}$$

Since there are Z such conductors, the total torque of the machine is

$$T = \frac{1}{2\sqrt{2}} B_{gs\ max} JZLD \times 10^{-1} \text{ dyne-centimetres} \quad . \quad . \quad (5)$$

$$= \frac{1}{\sqrt{2}} B_{gs\ max} JZ \frac{\pi}{60} DL \times 10^{-8} \text{ watts per rev. per min.} \quad . \quad (5a)$$

The previous expression for the continuous-current case was given in terms of the average density, $kB_{g\,max}$. On the basis of our assumed sinusoidal distribution of the field in the alternating-current case, $k = 2/\pi$, and we may equally well write—

$$T = \frac{\pi}{2\sqrt{2}} \cdot kB_{gs\,max} JZ \frac{\pi}{60} DL \times 10^{-8} \text{ watts per rev. per min.}$$

where k has the special value of $2/\pi$ for the case of B_{gs}. For $kB_{gs\,max} \cdot \pi DL = 2B_{gs\,max} \cdot DL$ may then be substituted $2p\Phi_{as}$, where Φ_{as} is the total flux of one pole-pitch when the field is sinusoidally distributed. Alternative expressions for the torque in terms of the flux per pole-pitch in the alternating case are, therefore,

$$T = \frac{p\Phi_{as} JZ}{2\sqrt{2}} \times 10^{-1} \text{ dyne-centimetres} \quad . \quad . \quad . \quad . \quad (5b)$$

$$= \frac{\pi}{2\sqrt{2}} \cdot \frac{p\Phi_{as} JZ}{4.25} \times 10^{-8} \text{ lb.-feet} \quad . \quad . \quad . \quad . \quad (5c)$$

$$= \frac{\pi}{2\sqrt{2}} \cdot \frac{2p\Phi_{as} JZ}{60} \times 10^{-8} \text{ watts per rev. per min.} \quad . \quad (5d)$$

A further approach to the conditions of practice[1] is given in Chapter VI, § 14, where it is shown that when the change of direction of the current in the conductors does not coincide with their position on the line where the field changes direction but lags behind it, allowance must be made for this by multiplying the above expressions by the cosine of the angle of lag. Lastly in practice the alternating current does not spacially vary strictly after a sine law over the double pole-pitch, even if in any one conductor it varies temporally after a sine law: the conductors are, in fact, divisible into a comparatively small number of groups, in each of which the current at any moment has one and the same value, since the conductors of which it is composed are in series. To take account of this departure from the conditions assumed above, the expression must be multiplied by a differential factor k_d, which will be further explained later.

Both the continuous-current and alternating cases may then be brought under one general formula, as—

$$T = k_f \cdot k_d \cdot k \cdot B_{g\,max} JZ \frac{\pi}{60} DL \times 10^{-8} \text{ watts per rev. per min.} \quad (5e)$$

$$= k_f \cdot k_d \cdot \frac{2p\Phi_a JZ}{60} \times 10^{-8} \text{ watts per rev. per min.} \quad (5f)$$

[1] If the field-distribution is not purely sinusoidal, then as a first approximation and with but little error may be taken the fundamental of the flux-curve, and for $B_{gs\,max}$ and Φ_{as} will be substituted B_{g1} and Φ_1. In most cases this initial step in the complete process for the actual field is all that is required for practical purposes.

where in the continuous-current case k_f and k_a are both unity, and in the alternating case with a sinusoidally distributed field $k_f = \pi/2\sqrt{2}$ and $k = 2/\pi$, the former being the "form factor" or ratio $\dfrac{\text{virtual value}}{\text{average value}}$ and the latter the ratio $\dfrac{\text{average value}}{\text{maximum value}}$ of a sine function. If in the latter case the change of current-direction does not coincide with change of field-direction, T is the hypothetical torque, that would result if there was coincidence, measured in volt-amperes per rev. per min., and the real torque in watts per rev. per min. will be given by the product of the hypothetical torque and the angle of lag $\cos\phi_a$.

§ 7. The toothed or tunnel armature and its magnetically shielded conductors.—Next let the current-carrying conductors be embedded in slots on the surface of the armature, as in Fig. 6, or be actually threaded through closed tunnels. Thereby they become *magnetically shielded*, so that the flux-density B_s within the slot or tunnel is only a very small fraction of B_g, the average flux-density in the air-gap, and a still smaller fraction of B_t, the flux-density in the teeth. The mechanical force on a conductor is then strictly proportional to B_s, the density of the field in which it is immersed in the slot, and obeys the fundamental law[1] of equation (2). For the same total flux therefore the mechanical torque on the conductors of the toothed armature is very much reduced as compared with its value for the same conductors and current on the surface of a similar smooth-core armature. Yet actually it is found by experiment that the total torque on the toothed armature as a whole is precisely the same as on the conductors of the similar smooth-core armature.

The explanation of this apparent paradox is again to be sought in the interaction of the magnetic field of the conductors in the slots with the main field. The influence of the magnetomotive forces of the conductors extends into the air-gap and causes the flux to take a slanting direction across the gap from the iron surface of the pole-pieces to the iron surface of the armature. Its path is therefore longer, and the tension of the lines causes a magnetic pull between the tips of the teeth and the iron pole-faces which has a tangential component.

Fig. 26[2] shows the path of the lines in the air-gap after emerging from a block of iron, which is immersed in a uniform magnetic field and in which is embedded an insulated conductor carrying a

[1] For experiments in proof of this, cp. W. M. Mordey, *Journ. I. E. E.*, vol. 26, p. 566; cp. also P. M. Heldt, *Elec. World and Engineer*, vol. 33, p. 699 (1899).

[2] Taken from Mr. Searle's paper above quoted (§ 5), to which the reader is again especially referred.

current directed away from the observer. In drawing the diagram, in order to show the distortion of the lines, the strength of the original uniform field H_0 in the air has been made small as compared with the magnetic force due to the current at the points in the air nearest to the current; in an actual dynamo it would be greater, and the distortion less, although always present. From the tension and pressure mentioned in § 5, there results across any plane in the ether parallel to the face of the block and above it not only a tension tending to lift the block, but also a shearing stress tending to make all the system below the plane move towards the right. The sloping direction of the lines as they leave the iron shows that the iron will experience a force towards the right. The whole of the force parallel to the face of the iron iH_0 would in any case finally come upon the block of iron, but now if the tunnel is circular and small the force on the conductor parallel to the face of the iron, per cm. length, is only $2iH_0/\mu$ dynes, and the remainder $iH_0(1-2/\mu)$ falls *directly* on the iron.

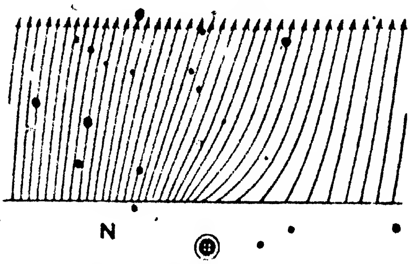


FIG. 26.—Force on iron containing an embedded conductor.

Fig. 27 shows roughly, and to an exaggerated scale, the path of a line through a double air-gap and cylindrical iron core, when the latter has a current-carrying loop embedded in it near its surface, in the position of greatest torque. Again by far the greater part of the mechanical torque falls *directly* on the iron.[1]

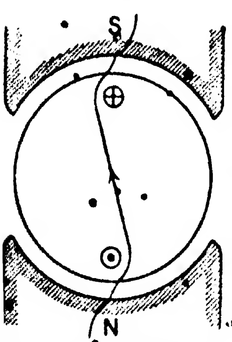
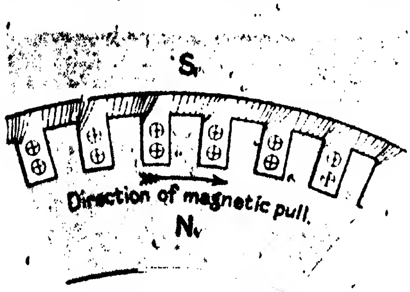


FIG. 27.—Approximate path of lines giving rise to torque on iron cylinder containing an embedded loop.

When the armature is toothed with open or semi-closed slots, an accurate pictorial representation and quantitative calculation of the relative forces on iron and conductor becomes more difficult. The qualitative effect is therefore only roughly indicated in Fig. 28. The flux of the field after passing through the air-gap proper divides

[1] See especially "The Magnetic Field in Tunnel Armatures," by Prof. F. G. Baily, *Electrician*, vol. 39, p. 810 (15th Oct., 1897), with subsequent correspondence, and "On Magnetic Shielding," by Prof. H. Du Bois, *Electrician*, vol. 41, p. 108 (20th May, 1898).

between the alternative paths offered by the teeth and the slots in proportion to their permeances since the two paths are, roughly speaking, in parallel; the relative densities within tooth and slot are therefore $B_t/B_s = \mu$, where μ is the permeability of the iron teeth for the induction B_t within them. Since the permeability of good iron stampings even at as high a flux-density as 20,000 lines per square centimetre is to that of air as 66 : 1, it is evident that if the width of the teeth is comparable with that of the slots almost all the lines will follow the iron path. But the lines, instead of being nearly uniformly distributed over the face of each tooth, become denser on the trailing than on the leading side; by the "leading side" is indicated that side which during rotation as a dynamo first enters or leaves the gap between two poles, (the "trailing" side being that which is the last to enter or leave the interpolar gap), and the difference between the densities of the two edges increases as the current is increased. The lines tend to shorten and straighten themselves, so that the result of the unsymmetrical distribution is to produce directly on the armature core a magnetic pull dependent upon the amount of current that is flowing in the conductors. The extent to which the driving force that the conductors would experience on a smooth core with the same total flux is transferred to the iron of the toothed armature depends upon the degree to which the teeth are unsaturated, *i.e.* upon the average density in the air-gap and the relative proportions of the widths of tooth and slot. If $B_g = 6900$, and the width of the slot is about twice that of the tooth, the above-mentioned relative densities would be obtained, namely, $B_t/B_s = 20{,}000/300 = 66$; the force on the active conductors is then $B_s/B_g = 300/6900 = \frac{1}{23}$rd of that to which they would be subjected on a smooth core with an equal average density, and $\frac{22}{23}$rds of the total pull is transferred directly to the iron. The conductors themselves, therefore, and the insulation between the conductors and the walls of the slot are relieved from the greater part of the driving stress—a fact of great importance to the dynamo designer.



← Direction of rotation as generator
→ „ „ „ „ motor.

FIG. 28.—Distribution of flux in toothed armature carrying current.

Thus *the toothed armature*, which in an elementary form was first

employed in the ring armature of Pacinotti, not only gives good mechanical protection to the conductors, but affords a perfect system of driving. Largely on this account it has displaced the smooth-core armature formerly in vogue, and is now almost universally employed for all classes of machines.

But whether the armature be smooth or toothed, the importance of sufficient mechanical strength to core, shaft and framework in general in view of the forces described in the present Chapter will be clear. It only remains to be added that in spite of the transference of the major part of the torque from the conductors to the iron in the toothed armature, the total torque on the armature as a whole remains discoverable by equations 4·5, when B_g is interpreted as the density in the air-gap, and the conductors are imagined to be situated therein as they would be on an equivalent smooth armature.

CHAPTER V

THE PRODUCTION OF AN ELECTROMOTIVE FORCE

§ 1. The laws and the causes of induced E.M.F.—The theory of the induced E.M.F. of the dynamo as presented in textbooks has usually been based according to the preference of the author either on a line-cutting law or on the rate of change of line-linkages, the former being a concept which may be connected especially with the work of Faraday, and the latter with that of Clerk Maxwell. It is further usually stated that the two laws may be regarded as mutually convertible, and that either may be chosen as convenience dictates. Although this is very largely true, it may be questioned whether the matter is really so simple as is commonly supposed. So long as only smooth armatures were in general use, no difficulty was felt since the theory of their action is equally well explained on either basis, but the introduction of the toothed armature for almost all classes of machines raises a problem which calls for a re-examination of the fundamental theory and possibly for its re-statement on somewhat different lines.

Before proceeding to discuss the validity of the two *laws* based on the concepts of line-cutting and of rate of change of linkages, each will in turn be given some further consideration, with a view to bringing out their essential natures and differences. Certain suggestions in regard to the *physical causes* of an induced E.M.F. will then be tentatively put forward, the grounds for which follow, and finally in the light of these suggestions a return will be made to the two laws, in order to test their validity and connection.

I. The Line-cutting Law

§ 2. Movement of a body in a stationary magnetic field.—It has been shown in Chapter II that in the presence of a permanent magnet, electromagnet or current-carrying conductor the surrounding space possesses certain distinctive properties. It has further been shown that the properties of the magnetic field of induction at any point may be completely defined by a vector having a certain direction, magnitude and sense.

Let a straight rod or an element so small that it may be regarded as straight, of any material substance, conductive or insulating, be moved in a stationary magnetic field (*cp.* Fig. 29). Three directions are then concerned, viz., those of the field, of the length of the moving body, and of the movement. (*a*) If all three coincide, no difference of state can be detected and no E.M.F. is induced.

If two of them are inclined to one another at some angle, a plane containing them is thereby defined, and two further cases then arise. Either (*b*) the third direction falls in the plane defined by the other two, or (*c*) the three directions are not in the same plane. In the former case again no E.M.F. is induced, and nothing distinguishes the movement from movement through unmagnetized space; it is, in fact, perfectly indifferent whether the magnetic field surrounds the moving body or not. But in case (*c*) an electrometer applied to the ends of the moving body will show that an E.M.F. is induced during the movement, the potential of one end being raised above that of the other.

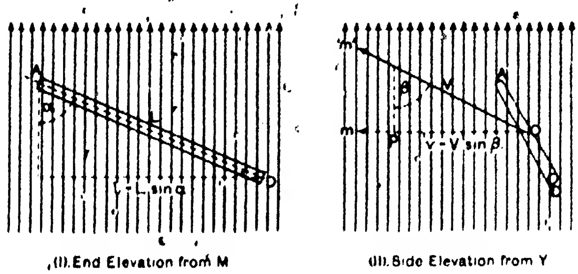
The feature which distinguishes the third case from the other two is that the body now does not simply move in the field, but is caused to *cut across* the magnetic field, its length intersecting or passing athwart the lines which as we have seen mark the direction of the field. The cutting action is at once rendered evident, if the reference plane is taken as formed by the length of the element and by the direction of its movement; then by the definition of the third kind of movement the direction of the magnetic field must either be perpendicular to the above plane or have a component at right angles to it, so that the lines or resolved components of them are directly cut by the moving element.

To this law there is no exception, and it matters not what may be the material of the body, whether it be a solid conductor or a liquid electrolyte or an insulator. If the body is a conductor, and its ends are electrically joined (as *e.g.* in Fig. 22) so as to complete a closed circuit of conducting material, part of which is outside the magnetic field, a current of electricity begins to flow under the induced E.M.F. and continues to flow until after the conductor has ceased to cut through the field. The conductor within the magnetic field may be appropriately regarded as having become *active* owing to the action having its origin within its length.

That the origin of the E.M.F. causing the current is located in the length of the body moved across the magnetic field may fairly be concluded from certain elementary cases, such *e.g.* as a single conductor rotated in a uniform homopolar field (*cp.* Fig. 8). It is true that until some more or less definite circuit is applied to the ends of the conductor through an indicating instrument, we can have no knowledge as to whether there is any difference of potential between its ends or E.M.F. induced in it; thus even when an electrometer is employed as the indicating instrument, its circuit, though an imperfect one, is virtually closed through the ether between the plates of the electrometer, and we only become aware of the presence of an induced E.M.F. by reason of the charging or "displacement" current that flows through it. Hence the effect of the movement of the conductor can only appear in the circuit as

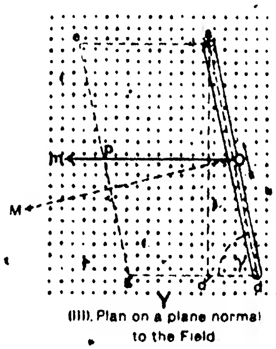
a whole. Yet, in such a case as that cited above, it is the only reasonable assumption that the E.M.F. is induced in the moving conductor, since everything else remains entirely unchanged and unaffected.

§ 3. Simple translation of a straight element through a magnetic field.—The fundamental idea of line-cutting to which reference



(i). End Elevation from M (ii). Side Elevation from Y

will frequently have to be made may best be illustrated by further consideration of the simple case of a straight element when its movement through a magnetic field is one of translation only. In the previous statement the two directions contemplated as forming a reference plane were the length of the element and the direction of its movement, and the direction of the field relatively to that plane decided the question of whether an E.M.F. was induced. But a plane



(iii). Plan on a plane normal to the Field

Fig. 29. Simple translation of straight element.

normal to the field is at once defined by the direction of the latter, and by reference to this normal plane, the case can be put in an alternative form. Let the length of the active element AD be inclined to the direction of the field at some angle α (Fig. 29i); then the projection of its length on the plane normal to the field is ad (Fig. 29 iii). Next let the direction of the motion Om' (Fig. 29 ii) be inclined to the direction of the field at some angle β, and let the projection of Om' on the normal plane be Om; then if these two projections ad and Om enclose any angle, the movement of the conductor causes it to cut across lines along its length. If the three directions coincide as was first supposed in case (a) of § 2, the projections of the active conductor and of the direction of

movement on the normal plane coincide at a point, and the conductor merely passes end on through the field. If the plane formed by the length of the conductor and the direction of movement also contains the direction of the field (case (*b*) of § 2), the projections of the two former on the normal plane fall in one straight line, and do not enclose any angle; in this case either the conductor simply slides up and down the lines, or its motion is partly endwise through the field and partly sliding along the lines, but no lines are cut on either alternative.

But when the above expressions are used, it must always be borne in mind that no question of the avoidance of discrete lines by the moving conductor is in reality involved; the field is everywhere present in the space surrounding the moving body, and it is structureless although it has direction. The appearance of separate and distinct lines is only an accompaniment of the pictorial representation of the field and is simply due to the convention by which the strength of the field is indicated by the number of lines per square centimetre.

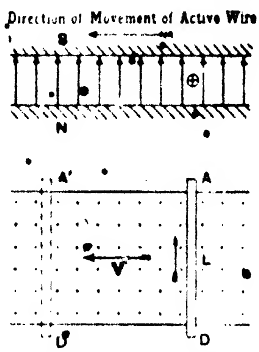


FIG. 30.—Maximum E.M.F. from conductor moved in magnetic field.

§ 4. The direction of the E.M.F.—When the length of the conductor lies in the plane normal to the field, and it is moved at right angles to its length in that plane (Fig. 30), then if the field is directed as shown in that figure, the "sense" or direction in which the E.M.F. acts along the length *AD* is from the observer, so that the end *A* is at a higher potential than the end *D*, as shown by the arrow in the plan.

In this, the simplest case of electromagnetic induction, it will be seen that the directions of the field, of the length of the active element, and of the movement are all at right angles to each other, and form co-ordinate axes, as indicated in perspective in Fig. 31; further, the direction of the induced E.M.F. is most easily remembered and discovered by means of the following rule. *Place the* **right** *hand outstretched in the line of the active conductor, so that the thumb points along the direction of the field,* i.e. *towards the S. pole, and further so that when relative movement takes place the lines of induction pass across the hand from the palm to the back; then the induced E.M.F. is directed from the wrist to the tips of the fingers* (as shown in Fig. 31). It will be found that, according to this rule, if it is the conductor which is moved while the field is stationary, the palm of the hand must face in the direction towards which the movement

is made, as in the case illustrated; but the direction of the E.M.F. would be exactly the same if a moving field crossed the conductor in the opposite direction, the conductor itself remaining stationary, in which case it is the back of the hand which must face in the direction of movement. Or again, both field and conductor may move in opposite directions; but in all cases the lines of the flux must by the movement traverse the right hand from the palm to the back, if the direction of the E.M.F. is to be found by the above rule.

In the general case of Fig. 29, it is only necessary to apply the same rule to the projection of the actual length of the conductor on the plane normal to the field. By the rule of the right hand, when laid outstretched along the projected length *ad* of the active

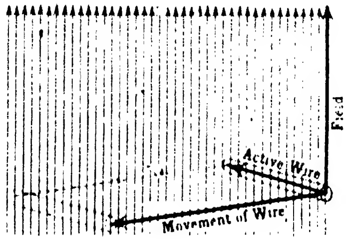


FIG. 31. Co-ordinate axes of length of conductor, and directions of movement and field.

conductor, so that the lines traverse the hand from the palm across to the back, the E.M.F. is directed from the wrist to the tips of the fingers.

§ 5. The magnitude of the E.M.F., and its average and instantaneous values.—Just as the nature of the material makes no difference to the inducing of an E.M.F. when the three directions are not in the same plane, so likewise it does not in any way affect its magnitude. A crucial experiment in proof of this was made by Faraday when he showed that an iron wire and a copper wire twisted together, joined at one end and connected at their other ends through a galvanometer, gave no deflection when the pair was passed across a magnetic field.

The actual magnitude of the E.M.F. induced along the length of the element considered is in all cases proportional to the *rate* at which the lines giving the strength of the field are cut along that length. The larger the number cut in a given time, or the shorter the time taken to cut a given number, the greater is the E.M.F.

The average rate of cutting to which the average E.M.F. is proportional is the ratio of the number of lines cut in any time to

the time taken to cut them; so that if Φ is the total number *cut* along the length of the body under consideration in time t, the average E.M.F. induced between the ends of that length is proportional to Φ/t, or $E_{av} = k\Phi/t$, where k is some constant, the value of which depends on the system of units according to which the field is mapped out in lines and time is reckoned. On the C.G.S. absolute system the magnitude of a unit of induction is so chosen that one line cut per second generates the absolute unit of electromotive force, and k is 1. As, however, unit E.M.F. on the absolute system is inconveniently small, the practical unit of the *volt* is one hundred million times greater, and consequently 100,000,000 C.G.S. lines cut per second produce one volt. If, therefore, Φ be reckoned in C.G.S. lines, and t in seconds, and E is to be measured, as is usual, in volts, $k = 10^{-8}$, or

$$E_{average} = \frac{\Phi}{t} \times 10^{-8} \text{ volts.}$$

The C.G.S. unit line is therefore as inconveniently small as the absolute unit of E.M.F., and a "kiloline," equivalent to 10^3 C.G.S. lines, or a "megaline" equivalent to 10^6 C.G.S. lines, have been sometimes adopted as the practical unit of flux, so that the fluxes and densities met with in dynamo practice may be expressed by more manageable quantities.

But if the rate of cutting is variable, then (as in the case of all phenomena which can be expressed in terms of a time rate) a distinction must be drawn between its average and instantaneous values. To approximate to the actual rate of cutting when this is variable, the number of lines cut in a very short time must be taken and must be divided by that small time, until finally for complete accuracy the actual rate of cutting at any instant is the limiting value obtained when the infinitesimally small number of lines cut is divided by the infinitesimally small time in which they are cut, or in Leibnitz's notation of the calculus, the actual E.M.F. induced at any moment is –

$$e_{instantaneous} = \frac{d\Phi}{dt} \times 10^{-8} \text{ volts} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

§ 6. The magnitude of the E.M.F. induced by simple translation of a straight element in terms of the virtual cutting length and velocity in the plane normal to the field.—Returning to Fig. 30 representing a straight wire being moved parallel to itself across the lines of a uniform field, it will be seen that the plane containing the length of the wire and the direction of movement is itself normal to the field; and that the direction of movement in that plane is at right angles to the length of the wire. Let L be its active length within the field in cm., and upon the line giving the direction

of movement let a length V be marked off equal to the velocity in cm. per second; then the area in sq. cm. swept through in one second is $ADD'A' = LV$. Hence if B_g is the uniform density of the flux in the air-gap in C.G.S. lines per cm.², the E.M.F. induced between the points A, D is

$$e = B_g LV \times 10^{-8} \text{ volts} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

But in the more general case of Fig. 29 when the three directions of field, active element and movement are not all at right angles to each other, and considering the plane normal to the field as in § 3, the projection of the length L of the active element on this plane is $l = L \sin \alpha$ (Fig. 29 i), and this value must be taken instead of L as being the effective cutting length. The velocity V inclined at an angle β to the direction of the field may be resolved into two components $V \sin \beta$ in a plane normal to the field, and $V \cos \beta$ in a plane parallel to the lines of the field. So far as the latter component is concerned, the movement is merely a sliding one along the lines, so that it is only the former component $v = V \sin \beta$ which is effective in inducing an E.M.F. In the projection Om of the direction of movement on the normal, let $Op = dg = v$ (Fig. 29, iii). The parallelogram $a\,d\,g\,e$ formed by the effective length and the effective velocity is then the projection on the normal plane of the actual area $ADGE$ swept through in one second. If γ be the angle of inclination of l and v to each other, the projected area $= l\,v \sin \gamma$. Over this area the full density B_g holds, and the number of lines cut per second is $B_g\, l\, v \sin \gamma$. Substituting the previous expressions for l and v, the induced E.M.F. is therefore—

$$e = B_g\, LV \sin \alpha \,.\, \sin \beta \,.\, \sin \gamma \,.\, \times 10^{-8} \text{ volts} \quad . \quad . \quad . \quad (8)$$

The E.M.F. induced in a rectilinear element by simple translation through a field is thus proportional to the virtual area swept through on the normal plane when the length and velocity of the rectilinear element are projected thereon, multiplied by the density of the lines assumed uniform over that area.

Such is the complete law of the E.M.F. induced in a straight conductor moved parallel to itself through a magnetic field, and the close analogy of (8) with the expression (3) previously obtained for the magnetic pull on a straight current-carrying conductor immersed in a magnetic field will be noted. They may be put into similar words as follows—

With any position of the conductor relatively to the lines of field and the line of action, or line of movement in the E.M.F. case, either the projections of the conductor's length and of the line of action or movement are in the same straight line, or they are inclined to each other at some angle other than 180°. In the first case there is no magnetic pull acting on the conductor tending to

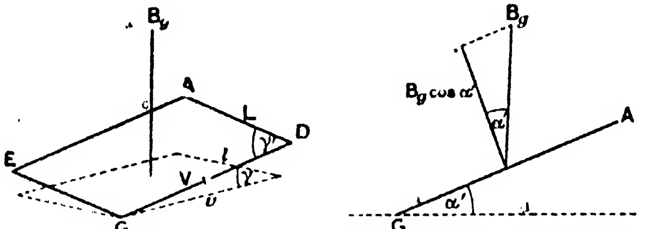
move it in the given line; nor is there any E.M.F. through movement along the given line. In the second case there is a magnetic force or component of a pull along the line of action, and a certain E.M.F. induced by movement along the line of movement. Whenever a conductor is subjected to a magnetic pull, the line of action along which the force is a maximum is at right angles to the projected length and in the normal plane, and it is along a similar line of movement that the greatest E.M.F. is induced for any given inclination of the conductor to the field.

§ 7. Lenz's Law.—By comparing Figs. 22 and 30 it will be seen that the direction of greatest magnetic pull is exactly opposed to the direction in which the conductor must be moved in order to give the greatest E.M.F. Again, by comparing Figs. 21 and 29 it will be seen that there is a component of the magnetic pull along a line of action which in Fig. 21 was chosen so as to be directly opposite to the line of movement of Fig. 29. Yet in both cases the current has been shown directed similarly to the E.M.F., *i.e.* from *D* to *A*. This opposition of the pull to the movement giving the E.M.F. is embodied in the rule that in determining the direction of an E.M.F. along a given line of action the *right* hand is used, while in determining the direction of pull on the conductor the *left* hand is used. Suppose that the conductor conveying a current from *D* to *A* is free to move under the action of the pull; then at once it cuts the lines of the field, and the induced E.M.F., as shown by the application of the right hand, is from *A* to *D*; it is therefore a *back* E.M.F. opposing the flow of the current. This is the case of the *electric motor*. Whenever a conductor is free to move under the magnetic pull on it, or under some component of it, it always cuts lines along its length, and induces an E.M.F. so directed as to oppose the flow of current to which this movement is itself due. That it must so oppose the current is evident if we consider what would be the consequences were it to assist the flow of current; the latter would be increased, the pull would proportionately increase, and the conductor would move with ever increasing velocity, which in turn would induce a continually increasing E.M.F. in the same direction as the current; this process would then go on for ever until both the E.M.F. and current were infinitely great. An indefinite amount of energy in an electrical form would thus be obtained without the expenditure of any, in contravention of the law of the conservation of energy.

Next, let the current in *AD* be itself due to movement of it through the magnetic field; that is, an E.M.F. is being generated in it, acting from *D* to *A*, and the case is that of the *dynamo* or *electric generator*. The direction of movement must be towards the left, but the direction of the force acting on the conductor, due to its carrying a current from *D* to *A*, is towards the right, and so is

directly opposed to the movement which produces the E.M.F. Hence the movement of the conductor is resisted by a mechanical drag in the opposite direction, and this fact may conversely be used to determine the direction of the E.M.F. along the length of the active wire. For, suppose a current to flow under the induced E.M.F., the direction of the two being the same; then this direction must be such that there shall be a force acting on the wire opposing the movement which induces the E.M.F. In all cases, therefore, of electromagnetic induction the direction of the induced E.M.F. must be such that a current flowing under it tends by its electro-dynamic action to stop the motion which produces the E.M.F. This general statement was formulated by Lenz in 1834, and is known as *Lenz's law*.

§ 8. The E.M.F. in terms of the component of the flux-density normal to the plane containing L and V and the angle between L and V in that plane.—If it is preferred to consider the reference



View from above the normal plane

Plane normal to field and containing l and v.

Fig. 32.

plane containing L and V as in § 2, let L and V be inclined to one another at the angle γ'. The area of the complete parallelogram swept through in one second is then $ADGE = LV \sin \gamma'$ (Fig. 32). If the direction of B_g is not at right angles to the plane of L and V, let it be resolved into two components, one parallel to the plane and the other at right angles to it. The value of the latter component is $B_g \cos \alpha'$, where α' is the angle between the direction of the field and the normal to the plane of LV, and it gives the virtual density over the area $LV \sin \gamma'$ at right angles thereto. The induced E.M.F. is then—

$$e = B_g LV \cos \alpha' . \sin \gamma' . \times 10^{-8} \text{ volts} \quad . \quad . \quad . \quad (9)$$

The E.M.F. is thus from this point of view proportional to the area swept through in one second multiplied by the component of the flux-density which is at right angles to the plane of L and V.

The equivalence of the two modes of expression (9) and (8) follows at once from the proposition of solid co-ordinate geometry that if A_a is the area of any closed plane figure and A is the area

of its projection on another given plane, $A_1 = A_0 \cos \alpha'$, where α' is the angle between the planes, *i.e.* the angle that the direction of a normal to one makes with the direction of a normal to the other. In the present case if AG is the diagonal of the parallelogram formed by L and V, it is inclined to the normal plane at the angle α' which is the same as the angle between R_a and its component perpendicular to the plane of L and V, and the area formed by L and V, viz., $LV \sin \gamma'$, when projected on to the normal plane, is $LV \sin \gamma' \cos \alpha' = l \cdot v \sin \gamma$. Hence $\cos \alpha' \sin \gamma' = \sin \alpha . \sin \beta . \sin \gamma$.

§ 9. Effect of width of active element.—In the production of an E.M.F. by line-cutting, any line of action joining two points A and D in a conductor may be considered, and this line need not coincide with the length of the conductor, as tacitly assumed in the previous cases. It might *e.g.* be necessary to find the E.M.F. acting across or slanting from one side to the other of a wide conductor; it would only then be necessary to consider a thin lamina within the substance of the conductor and across it in the required direction, and to treat it as the length of the active element in the same way as before, by *e.g.* projection of it on to the plane normal to the field.

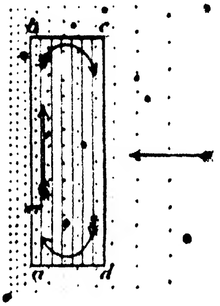


FIG. 33.

Again, when the active element has width, as AD in Fig. 29, and it is required to find the E.M.F. acting between its two ends A and D, in strictness it would be necessary to divide it mentally along its length into a number of very thin laminae lying side by side; the area traversed by any of these would in any time be equal since they move with equal velocity in the same direction, and therefore with a field of uniform density the E.M.F. induced along each would be equal, and they may all be regarded as in parallel. But if the one edge of the whole bar were moving in a field of different density, as in Fig. 33, where ab is moving in a stronger field than cd, then ab will induce a greater E.M.F. than cd. Consequently, unless the thin laminae are electrically separated by some insulating material, a current will flow round the bar as a whole in the direction shown by arrows. The case is introduced owing to its importance in the manufacture of dynamos, as will subsequently appear; it will only here be added that the difficulty is overcome, if midway in the length of the bar the positions of the insulated laminae are transposed by twisting the whole through 180°, so that *e.g.* the lamina which is on the right outside edge for half its length reappears on the left outside edge: the E.M.F.'s of all laminae are

thereby equalized, and they can be joined in parallel by soldering at the ends ad, bc without fear of any parasitic current flowing up one side and down the other.

II. THE "LINE-LINKAGE" LAW

§ 10. Time-variation of magnetic flux linked with an electric circuit.—In place of a small portion of a possible circuit, let a loop forming a closed circuit be considered, and let it be linked with a certain flux Φ_l. The classical experiments of Faraday then showed that if the flux linked with the loop be varied, an E.M.F. is induced in the closed circuit, and that its magnitude is proportional to the time-rate of change of the lines of the flux linked with the loop. In the practical unit of the C.G.S. electromagnetic system

$$e = -\frac{d\Phi_l}{dt} \times 10^{-8} \text{ volts} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

the negative sign indicating that if the linked flux increases the E.M.F. induced thereby is a back E.M.F. so directed that if a current flowed in its direction, it would always by its magnetic effect oppose the increase of the linked flux. This leads to the rule that if an observer is looking along the lines of flux in their positive direction as they pass through the circuit, and the number linked with it increases, the direction of the E.M.F. round the circuit is counter-clockwise.[1] The relation between the circular direction of the E.M.F. round the circuit as viewed from the side which the lines enter, and increase or decrease of their number is therefore exactly the opposite of that between the rotation and the forward or backward movement of a right-handed screw.

If the circuit has several loops or turns T_c in series, and each of these T_c loops were interlinked with the same flux, the time-rate of change of the flux will induce the same E.M.F. in each loop, and the total E.M.F. may be written indifferently either as $-T_c\, d\Phi_l/dt$ or as $-d(T_c \,.\, \Phi_l)/dt$. But in general the same flux will not thread through all the loops, and in such a case for the circuit as a whole

$$e = -\sum_1^{T_c} \frac{d\Phi_l}{dt}$$

Φ_l being the linked flux in relation to each loop separately, and any fraction of a loop being taken into account while T_c need not be a constant.

If the equation be written $e = -dN/dt \times 10^{-8}$, where N is the total number of linkages of lines with the circuit (any line which

[1] A circuit being now in view, the negative sign is inserted and bears a definite meaning, whereas it would be inappropriate in equation (6), which deals solely with line-cutting by an isolated conductor which only potentially forms part of a circuit.

passes through n turns and so is linked n times with the circuit being reckoned as giving n linkages), the E.M.F. is made to be proportional to the time-rate of change of the number of linkages. But when the law is so expressed and N, which is virtually a composite symbol for the number of linkages $= \sum_{1}^{T_t} \Phi_l$, is employed (as in almost all cases it may be), it must be borne in mind that it is essential under the present law that in any loop in which an E.M.F. is induced there must be a time-variation of Φ_l. It may in exceptional cases be possible for the number of linkages to be varied by alteration of the number of loops in the circuit without alteration in the values of Φ_l in those which at any instant are present, and therefore without any induced E.M.F.[1]

To this second law of electromagnetic induction based on the time-rate of change of the flux threading through a circuit, there is again no exception.[2]

§ 11. The line-linkage law, not limited to cases when mechanical motion is absent.—The above law of electromagnetic induction is evidently especially appropriate to the case of the *static transformer* in which mechanical motion is absent. Under this second law electrical energy supplied to a primary conducting circuit can be transmitted through space to a secondary conducting circuit, where it still appears as electrical energy. But though the rates of supply

[1] As in the experiment devised to test the question and tried by A. Blondel, *Comptes Rendus*, vol. 159, pp. 67 679, abstracted in *Electrician*, 11th June, 1915, Vol. 75, p. 344.

[2] The ingenious experiment due to Carl Hering (*Trans. Amer. I.E.E.*, vol. 27, part II, p. 1341, and *Electrician*, vol. 75, p. 559, 16th July, 1915, with subsequent correspondence) only contradicts the shortened form of words in which the law is often expressed and not its true meaning; it admits of a simple explanation upon which no E.M.F. should be expected from it. As soon as the spring clip in the original form of the experiment touches the magnet and opens, there are no longer two definite circuits, one electric and the other magnetic, simply interlinked, and the law in its simple form does not directly apply. The real connection across the blades of the clip is the whole of the magnet section—not the skin of the upper part of the magnet any more than the skin of the lower part, nor an imaginary shortest line between the contact surfaces of the springs which would in truth make the linked flux progressively decrease. The section of the magnet then forms to all intents and purposes a second closed electric loop embracing a certain amount of flux. By the operation of drawing down the spring clips, this second loop is forcibly thrust against and finally into the first loop of the clips, electrical continuity being maintained. The circuit therefore at the commencement of the drawing-off process is not the simple one which embraced the flux originally, but a composite one, consisting of a smaller loop inside and touching the larger loop, and the flux which is supposed to be linked with the latter is really contained within a convolution temporarily in contact with the large loop. All that is proved is that with two closed electric loops, the one inside the other, and the inner embracing flux, the inner with its flux can be withdrawn through an opening in the outer without the latter being necessarily opened electrically.

of electrical energy to the primary on the one hand, and of its development in the secondary on the other hand are equal (under ideal conditions and apart from certain secondary effects), the transmission is usually accompanied by a *transformation* of the voltage, so that the voltage and current forming the factors in the two rates are different in the primary and secondary system respectively.

But in order that there may be any *conversion* of mechanical into electrical energy, there must be actual mechanical motion of one portion of the total system relatively to another portion, as called for by the definition of the dynamo (Chapter I, § 1), and the application of the line-linkage law is in no way limited to cases in which mechanical motion is absent. This is shown by Faraday's original experiment, in which the increase or decrease of the linked flux was produced by the insertion of a magnet into, or its withdrawal from, a helix of wire. The change of linked flux which the law presupposes need not be due to alteration of exciting current-strength, but alternatively may be due to mechanical movement. In the latter case, the direction of the E.M.F. is such that if a current flows under it, the magnetic pull that would arise between conductor and magnet, or between the two portions of the magnetic system that are moved relatively to one another, will always oppose the movement.

§ 12. The two causes (A) and (B) of induced E.M.F. — Now though each of the above two laws may be true and may hold over a wide field or even universally, it does not necessarily follow that either must be the best or only expression for the true physical cause of an induced E.M.F. When the analysis is pushed far enough, the physicist finds that there is still much to be learnt as to the ultimate causes. Hence even at the cost of introducing greater complexity and with it perhaps a loss of lucidity, something more must be added from the stand-point of the engineer in regard to the confessedly obscure problem[1] of the causes of induced E.M.F.

Making use of modern views of electromagnetism, the writer believes that two closely related but distinct causes can be formulated by either of which or by both simultaneously an E.M.F. is induced. They are as follows—

(A). Motion of a material body through magnetized ether, in such wise that the direction of movement is at right angles, or has a component at right angles, to the direction of the magnetization. This yields what has been called a "motional" electric intensity

[1] See S. J. Barnett, "Report on Electromagnetic Induction," *Trans. Amer. I.E.E.*, vol. 38, part II, p. 1495, and W. F. G. Swann, "Unipolar Induction," *Physical Review*, 2nd series, vol. 15, p. 365.

at a point in the moving body, and a *motional* E.M.F. along any line of action in it.[1]

(B). The propagation of a wave of electric force from each and every spot at which the flux-density B in the ether is changing in time. When such a wave, proceeding with the velocity of light from any and all spots at which there is disturbance of the ether's magnetic state, meets the whole or a portion of an electric circuit, it *is* the E.M.F. induced therein. The cause is here the rate of change dB/dt in the ether, and to the E.M.F. yielded by it the term *induced* is sometimes confined.[2]

The first cause (A) presupposes motion of the material body in which the E.M.F. is induced; the second or (B) process does not necessarily imply any motion of a material body, although often the magnetic disturbance of the ether may be due to such motion. The first is typically the case of the heteropolar or homopolar machine, each with revolving smooth armature and stationary field-magnet. The second finds its most striking example in the static transformer in which there is no motion of any material body.

If the two causes have been correctly differentiated and formulated, it is evident that the absence of any disturbance of the magnetic state of the ether or change of the spacial flux-density in time in the former case with a stationary field-magnet prevents it from being brought under cause (B), and the absence of mechanical motion in the transformer case prevents it from being brought under cause (A). It may, however, be that by some widening or modification of the statements the E.M.F. of the one class may successfully be explained as being due to the same cause as that which acts in the other class. In this direction the attempt has mostly been made to bring the static transformer under a widened line-cutting law by the ascription of movement to the lines of flux in it. The two causes (A) and (B) must, therefore, each in turn be further considered, in order to bring out their distinctive differences, and as a result to show that it is questionable whether any true explanation of the physics of the static transformer and toothed armature can be obtained on the supposition of moving lines of flux.

(A) Motion of a Material Body Through Magnetized Ether

§ 13. The "drift" of electrons.—The properties of the magnetic field cannot attach to the empty space of geometry, so that even though ordinary ponderable matter may be absent, they may for our present purpose be assigned to an all-pervading ether. The

[1] No origin by which to judge of the relative movement of body and ether is stated or is needed, for a reason which, it is hoped, will be made clearer in § 15.

[2] *Cp.* S. J. Barnett, *loc. cit.*, p. 1152.

progress of physical science may necessitate a radical modification or even abandonment of former theories as to the nature of the ether, but such possibilities of the future need not affect the elementary use of the conception that is alone required here.

When a material body is moved mechanically through a stationary magnetic field in such a way as to cut across the lines marking the direction of the flux in the magnetized ether (*cp.* Fig. 30), there seems no doubt that the reason for the inducing of the E.M.F. is to be explained as follows: by such movement the electrons of the ponderable matter are definitely carried *across* the magnetized state of the ether, and whenever this is the case, a regular "drift" is given to them[1] along the length of the body, negative electrons moving towards one end and making it negatively charged, while a corresponding positive charge appears at the other end. The only difference between an insulator and a conductor is that in the former the degree of freedom with which the electrons can move is much restricted. If the body is a conductor and its circuit is closed, as *e.g.* in Fig. 22, the flow of electrons becomes continuous so long as the motion lasts, forming a current of electricity.

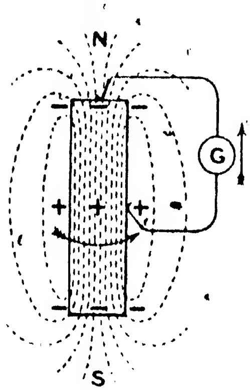


FIG. 34.

§ 14. Rotation of a homopolar magnet of Class I, i, giving a steady magnetic field which is uniform in the direction of movement.—Before proceeding to the ordinary homopolar machine of class I, i, let us consider the case of a permanent magnet, the shape of which is a solid of rotation, arranged as in Fig. 34 with a conducting loop connecting the axis of the magnet with its central neutral zone by contacts permitting the loop or the magnet to be rotated. When the loop is rotated, its electrons are moved through magnetized ether, and an E.M.F. is induced as shown by a current through the galvanometer, the case being similar to that mentioned in § 13. But now, when the loop is stationary, and the bar magnet of circular section is rotated, an E.M.F. is induced and current flows, although the loop is at rest in a magnetic field which at each point is constant in its magnitude, direction, and sense. The state of magnetization of the ether cannot be said either to revolve or to stand still, rest or motion as applied to such magnetization being meaningless.[2] Yet the

[1] As explained by Prof. G. W. O. Howe, "Some Problems of Electromagnetic Induction," *Electr.*, Nov. 5, 1915, vol. 76, p. 169.

[2] *Cp.* Steinmetz, *Trans. Amer. I.E.E.*, vol. 27, part II, p. 1353, and H. Poincaré, *Eclairage Electr.*, vol. 23, p. 41.

steel is being moved through magnetized ether and there is an E.M.F. induced. The explanation is that the electrons of the steel magnet are being forcibly moved through the region of magnetized ether which exists within the magnet itself; they separate, and so far as the flux passes outwards through the polar sides of the magnet and not from the ends, those of one sign move to each end of the magnet, leaving the centre oppositely charged.[1] If then the centre of one end of the magnet is connected to the surface at the centre of its length by a stationary external wire, a continuous current will flow.

A feature special to the above case is that the rotating magnet itself forms part of the electric circuit, and the torque arises from the mutual force between the two portions of the electric circuit which move relatively to one another.

Passing to the ordinary homopolar type of Class I, i as shown in Fig. 8, let the armature be stationary, and let the magnet be rotated. The presumption[2] then is that as in the preceding case, the carriage of the electrons of the magnet through the ether in its magnetized state will cause minute currents to flow in the iron as the magnet is brought up to speed, and these currents will cause a re-distribution of electric charges, so that the whole pole-face on both sides of the air-gap becomes, say, positively charged, and the centre of the magnetic circuit oppositely charged. The static charges on the pole-faces then induce an opposite charge on the stationary cylindrical armature. The action of the machine, so far as the production of a difference of potential between the ends of the armature is concerned, is thus reversible, but in this and similar cases when the magnet rotates, no use can be made of the static separation of charges. When the potential difference set up thereby is equal and opposite to the inducing E.M.F., the action ceases. It is only when the circuit is closed as in Fig. 8 that a continuous current flows, but now the external circuit being taken through the magnet must rotate with its brushes B and A, so that again there is relative movement between the two portions of the electric circuit, and the torque eventually falls on the iron magnet.

§ 15. Movement or otherwise of the ether itself immaterial.—In the case of a homopolar field perfectly uniform in the path of the movement and perfectly constant, rest or motion cannot be predicated of the lines of the field. The question of whether they move with the magnet when this rotates or remain stationary is only asked owing to the convention by which the magnetic field

[1] The writer again follows the above-quoted article by Prof. Howe, *qu. v.*

[2] Although not, so far as the writer is aware, indisputably proved nor indeed easily susceptible of a clear experimental proof. Any testing lead, when carried *through* the rotating magnet to the far end of the armature, itself becomes the seat of an induced E.M.F.

is pictorially represented by discrete lines of induction: one is led thereby to argue as if they could be labelled and identified, and would then be found to be either attached to the iron faces of the pole-pieces or to the armature, whereas the magnetic field is structure-less.

But this granted, the question of whether the ether itself, in contrast to its magnetized *state*, takes or any rotation or is at rest has not been disposed of. It has been tacitly assumed above in §§ 12 and 13 that the ether is at rest and cut through by the material particles of the conductor in the first case or of the iron magnet in the second case. But now it will be noted that if the ether rotates with the rotating conductor and at the same speed, the pole-face will, as assumed in the second case, become charged and again induce an E.M.F. in the circuit of the closed conductor; if the ether rotates with the rotating pole-pieces, the E.M.F. becomes induced in the conductor as assumed in the first case and calls for a similar charge on the pole-face, while for any intermediate speed of the ether, the E.M.F. in the conductor will be due partly to the first and partly to the second cause. On all three suppositions, it may reasonably be assumed that the resulting E.M.F. would be identical. The consequence is that no experiment on the lines of Fig. 8 can afford any information as to the movement or otherwise of the ether. Without entering into the question of relativity and a deeper analysis of the physical and philosophical problems involved therein, the view taken here is that the physics of the ether for the reason stated above become for our purpose immaterial, and therefore under cause (A) no definite statement is made as to whether the ether in its magnetized state is stationary or moving. In either case there is in the dynamo movement of *a* material body through magnetized ether, because there is *relative* movement between two portions of the combined magnetic and electric system, and in the end between two portions of the electric circuit, although the torque may by the mechanism of the ether be transferred from the latter to the iron. In any actual experiment, V in the equation $e = BLV$ finally turns out to be the relative velocity between two material parts, and the induced pressure as measured, being the sum (or difference) of an electromagnetically and an electrostatically induced pressure, is independent of the speed of the ether. In any explanations or calculations of an E.M.F. that may be required, the ether can therefore in all cases be assumed to be at rest.

§ 16. The line-cutting law as applied to cause (A).—Cause (A) includes (among others as will be seen later) all cases of smooth armatures rotating in a heteropolar or homopolar field due to a stationary magnet system, and to all these cases the line-cutting law applies. Its immediate applicability to the case of an armature rotating in a stationary heteropolar field is evident. It is equally

evident that it applies also to a smooth armature rotating in a homopolar field when this is due to a stationary field-magnet system, and is not uniform in density in the path of the motion, *i.e.* in cases where the polar surface is broken up into definite polar projections of the same sign and the armature, if so desired, can be truly *wound*. The line-cutting law may also be extended to the smooth armature rotating in a stationary homopolar field-magnet in which there are no polar projections and in which the flux-density is perfectly constant and uniform in the path of the movement. It is true that in this latter case it cannot be asserted that the flux, whether in the shape of discrete lines or as a structureless state, stands still in order to be cut by the rotating conductor. But the essential idea being motion at right angles to the direction of flux as causing an E.M.F. at right angles to both, there is but little objection to "line-cutting" upon the arbitrary convention that the lines of the stationary field-magnet are attached to it and stand still with it, provided that it is clearly borne in mind that the true cause is the motion of the conductor through and at right angles to the magnetized ether.

(B) THE PROPAGATION OF A WAVE OF ELECTRIC FORCE BY REASON OF CHANGE IN THE MAGNETIC STATE OF THE ETHER

§ 17. The line-cutting law under cause (B).—But it must not be supposed that the truth of the line-cutting law as a quantitative expression of the E.M.F. is confined to cases under cause (A). We therefore pass to cases in which the cause of the E.M.F. is to be found under (B), and the first of these is the case of the stationary smooth armature when a field-magnet which is either heteropolar or homopolar but of non-uniform flux-density in the path of the motion, revolves about it. Under (B) the cause of the E.M.F. is dB/dt, and this rate of change must be considered for each spot in relation to the ether permeating air and iron, and not in relation to any unit volume of iron in which, as it moves, B may perhaps remain constant.

§ 18. Movement of a steady magnetic field which is non-uniform in the path of the movement as a cause of E.M.F. in a stationary conductor.—Returning to the case of Fig. 30, let the pole-pieces now be of finite length so that at the edges the strength of the magnetic field must taper off, or, to go still further, let the field become reversed in direction owing to the presence of a second pair of pole-pieces of opposite polarity. Let the inducing element be at rest, and let the pole-pieces be mechanically moved. As the pole-pieces move, at their edges and especially at the centre of an inter-polar gap the ether undergoes a magnetic disturbance which progresses in space with the motion. Thereby a wave of electric force is propagated with the velocity of light from each centre of

disturbance opposite an interpolar gap up to the pole-centre, where the two waves reinforce one another and produce a maximum electric force in a direction at right angles to the magnetic field and parallel to the stationary element. When, therefore, the centre of the pole-pieces passes the stationary element, the latter experiences the maximum induced E.M.F. which is primarily due to action elsewhere owing to the field not being uniform, but tapering or reversing in the interpolar gap. When the armature is smooth, the waves simply result in a distribution of electric force in the space between the poles and the armature and in the surrounding space, which is constant and moving with the poles. The conductor, being stationary, is not cutting through magnetized ether, so that the source of the E.M.F. cannot be looked for under cause (A) by its definition. The magnetic state of the ether in which it is immersed is, however, progressively changing, and this *changing state of magnetization* does in fact traverse across the conductor, so that the line-cutting concept has meaning as applied to the case. Owing to the field being non-uniform in the path of the rotating field-magnet, the vector defining it varies at each point of space as movement proceeds, but since the field always remains similar to itself or constant, the vector system although occupying different positions in space, remains otherwise unchanged, and may truly be said to *move* at the same speed as the pole-pieces and to cut the conductor transversely along its length. With this limitation then to a constant magnetic field which is non-uniform in the path of the movement, it becomes legitimate to say that the magnetic field moves with the poles, although the magnetic state at each point including the position occupied by the conductor is more truly conceived as alternately dying away and growing again.

It must now be laid down that in such a case of a moving non-uniform field which retains under movement its configuration and the lengths of path of its flux-lines (when so represented), if any loop is formed to embrace a part of the flux, the total effect of the wave of electric force from the changing magnetic state of the ether, as it meets the two sides of the stationary loop wherever these are situated, is precisely equal to the algebraic difference in the rates at which the two sides of the loop are cut by the lines expressing the density of the fields in which at the moment they are actually situated. That this equality must hold and that the line-cutting quantitative law in the above form remains true, although the E.M.F. is due to cause (B), is supported by consideration of the case of a horseshoe magnet with a conductor in its air gap, when the whole, including the conductor, is moved bodily through space. The circuit of the conductor can be completed as a loop, of which the second side can be arranged so as to be virtually outside the field in unmagnetized air. There is no net E.M.F. induced in the loop,

yet the change of the flux-density in the ether through which the magnet is moved especially at the edges of the air-gap will from cause (B) yield a wave of electric force and an E.M.F. in the conductor in the air-gap. The presumption then is that it is exactly balanced by the E.M.F. from cause (A), due to the fact that the conductor is being forcibly moved through ether that is magnetized, at the spot where it is situated. If, therefore, the conductor remains stationary and there is no such balancing effect, there is left an E.M.F. expressible as a rate of cutting flux-lines, although it is now the varying magnetized state of the ether that is cut through and not the ether itself.

To distinguish the present case the E.M.F. due to it will be termed "quasi-motional."

§ 19. The connection between the "rotational E.M.F.'s" above considered and the mechanical force on the conductor. In all the cases above considered the E.M.F.'s, whether "purely" motional" under cause (A), or "quasi-motional" and induced under cause (B) as having their physical origin in dB/dt in the ether, may be grouped together as "rotational E.M.F.'s," and to them the line-cutting law applies.

It will now also be noticed that throughout the above cases the rotational E.M.F. for a given speed of mechanical movement is a simple linear function of the flux-density in which the conductor is actually situated at each moment. So also is the mechanical force on the conductor for a given current. The one is therefore proportional to the other, and it is by reason of this fact that the rate of absorption or development of mechanical energy *in the conductor* as a dynamo or motor can be equated to the rate of development or absorption of electrical energy in it, so far as the rotational E.M.F. enters as a factor.[1] The establishment of this fundamental relation as required by the principle of the conservation of energy is as follows.

When the conductor of Fig. 30 is mechanically moved towards the left with velocity v cm. per sec., and the E.M.F. induced in it is $e = B_g Lv \times 10^{-8}$ volts, let i be the current in amperes flowing through it under this E.M.F. when its circuit is closed as in Fig. 22. The rate of development of electrical energy in the circuit is then $ei = B_g Lvi \times 10^{-8}$ watts. But the conductor is acted on by a force $F = B_g Li \times 10^{-1}$ dynes tending to pull it towards the right; this force is overcome through a distance v in unit time, so that mechanical energy has to be expended in moving the conductor as a dynamo at the rate of Fv ergs per second $= B_g Lvi \times 10^{-1}$ ergs per second $= B_g Lvi \times 10^{-8}$ watts, which is identical with the expression for the rate of development of electrical energy.

[1] The bearing of the latter proviso will appear when the case of the toothed armature is considered.

Or again in the more general case of Fig. 29 the E.M.F. induced by movement of the conductor in the direction Om' is $B_g LV \sin \alpha . \sin \beta . \sin \gamma \times 10^{-8}$), directed from D to A. If this produces a current of i amperes round the circuit when closed, the conductor is acted on by a magnetic pull, $F = B_g Li \sin \alpha \times 10^{-1}$ in the direction OM (Fig. 21). But it is only the component of F along the line Om' which resists the movement, and this is equal to $F \sin \beta . \sin \gamma$. The product of the resistance overcome and the velocity V is therefore $B_g LVi \sin \alpha . \sin \beta . \sin \gamma \times 10^{-1}$ ergs per second $= B_g LVi \sin \alpha . \sin \beta . \sin \gamma \times 10^{-8}$ watts, which is again identical with the rate of development of electrical energy.

§ 20. Can the line-cutting concept be extended to other cases under cause (B)? It has been shown that the quantitative line-cutting law affords a connecting link between cause (A) and the quasi-motional cases in which the cause of the E.M.F. falls under (B), and this fact, even if it stood alone, bears witness to some real and close connection that exists between the two causes that have been differentiated as (A) and (B), and to the undoubted probability that at bottom they must be traceable to some one more general cause. It has therefore next to be examined whether the line-cutting law can also be extended to other cases which have so far been placed inferentially under cause (B)—in especial to the striking case of the static transformer, and to the toothed armature.

§ 21. The line-cutting hypothesis as applied to the static transformer. To explain the cases of an induced E.M.F. when mechanical motion is absent on the "line-cutting" hypothesis, it remains to assign movement in space to the field as mapped out by lines of flux, such movement arising from variation of the exciting current round an electromagnet or ring core to which the field is due or to variation of the current in the circuit itself which is under consideration. The theory based thereon is then as follows.

When a current starts to flow in a conductor, the magnetic field surrounding it has to come into existence, and the growth of this field is assumed to take place by a process of closed line-curves or flux-rings (such as those of Figs. 2, 3, or 4), expanding outwards from a point in the central axis of the conductor or from the centre of a uniformly-wound toroid, like the circular ripples caused by the dropping of a stone into still water, and cutting the conducting circuit as they expand. As the ring-lines first formed expand, new ones start and take their place and themselves in turn expand outwards. Whenever the exciting current increases its strength, the newer flux-rings drive further outwards and closer together the initial flux-rings, so that the field becomes denser, *i.e.* increases in strength. When the exciting current is decreased, all of the flux-rings contract inwards, and some disappear by absorption at the centre; they can never be broken or opened, but disappear by

collapsing to a point. The magnetic lines are thus regarded as behaving in fact like elastic rings stretched outwards by some internal pressure, and when the current diminishes, being unsupported, as it were, to the same extent by the falling current, they collapse inwards.

The explanation of the induced E.M.F. thus appears complete and consistent. The expansion and contraction of the flux-rings on the above view may conveniently be applied to determine by the hand rule the direction of an E.M.F. in a circuit, and especially of a self-induced E.M.F. in a portion of a circuit, even when there is no exact knowledge as to the paths of the lines and the parts of the circuit cut by them. In such cases some assumption which appears the most natural must be made. More than this, given complete knowledge of the static magnetic qualities of the circuit linked with the electric current, the speed of movement of the lines can be predicted and has mathematical truth.

§ 22. The objections to the theory and the alternative view.—Yet as a physical explanation of the real mechanism, a difficulty remains. The velocity with which the flux-rings must spring out from the conductor and travel is not a fixed quantity dependent on the specific properties of that medium in which they start, but varies with the quality of the whole of the magnetic circuit which is to be filled with flux. The analogy from wave-motion due to a stone dropped into still water is therefore illusory. From the kinetic energy possessed by the stone when it meets the water, the velocity of the initial propagation of the water waves is definitely determinable by hydromechanical theory without any prior knowledge of what they may subsequently meet with as they progress from point to point. And so also in the case of sound waves proceeding from some source, but which may in their progress pass from one medium to another. Whereas the rate at which lines of flux are shed off by a growing current and pass through an intervening air-gap, say, to an iron core, will be different from the rate if the iron-core be removed. There is, in fact, no independent evidence in physics for the supposed movement of the lines,[1] so that it does not appear that their velocity can be determined apart from a knowledge of dB/dt over the area of the circuit. This is virtually to know the induced E.M.F. so that if the view here put forward is correct, no additional knowledge is brought to bear on the determination of the E.M.F.

The difference from the preceding quasi-motional case of § 18 is that even when the configuration of the magnetic field remains the same, as *e.g.* circular lines round a straight wire, their length changes with their supposed movement, and it then appears to

[1] As pointed out by Fritz Emde in an article on the "Induction Law," *E. u. M.*, vol. 26, p. 997, ff., 15th Nov. 1908.

be truer, physically, to imagine the growth or decay of the magnetic state which is described as a flux, as taking place, so to speak, *in situ* in its original paths. *E.g.* in the case of a uniformly and closely wound toroid, it is not difficult and perhaps even simpler to regard the denseness of the magnetic state as growing or dying away in each infinitesimal circular ring within the coils without concentrating attention on a particular value of B as moving inwards and outwards with consequent alteration of the length of path over which it holds. The closely allied natures of the phenomena is, however, borne witness to, if the stator winding of a polyphase induction motor or the armature reaction of a polyphase alternator be considered; by the addition of two or more phases, an effect for which the explanation in the case of a single phase must be sought under the present section can be made to pass into the quasi-motional category.

Finally, therefore, in cases such as that of the static transformer where the change of flux-density is due solely to variation of the exciting current, according to the view here maintained, the wave of electric force summing up the whole effect of dB/dt over the area within the primary coils meets primary and secondary, and is the E.M.F. induced therein. Its value follows the change of B practically instantaneously, since the velocity of propagation of the wave is that of light. Mechanical motion being absent, it will be called a "transformer E.M.F. of the first kind."

§ 23. The toothed armature.—The extension of the line-cutting law to explain the whole of the E.M.F. of the toothed armature rotating within a stationary field-magnet system by ascription of movement to the lines of flux is made in much the same way as in the static transformer above considered.

For a given flux the E.M.F. of the rotating toothed armature (when minor pulsations are neglected) is the same as that of a similar smooth armature. The virtual equivalence of the two would best appear in the case of an armature having a single conductor embedded in a closed slot or tunnel close to the periphery and rotating in a homopolar field uniform in the path of the motion. There would then be no minor pulsations to confuse the issue. But as explained in Chapter IV § 7, the densities within the closed slot or tunnel and in the iron beyond it would be as $B_s : B_t$, where B_s is only a very small fraction of B_t, say $\frac{1}{100}$, and but little larger in proportion to B_g in the air-gap. If, therefore, the conductor situated in the weak field of density B_s is to yield the same E.M.F. and *for the same reason* as the conductor on the surface of the equivalent smooth armature, nothing is easier than to assume that the lines of flux snap across the tunnel against the direction of rotation, yielding an increased relative velocity V_s which so far exceeds that of the rotating armature that its product with B_s is equal to the product

of B_s with the peripheral velocity V of the armature, or $V_s B_s = V . B_g$ and $V_s = V \frac{B_g}{B_s}$

As quantitatively true over a tooth-cycle or any fraction of a tooth-cycle, no exception can be taken to the above supposition of an increased relative velocity of the lines, but to its truth as a physical explanation there remains a fatal objection. The exact correspondence between the rates of development or absorption of electrical and mechanical energy in dynamo or motor is the foundation stone of dynamo-electric theory. In the case of the toothed or tunnel armature experiment has conclusively shown that the force on the conductor is only that corresponding to the weak field in which it is situated. By no possibility then can the conductor absorb or develop in dynamo or motor mechanical energy at a rate greater than that corresponding to the product of the mechanical force acting on it and its own mechanical velocity. Yet this does not account for the whole of the electrical or mechanical power. The current as one of the factors of the electrical energy cannot be divided, and if the whole of the induced E.M.F. were *the same in kind*, due to the cutting of lines at an increased speed, the whole of the mechanical force or torque must fall on the conductor. It remains then to recognize that the E.M.F. is divisible into two parts, and that the division corresponds to the two different causes to which they are respectively due, as now to be shown.

In the homopolar field of uniform density, whatever the position of the conductor in the slot or tunnel, it is always situated in a weak field of definite density. In the heteropolar field, the field in the slots at any moment consists of isolated portions, as many as there are slots, but, whether there be one or many slots and whatever the position of the conductor in the slot, the conductor is always situated in the same density at the same spot relatively to the stationary poles. The density in which it is immersed is therefore a continuous function of the space moved through and forms a steady continuous field, constant in time. In either case, therefore, the conductor is in fact being driven forcibly through ether which is weakly magnetized either to a uniform or to a spacially varying degree. It has therefore set up in it by cause (A) an E.M.F. which is subject to the line-cutting law; this is a pure "E.M.F. of rotation," and is in strict correspondence to the mechanical force on the conductor.

But in addition at each edge of the moving slot or tunnel the ether is undergoing a marked magnetic disturbance from a strong to a weak magnetization or *vice versâ*. Therefrom a wave of electric force is continuously propagated to the centre of the slot and meets the conductor, causing in it an additional E.M.F. which may be called a "transformer E.M.F. of the second kind," as being due to

relative motion between parts of the magnetic system. Exactly in so far as there is such a transformer E.M.F., for a given current in the conductor the mechanical force falls solely on the iron portions of the structure.

A transformer E.M.F. which is not subject to the line-cutting law is therefore the accompaniment of the magnetic shielding of the conductor. If we start with a smooth armature and gradually embed the conductor within the iron, the E.M.F. gradually changes from a pure rotation E.M.F. to a composite of which the rotation part becomes less and less in proportion to the transformer part, and ends by being comparatively unimportant.

In the stationary toothed armature with rotating field-magnet the whole of the E.M.F. falls under cause (B), but is again divisible into the same two portions. So far as the magnetic disturbance is that due to the movement of the field as a whole without change in its configuration or in the lengths of its flux-lines, the E.M.F. is quasi-motional or a "rotation E.M.F." identical in nature with that described in § 18, and of the same value as in the rotating toothed armature. But this disturbance does not exhaust the whole rate of change dB/dt in the ether; at the edge of the slots there is again a rapid change in its magnetization, yielding the same transformer E.M.F. as above described.

§ 24. The limitations of the line-cutting law.—The line-cutting concept is thus directly applicable to *all* cases under cause (A), but if the rotating member be toothed and the embedded conductor be thereby magnetically shielded, as a physical explanation of the cause of the E.M.F. it is only true of one portion of the E.M.F.; further, it is applicable to cases under cause (B) in which there is change of magnetic state and a wave of electric force from a rotating field-magnet, but again if the stationary conductor be shielded, it is only physically true as a cause of E.M.F. in so far as the weak magnetization in the slot or tunnel changes and is virtually cut by the conductor. Although the line-cutting *law* is quantitatively true in all the above cases, yet the extension of line-cutting as a cause to the whole of the E.M.F. of the toothed armature involves a somewhat artificial explanation and obscures a physical difference which is of vital importance to the dynamo designer, while its extension to the remaining cases under (B) in which there is no mechanical motion may again be mathematically true, but has no independent physical evidence to support it.

§ 25. The line-linkage law as the more general concept.—Line-cutting hardly, therefore, affords a satisfactory bridge from a physical point of view between the extreme cases of (A) and (B), *i.e.* between a smooth armature rotating in a uniform homopolar field and a static transformer. It may, however, still be possible to bring both types of E.M.F., though due to different causes, under

a more generalized concept, and this is in fact found in the line-linkage law based on the rate of change of the lines linked with the electric circuit. Although not free from certain difficulties to be mentioned later, the law holds over the whole range of practical cases, in the sense that with or without mechanical motion, the total E.M.F. induced, whether due to (A) or (B), can be quantitatively expressed through the time-rate of change of the line-linkages.

§ 26. The equivalence of the two laws in the case of smooth armatures in a heteropolar or non-uniform homopolar field. The immediate equivalence of the two concepts in the case of all dynamos with smooth armatures in a heteropolar or non-uniform homopolar field, is at once evident, and it is a matter of convenience only which concept is chosen. It has been already pointed out that the application of any form of E.M.F. indicator to the ends of the inducing element virtually amounts to the formation of a closed electric circuit, even though it may be an imperfect one which only permits of a displacement current. It will then be found that wherever the indicating instrument and its leads are placed, if the rate of line-cutting is $d\Phi/dt$ in any case coming under the above heading, the flux threading through the closed circuit is varying at a rate $d\Phi_1/dt$ which is precisely equal to $d\Phi/dt$, so that the same induced E.M.F. is given by either law. That is to say, when the lines linked with the circuit are being altered in number owing to the mechanical movement of the circuit and the latter is in no way magnetically shielded, some portion of it must have been cut by the lines as they passed into or out of its embrace, and this portion is the element or system of elements which has been assumed to be active. Or if it be the field-magnet which is moved while the conducting circuit is kept fixed, the magnetized state which is described as the flux-density is, say, growing up on one side of an inducing single element and dying away on its other side. The two concepts then become indistinguishable, since such growth and decay on the two sides is essentially the same as the passage of flux across the inducing element. But it must be noticed that in this case when the magnet is the rotating member, there is no change in the configuration or lengths of the lines, so that the identity of the two concepts is confined to the motional and quasi-motional cases, if the idea of the increased relative velocity of the lines in a toothed armature is abandoned as not physically true.

§ 27. The line-linkage law as applied to homopolar dynamos with smooth armature rotating in a field uniform in the path of movement.—The equivalence of the two concepts in the case of a smooth armature rotating in a homopolar field which is perfectly constant and uniform in the path of the movement is more difficult of establishment, but may still be maintained. The difficulty lies in the interpretation of the fundamental idea underlying the phrase "enclosure" of flux by the circuit in these cases. To assist in analysing the problem, take a single radial conductor, extending from a shaft to a slip-ring, which on the line-cutting hypothesis cuts the lines of a uniform

field running parallel to the shaft. Let an external circuit R lying in a vertical plane be applied by means of brushes BB', as shown in Fig. 35*a*. Then it will be recognized that, the electric circuit being in one portion of its length split, there are two loops to be considered: in the one, $DAEB'RB$, the enclosed or linked flux is diminishing and becomes zero when AD passes RB, after which it again increases; in the other, $DAE'B'RB$, the linked flux is increasing to a maximum when AD passes RB. The rate of change in each is the same, and the cyclic direction of the E.M.F. in each is such (as shown in Fig. 35*a*) that it is directed radially outwards along DA and radially inwards from B' to B through E. As soon as AD passes RB, the first loop becomes the second loop, and *vice versa*, and the flux in each is equal when AD completes

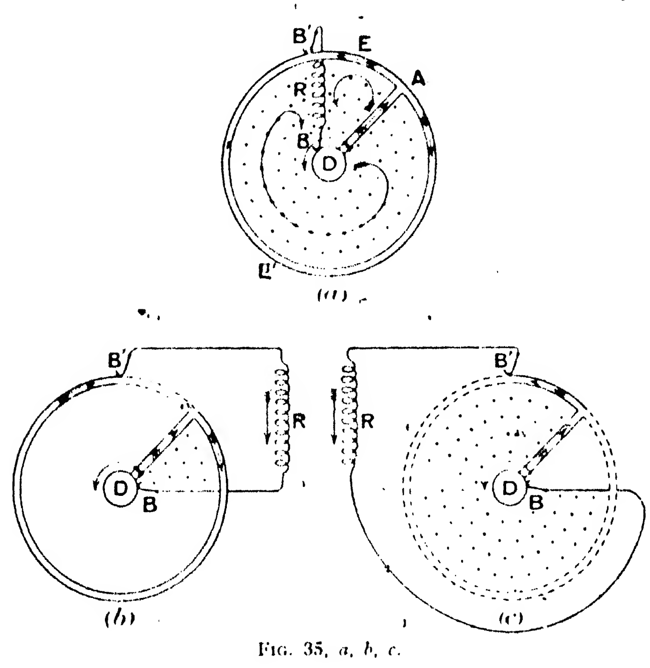


FIG. 35, *a*, *b*, *c*.

the diameter with RB. The two E.M.F.'s are not additive along DA, but are complementary, the one to the other, in the sense that the radial conductor AD could be mentally divided into two portions in parallel, one for each loop.

But now let the external circuit be arranged to one side of the collector ring (Fig. 35*b*). Then wherever the connecting lead BRB' is placed, the tendency is to regard the linked flux as that shown increasing in Fig. 35*b*, and to ignore the complementary truth which would appear if the external circuit were removed to the other side as in Fig. 35*c*. In the case of a homopolar dynamo with uniform field, a single active element and collector ring, a cyclic E.M.F. arises as much from what at first sight is outer flux as from the inner which appears more evidently embraced within the circuit. Consider a pair of parallel conductors, joined by a conductor AD, and the whole sliding under a pair of brushes BB' attached to an external circuit, so that DA cuts across the lines of a uniform field (Fig. 36*a*). Then as soon as the collector bars are bent up into closed circles (Fig. 36*b*), so as to permit of continuous rotation, and the passage of the lines correspondingly becomes radially outwards,

the flux to the right of DA round to the radial plane in which BB' fall is as much enclosed in the second loop as the flux to the left of DA up to the same plane of BB'. If R lies, as it may, very close to the ring, so that $B'RB$ also encloses the radial flux, the enveloping sheet representing the area of the one circuit must be carried from D clockwise to BB', and then fold back on itself to R, so that the flux from R to BB' cancels out, while correspondingly the sheet for the other circuit proceeds counter-clockwise from D to BB', and then must be extended up BB' to R.

To the homopolar dynamo with single active element, whether of the axial or of the radial type, let a second active element now be added, say at the opposite side of the circle. Nothing in principle is thereby altered, and each may be credited with its own two loops. Carried then to the limit when the entire cylindrical or the entire radial face has been filled with active elements, we reach the homopolar dynamo with complete cylinder or disc, which is thus to be regarded as made up of an infinite number of pairs of loops, such as have been described. Wherever and however these are placed and paired, so long as they conform to the required conditions, the same axial or radial E.M.F. is inevitably reached over the whole face of the disc or surface of the cylinder.[1]

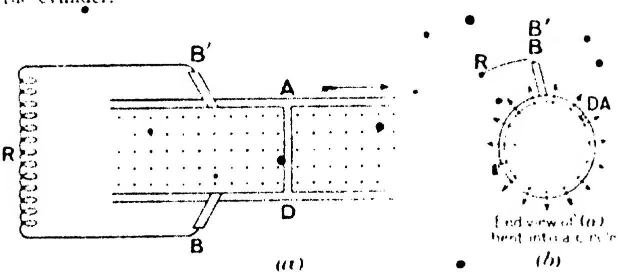


FIG. 36, *a* and *b*.

So far then the "line-linkage" law cannot be said to fail. Yet in its application to homopolar machines with field uniform in the path of the motion it loses much of its own initial simplicity, and does not afford such a convincing explanation of the natural phenomenon as that which is given by the line-cutting law. Further the province to which the concept of $d\Phi_1/dt$ naturally applies is that of all cases in which it sums up and expresses the integrated effect of dB/dt over the area of the circuit, but in homopolar machines of the present kind there is no quantity dB/dt existent at any spot, whether B is considered in relation to the ether or air or iron.

§ 28. The line-linkage law as applied to the toothed armature. If the line-linkage law is not to obscure the important physical characteristic of the toothed or tunnel armature and therefore is not to be open to the same objection as that already urged in § 22 against the supposed movement of all flux-lines across the slots, it must be interpreted in the following way.

The "true rotation E.M.F." of a loop on the toothed armature will be given by considering the effect of an infinitely small movement of the loop within the slots when the slots, armature iron and poles are held stationary,

[1] It is suggested that the line of explanation here adopted adheres more closely to the real nature of the physical facts than the alternative explanation which has frequently been given. This concentrates attention only on one loop and treats its area as expanding indefinitely so as to embrace an ever-increasing amount of flux. But the analogy to a trigonometrical angle of unlimited magnitude appears to the writer to partake too much of the nature of a mathematical subterfuge to be accepted as a satisfactory explanation. The simple fact is that after N revolutions the flux embraced by the loop, say of Fig. 35*b*, is not larger by N times the total flux but is the same as before.

and the exciting ampere-turns are unvarying, *i.e.* by taking the partial differential of the linked flux with respect to the angle α expressing the relative position of loop to poles. The "transformer E.M.F." can itself be resolved into two components, according to whether an infinitely small movement of the armature iron relatively to the poles with the loop imagined stationary alters the linkages when the exciting ampere-turns are constant, or whether the effective ampere-turns of excitation vary and therefore the total flux of a pole-pitch would pulsate even when poles, armature iron, and loop were stationary. That is, in addition the partial differentials of the linked flux must be taken with respect to the angle β expressing the relative position of armature teeth to the pole, and with respect to time. Thus—

$$d\Phi_l = \frac{\partial\Phi_l}{\partial\alpha} . d\alpha + \frac{\partial\Phi_l}{\partial\beta} . d\beta + \frac{\partial\Phi_l}{\partial t} . dt$$

Since in fact the loop and armature iron are rigidly connected,

$$d\alpha = d\beta = \omega dt$$

whence [1]

$$\frac{d\Phi_l}{dt} = \underbrace{\overset{\text{True rotation E.M.F.}}{\omega\frac{\partial\Phi_l}{\partial\alpha}} + \rlap{$\displaystyle\overbrace{\phantom{\omega\frac{\partial\Phi_l}{\partial\beta} + \frac{\partial\Phi_l}{\partial t}}}^{\text{Transformer E.M.F.}}$}\omega\frac{\partial\Phi_l}{\partial\beta}}_{\text{Rotation E.M.F.}} + \underbrace{\frac{\partial\Phi_l}{\partial t}}_{\text{Pulsation E.M.F.}} \quad . \quad (11)$$

Since the first two E.M.F.'s arise from rotation and correspond to mechanical torque on conductor or iron, they may be and frequently are grouped together (as shown in the lower line) as the "rotation E.M.F." in a different sense to that used above, and in contrast with a "pulsation E.M.F." which is not related to any mechanical torque.

Thus at least theoretically the line-linkage law does not labour under the same disadvantage as the line-cutting law that it cannot predict the true instantaneous E.M.F. of the toothed armature and differentiate its causes.

§ 29. Summary and conclusion.—But though the more generalized concept of the line-linkage law may even be held to be universally true, it does not appear to afford so clear an insight into the physical processes of nature as the separate causes which have been classified as (A) and (B).

A chief difference between the two causes and the two laws is that in the case of (A) when the magnetic field is undergoing no change, under the line-cutting law as applied to that case the electric force or intensity induced at a point has immediate reality and is definable. In case (B) when the magnetic state is changing, our lack of real knowledge as to the contour of the wave-front of electric force which results from the integrated effect of the regions of magnetic disturbance renders it difficult to determine the electric force at a point.[2]

[1] *Cp.* Fritz Emde, *E. u. M.*, 1909, No. 34, and R. Rüdenberg, *E. u. M.*, 1907, p. 600.

[2] Except perhaps by a generalized expression based on the vector potential of the moving electrons which themselves produce the magnetic field but without using the magnetic field as an intermediate link. *Cp.* S. J. Barnett, *loc. cit.*, p. 1160, and W. F. G. Swann, *loc. cit.*

TABLE I

Cause (A). Motion of material body through magnetized ether	(1) Stationary homopolar field uniform in path of movement.	Motional E.M.F.	Line-cutting Law. Line-linkage Law.
	(2) Stationary hetero-polar or non-uniform homopolar field	Motional E.M.F.	Line-cutting Law. Line-linkage Law.
Cause (B). Wave of electric force from change of magnetization of ether.	(1) Constant non-uniform field moving without change in configuration or lengths of lines	Quasi-motional E.M.F.	Line-cutting Law. Line-linkage Law.
	(2) Mechanical motion causing change in configuration of field or in length of lines if supposed to move	Transformer E.M.F. of 2nd kind	Line-linkage Law.
	(3) Mechanical motion absent.	" " " 1st "	Line-linkage Law.

Smooth Armatures	Rotating field-magnet stationary	Cause A.	Rotation E.M.F., motional.
	Stationary hetero-polar or non-uniform homopolar field rotating.	Cause B.	" " quasi-motional.
Toothed Armatures	Rotating	Cause A and Cause B.	E.M.F. partly rotational (motional) and partly transformer.
	Stationary	Cause B.	E.M.F. partly rotational (quasi-motional) and partly transformer.

We are thus compelled under cause (B) to fall back (in practical cases) on the total effect in the circuit as a whole given by the rate of change of the line-linkages, and since this is dB/dt integrated over the area enclosed by the circuit, it is essentially a function of that area: consequently the effect remains a property of the circuit as a whole. The electric force at a point cannot then be laid down, and the induced E.M.F. cannot be located except to the extent that in some closed circuits considerations of exact symmetry as between different portions can be applied.

A summary of the views that have been here put forward may assist the reader, and is given on page 83 in tabular form. The dotted bracketed portions in the first part indicate the less defensible extensions of the two laws. The second part is merely an additional classification based on the first part and intended to bring out the difference between the causes acting in smooth and toothed armatures respectively.

Fortunately the fact that the two causes do not appear to admit at present of ultimate synthesis into a single universal law resting on a physical foundation does not lead to any difficulty in the practical cases that meet the dynamo designer. Indeed the two causes need not have been considered at such length, were it not for the distinction mechanically between the cases of the smooth armature and the toothed armature.

CHAPTER VI

SELF-INDUCTION AND ALTERNATING-CURRENT PHENOMENA

§ 1. Second consequence of flow of current.—One of the consequences which follow from the magnetic field that surrounds a current-carrying conductor, namely, the pull on it when immersed in an external field, has been considered, and in explaining the interaction between the two we have spoken of its own magnetic field as being superposed upon the external field. A second important consequence of its magnetic properties remains to be traced, and with especial reference to dynamo work when the current in the conductor is itself due to an external field it has now to be asked, What is the effect of the co-presence of the two fields as regards the E.M.F. induced in the active conductor? In actual nature at any point in space there can never be more than one magnetic field as mapped out by lines of induction, and this will be the resultant arising from the magnetomotive force of the winding that produces the original inducing field combined with the M.M.F. of the current-carrying circuit. But it is often possible and legitimate to regard each of these M.M.F.'s as producing its own field or set of lines, and to consider each set as existing separately and thus to arrive at the actual resultant E.M.F.

§ 2. The E.M.F. of self-induction and the inductance of a circuit or conductor.—The current-carrying circuit has lines of flux linked with it, and so also has the conductor as forming part of a circuit. In the case therefore of the conductor as much as in the case of the circuit as a whole, if a current in it begins, or ends, or varies in strength, its line-linkages vary and an E.M.F. is induced in it which being induced by the current itself is called the *E.M.F. of self-induction*. It is always so directed as to oppose the change of current of which it is itself the effect, and all changes from one definite value of current to another definite value must take a certain time and cannot be instantaneous.

The well-known analogy of the effects from electrical "self-induction" to those arising from mechanical "inertia" is obvious, since in virtue of the inertia attaching to any mass of matter, a finite velocity cannot be given to it or taken from it instantaneously, nor can it be instantaneously altered to another value, as shown by the familiar instance of starting from rest, or stopping, or altering the speed of a heavy flywheel. It must, however, be clearly understood that the current itself has no quality analogous to inertia; it is only to the current as producing a magnetic field that the property attaches. For with the same current flowing round a circuit, the self-induction can be altered very greatly by any change

which affects the magnetic field surrounding the circuit; and it is therefore truer to regard every circuit as possessing a definite quality other than its electrical resistance, but which in conjunction with its electrical resistance determines the current flowing through it at any instant under any E.M.F.; in this quality is included its capacity as a condenser, but of still greater importance is its property of electro-magnetic inertia, or as it is briefly called its *inductance* since upon it depends the E.M.F. of self-induction. Obviously the inductance of a circuit will be dependent on the magnetic conditions which determine the number of lines connected with it when a given current is flowing, and hence is governed, not only by the geometrical form or shape of the conducting path, but more especially by the presence of iron within or near it. By a further extension in cases where some portion of the total magnetic field of the circuit can be legitimately assigned to a portion of the conducting circuit as causing it, it becomes practically permissible to speak of the inductance of a portion only of the circuit or of a single conductor as forming a possible part of a circuit.

§ 3. The calculation of inductance.—The general formula for the inductance of a circuit or part of a circuit in C.G.S. absolute units is $\mathscr{L} = N_s/i$ where N_s is the number of linkages (each C.G.S. line encircling n loops or turns being reckoned as giving n linkages) and i is the current in absolute electromagnetic units. The practical unit of inductance, the *henry*, being 10^9 times the absolute,

$$\mathscr{L}\,(\text{henrys}) = \frac{N_s}{i_{(\text{abs.})}} \times 10^{-9} = \frac{N_s}{i_{(\text{amperes})}} \times 10^{-8} \quad . \quad . \quad (12)$$

Thus the inductance is one henry if one absolute unit of current flowing in the circuit or conductor gives rise to 10^9 linkages, or one ampere gives rise to 10^8 linkages.

In the presence of iron with its varying permeability, the inductance is not a constant quantity, but depends on the current and other conditions which must be specified: in fact, the term "inductance" then admits of more than one definition. In dynamo-electric machines iron is almost invariably present in parts of the circuit for which the inductance has to be calculated. Yet in most cases it usually suffices to consider $\mathscr{L}$ as constant, so that for purposes of calculation one absolute unit of current or one ampere may be taken and the linkages reckoned therefor.

Next, in the case of a coil of two or more turns, all the flux will not be linked with all the turns, although this condition will be the more nearly approached, the closer together the turns are wound. When it may be assumed that the total flux Φ_s is linked with all the T turns,

$$\mathscr{L}\,(\text{henrys}) = \frac{T\Phi_s}{i_{(\text{abs.})}} \times 10^{-9} = \frac{T\Phi_s}{i_{(\text{amps.})}} \times 10^{-8} \quad . \quad . \quad (13)$$

Φ_s being the flux due respectively to the current in absolute units or to the amperes. If $\mathcal{R}$ be the reluctance of the magnetic circuit concerned,

$$\Phi_s = \frac{4\pi Ti \text{ (abs.)}}{\mathcal{R}} = \frac{4\pi}{10} \cdot \frac{Ti \text{ (amps.)}}{\mathcal{R}},$$

so that

$$\text{(henrys)} = 4\pi \frac{T^2}{\mathcal{R}} \times 10^{-9} = \frac{4\pi}{10} \frac{T^2}{\mathcal{R}} \times 10^{-8}. \quad (14)$$

The current, therefore, no longer appears, yet its value is virtually implied in the reluctance $\mathcal{R}$ which will be variable if iron be present. But again if the assumption of the preceding paragraph may be made that the iron is of constant permeability or if the flux so largely passes through air or other non-magnetic medium that the iron becomes of negligible importance, $\mathcal{R}$ or its reciprocal $\mathcal{P}$, the permeance, may be treated as a constant. Finally, therefore,

$$= 4\pi \mathcal{P} T^2 \times 10^{-9} \text{ or } 1{\cdot}257\, \mathcal{P} T^2 \times 10^{-8} \text{ henrys} \quad . \quad . \quad (15)$$

Since under the given assumptions all the flux is linked with all the turns and for the same current the flux is proportional to the number of turns, the inductance varies as the square of the number of turns.

§ 4. Impressed, self-induced, and resultant E.M.F.'s distinguished. From the fact that an increasing current is directly opposed by the self-induced E.M.F. it is evident that the latter cannot be the cause of the flow; there must then be another E.M.F. in the same direction as the current, and greater than the back E.M.F. We must therefore, in cases where the current is altering in value, distinguish between (1) the *impressed* E.M.F. and (2) the counter E.M.F. of *self-induction*; together they yield the *resultant* or *active* E.M.F. which immediately causes the flow of current. In the absence of capacity or when the capacity is negligible, the resultant E.M.F. at any moment is equal to the algebraic sum of the values of the other two. Thus, if e_1 = the impressed E.M.F. at any moment, and e_s = the self-induced E.M.F. at the same moment, the resultant E.M.F. to which the current is then proportional is

$$e_r = e_1 + e_s,$$

the actual algebraic sign of e_s depending on the question whether the current is increasing or decreasing in strength. Or if e'_s is the E.M.F. consumed by the self-induction, and so is the exact opposite of e_s, i.e. $= -e_s$,

$$e_r = e_1 - e'_s \text{ and } e_1 = e_r + e'_s$$

At any instant, the current flowing is $i = e_r/R$, as given by Ohm's law, R being interpreted as an *effective resistance*, which includes not only the ohmic resistance but also takes into account any back

E.M.F. by reason of which chemical or mechanical or electrical work is *usefully* done in addition to the mere dissipation of heat over the ohmic resistance.

§ 5. Storage of energy in a magnetic field.—But when the impressed E.M.F. is so divided, it may be asked whether the division finds any actual counterpart in the physical phenomena.

The answer is that two different kinds of work are being done. Just as in the armature of a continuous-current motor taking I amperes under an impressed voltage V and developing a back E.M.F. e, the work done is divisible into two portions, the one corresponding to I^2R_a appearing as heat, and the other corresponding to eI appearing as mechanical work, so now the electrical energy developed is expended in two forms, of which the one appears as useful work or as heat at the rate i^2R or $e_r i$, while the other is energy stored or liberated in the magnetic field at the rate $e'_s i$ watts. The ether surrounding the metallic conductive circuit is an elastic medium which is magnetically stressed through the growth of a field round the conductor, and then acts as a bent spring which can again give back the energy expended in bending it.

The creation of a magnetic field demands energy, and the total amount absorbed in the process of bringing the current up from zero to its steady value I can be shown to be $\frac{1}{2}\ I^2$. This amount represents the electromagnetic energy of the field, corresponding to the kinetic energy, or $\frac{1}{2}mv^2$ of a moving body; it is measurable in ergs or joules according as the absolute or practical system of C.G.S. units is employed. When once the field is established, *i.e.* when a steady current is set up, no further expenditure of energy is required to maintain it. But the energy expended in establishing it is not irrecoverable; it is as it were stored up and can be liberated. For suppose e_i to be instantaneously withdrawn; then the self-induced E.M.F. tends to keep the current flowing, and does actually do so, since the current only falls to zero after a certain period of time. During this time work is being done in the circuit, the energy stored up in the magnetic field reappearing as heat, *e.g.* in the spark which occurs when the circuit is opened.

§ 6. The case of the continuous-current machine.—If a conductor is moving at a constant velocity between two pole pieces of infinite length, as in Fig. 30, and carries current due to the E.M.F. of movement, the E.M.F. and current are steady, and the magnetic effect of the current remains unvarying in intensity and position relatively to the conductor. The latter may then on the basis of two sets of lines be said to carry its own field along with it, and therefore can never as it moves cut its own lines. Ahead of itself the resultant lines are denser than the original field; behind it they are less dense, but the variation of the distribution travels with it, and only the lines of the original field are cut.

But it is also possible that a number of active conductors may be so arranged that at once they carry a steady current, and, although each moving, yet as a system yield a magnetic field that is constant and stationary in space. Such is the case of the continuous-current armature of Fig. 25. When therefore the steady value of the current has been established, the stationary magnetic field of the armature as a whole in combination with the inducing field will yield a resultant stationary field of different distribution which contains within itself the effect of both. In this case the rotating conductors do actually cut through the resultant field, *i.e.* through the original field after it has been reacted upon by the M.M.F. of the armature ampere-turns. The E.M.F. thence found is, however, a resultant E.M.F.; since the current is steady, there is no effect from self-induction. Yet in the resultant field causing the E.M.F. is stored the energy corresponding to the inductance of the armature ampere-turns, as well as the energy corresponding to the establishment of the original inducing field. Thus if the armature circuit is broken while the excitation of the field-magnet is unaltered, the former energy will be liberated and tend to maintain the current in the same direction as at the moment of breaking the circuit, while the field is reverting to its original distribution and original quantum of energy.

§ 7. **The case of the alternate-current machine.**—But in the alternate-current machine the current varies owing to relative movement: thus in a current-carrying loop rotated under pole-pieces, as in Fig. 18, the impressed E.M.F. alternates, and the inductance also varies, so that the distribution of the actual resultant field as containing the varying magnetic effect of the current varies. To the consideration of this case on the basis of two superposed sets of lines we now return.

To put it in its most elementary form, consider a loop of wire (Fig. 37) which is rapidly pushed up to the pole of a magnet from some position outside its field; by the cutting of the lines of the external field an E.M.F. is set up in the direction of the arrow on the loop: this increases in strength, since more and more lines are cut as it moves through the dense field near to the pole. Under the action of the E.M.F. in the closed loop a current begins to flow; but as soon as this current begins, its own lines of flux form loops linked with the electrical circuit. A second field is thus superposed on the first, and the direction of the lines of this second field as they pass through the loop is opposite to that of the first: this is roughly indicated in the diagram by the dotted lines lying counter to the full lines of the original field. It follows that the rising current tends to set up a field of lines opposite in direction over the area of the loop to those of the external field; or, in other words, it tends to reduce the flux-density through the loop, just at the time when

it is being increased by the motion. Consequently the rise of current in the loop is not so rapid as it would be if the current had itself no magnetic effect, or we may say that the current by reason of the magnetic qualities of its circuit reacts on the original field. At any moment the resultant field in which the loop is moving has a certain value and distribution, and the rate at which the lines of this field are cutting the conductor loop gives the E.M.F. to which the current at that moment is proportional; but this actual field may conveniently be resolved into two—the original field and the current's own field, the effect of the two being considered separately. At any instant the lines induced by the current itself are increasing at a certain rate, and their increase as linked with the loop causes a "transformer" E.M.F. in the negative direction round it. It is thus opposed to the impressed E.M.F. due to the cutting of the original field, and the rising current induces a counter E.M.F. opposing its own rise.

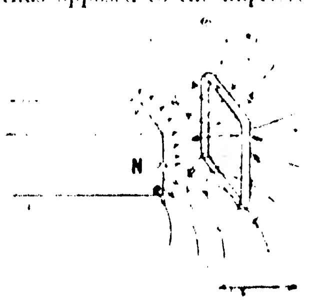


FIG. 37.

If the movement of the loop is reversed, and it is made to recede from the magnet, the direction of the impressed E.M.F. will be reversed, and so also eventually will the current. But not at once; the lines of its own field have to die away, to be succeeded by fresh lines passing in the reverse direction through the loop. The change of the line-linkages is continuous, and both the dying away of lines in one direction and their growth in the opposite direction cause a self-induced E.M.F. in the same direction as that in which the current was flowing; this for a time tends to counterbalance the new E.M.F. impressed on the loop by the external field, and hence the current only gradually sinks to zero and finally becomes reversed. Thus the magnetic effect of the current shows itself by modifying the external field, and such expressions as the inductance of the loop and the energy stored up in its field, although legitimate, are based on a mental separation of the actually existing field into two component parts. It is not that the loop first reacts on the field, and then cuts the resultant field by its own movement; the resultant field changes relatively to the conductor, partly by reason of the latter's own movement, and partly by reason of change in its current. To the movement of the resultant field relatively to the conductor the resultant E.M.F. is due; and if we were to consider this resultant field, we must not also credit the loop with self-induction or inductance; it must then be considered as a circuit possessing only resistance.

It is, however, more convenient in most cases to consider some part or parts of an alternate-current circuit as possessing inductance, and to take into account the effect of such separate inductance in combination with an impressed E.M.F.

§ 8. The impressed, self-induced and resultant E.M.F. curves in an alternating-current circuit.—The number of linkages of a conductor or circuit carrying an alternating current with its self-induced lines will vary in time, and this variation over a period may be due not merely to the variation of the current, but also to variation in the length or nature of the magnetic circuit that is

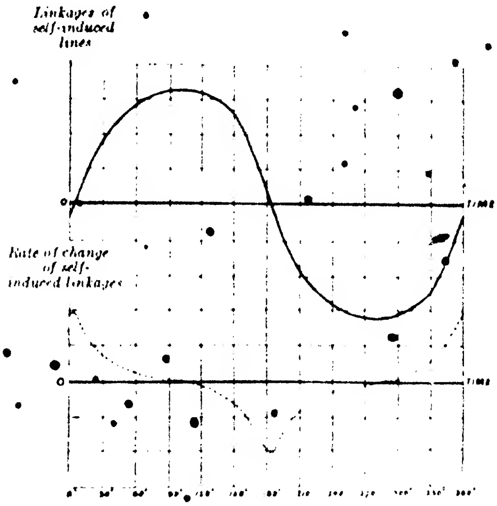


FIG. 38.

at any moment presented to it. In Fig. 38 let the full line be the linkage curve of the self-induced lines plotted on an axis of time over a complete period; this is not necessarily of the same shape as the current curve, since, as already mentioned, the number of self-induced lines may not vary directly as the current under all circumstances, nor need the phase of the current curve and of the curve of self-induced flux be the same; there may be a lag of the flux value behind the magnetizing current in respect of time, and, as will be shown hereafter, where a portion or the whole of the magnetic circuit through which the self-induced lines pass is composed of iron, there will be such a lag, although it may be very slight. Hence in Fig. 38 the full-line curve is shown passing through zero at an instant slightly later than zero time, which is reckoned here from passage of the current through zero. From the curve of self-induced

linkages by finding the "slope" of the tangent to it at each point, may be derived a second curve (shown dotted), representing to some scale the time-rate of change in the number of linkages, or dN_s/dt, where N_s is the total number of self-induced linkages; this may be unsymmetrical if the original curve is unsymmetrical, and may differ very greatly in shape and character if there are minor undulations on the original curve, even when these appear small, and almost negligible.

But the E.M.F. of self-induction is proportional to $-dN_s/dt$, so that if the derived curve of Fig. 38 is inverted, it will also represent to some scale the E.M.F. of self-induction at any instant, and at the same time indicate the direction of this E.M.F. When the number of self-induced lines is increasing in the positive direction, or decreasing in the negative direction, the E.M.F. is negative or in a direction opposing a positive current; and when they are decreasing in the positive direction, or increasing in the negative direction, the self-induced E.M.F. will be in the positive direction. Thus if the curve of linkages shows, as in Fig. 38, a continuous increase of lines, and then a continuous decrease of lines, although the rate of this increase or decrease may be very different at different points, the curve of self-induced E.M.F. will pass through zero when the total number of self-induced linkages is a maximum, and for a whole period will be divisible into two portions, below and above the horizontal axis.

If the magnetic circuit has a constant inductance $\mathcal{L}$, the self-induced E.M.F. can be derived immediately from the current curve, since $e_s = -\mathcal{L}di/dt$; but in the general case where the inductance may be variable, e_s must be taken as $= -dN_s/dt$, and the intermediate curve of N_s is required.

In Fig. 39 let the dotted curve E_s be the dotted curve of Fig. 38 when inverted, being, therefore, to some scale the curve of the self-induced E.M.F. in volts, and upon the same horizontal axis of time let the thick line (E) be the current curve. It will be seen that, roughly speaking, the phase of the self-induced E.M.F. lags 90° behind the phase of the current, but owing to the slight lag of the flux behind the magnetizing current the curve of self-induced E.M.F. does not reach its maximum until after the current curve has passed through its zero. But the thick-line current curve will also represent the curve of resultant E.M.F., provided that the scales be so chosen that the same height represents indifferently either one ampere or one ampere multiplied by the effective resistance of the circuit or portion of the circuit under consideration. In Fig. 39 this is supposed to be the case, so that the volts of resultant E.M.F. can be read off the curve E. We are now, therefore, in a position to deduce the curve of the impressed E.M.F., which must have acted on the circuit in order that with the assumed conditions

the curve of resultant E.M.F. should have the shape shown in the diagram. Since we have obtained curves which determine for us the signs at any moment of e_i and e_s the universal equation of §4, $e_i = e_r - e_s$ shows that we have only to subtract the ordinates of the self-induced E.M.F. curve from the ordinates at the same points of the resultant E.M.F. curve and plot their difference as a third curve, due regard being paid to the algebraic signs of the ordinates. The third-line curve so obtained (E_i) will be the required curve of impressed E.M.F. in volts. Or the curve $e'_s = dN_s/dt$, *i.e.* the dotted curve of Fig. 38, might have been plotted without

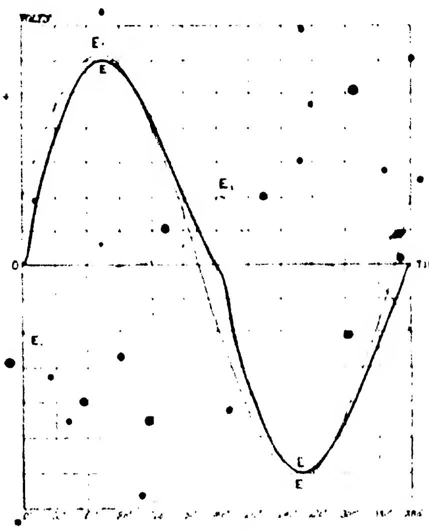


Fig. 39.

inversion, when we should have had to add the ordinates since $e_i = e_r + e'_s$ and e'_s is one of the components of the impressed E.M.F. But in either case if two out of the three curves are known, the third is also determined.

§ 9. Lag of resultant behind impressed E.M.F.—It is evident that the shape of the resultant E.M.F. curve may differ materially from that of the impressed E.M.F.; for example, in the assumed case of Fig. 39 the curve of impressed E.M.F. which has been deduced logically from the previous curves, is strictly a sine curve, yet the curve of resultant E.M.F. is not a sine curve, and is not even symmetrical on its ascending and descending sides. Next, it will be seen that, while the self-induced E.M.F. always opposes a rising e_r, it does not always assist a falling e_r. From this, two important consequences follow: (1) Since the self-induced E.M.F. does not

always assist a falling e_a but does always assist a falling current or falling resultant E.M.F., the curve of resultant E.M.F. cannot coincide in phase with the impressed E.M.F. curve, but, on the contrary, the current curve lags behind the curve of impressed E.M.F. by a certain time depending on the inductance of the circuit ; and further, this time is usually expressed as an angle or fraction of 360°, being measured by reference to the bipolar case when one period corresponds to 1 revolution, or 360°. (2) So long as the self-induced lines continue to rise, and their number of linkages to increase, however slowly, there is some self-induced E.M.F. opposing the impressed E.M.F. ; but at the instant when they are at their maximum, and the self-induced E.M.F. is zero, the curve of resultant E.M.F. cuts the curve of impressed E.M.F. Where this point of intersection is, entirely depends upon the curve of self-induced linkages ; and this depends, not only on the curve of magnetizing current, but also on the magnetic circuit through which at any instant it is inducing lines. Hence, if the curve of self-induced linkages continues to rise after the current curve has begun to fall, owing to a more than proportionate decrease in the reluctance of the magnetic circuit, the curve of resultant E.M.F. will cut the impressed E.M.F. at a point after it has reached its own maximum, as in fact is shown in Fig. 39. If, however, the maximum number of self-induced lines coincides in time with the maximum strength of the current, the highest value of the resultant E.M.F. will be its point of intersection with the curve of impressed E.M.F. Further, a given magnetic circuit always permits of an increase of the number of lines of flux through it, when the magnetizing current is increased. If, therefore, the impressed E.M.F. is never constant, but *always* altering in value, the curve of self-induced lines never becomes a straight line (unless the magnetic circuit be altered so as to exactly counterbalance the changing current—a rare possibility) ; hence there is a definite self-induced E.M.F. at the moment when the impressed E.M.F. reaches its maximum, and the point of intersection must be subsequent to the point of highest impressed E.M.F. ; in other words, E can never attain as high a value as E_i.

The whole may easily be illustrated by the case of a flywheel to which is applied a turning force, which not only alternates in the direction in which it tends to turn the wheel, but also varies in value from zero to a maximum, and thence, passing through zero, to a maximum in the opposite direction. Since the turning force is never steady, but varies continuously, the flywheel would never reach the maximum velocity corresponding to a steady turning force equal to the maximum value of the alternating force ; before reaching such a velocity the value of the turning force has already begun to decrease, and in just the same way the resultant E.M.F.

lags behind the impressed E.M.F., and its maximum value is less than the maximum value of the impressed E.M.F.

§ 10. Power in an alternating circuit as determined graphically.— In any portion of a circuit the rate of development of electrical energy at any instant is equal to the product of the current which is then flowing in that portion of the circuit and the E.M.F. which is impressed upon it at that instant. Thus, in Fig. 40, if the thick line represents the curve of current in amperes flowing through any portion of a circuit, and the thin line represents the corresponding curve of E.M.F. in volts which is impressed upon that portion, the rate of development of energy in watts at any instant a is equal

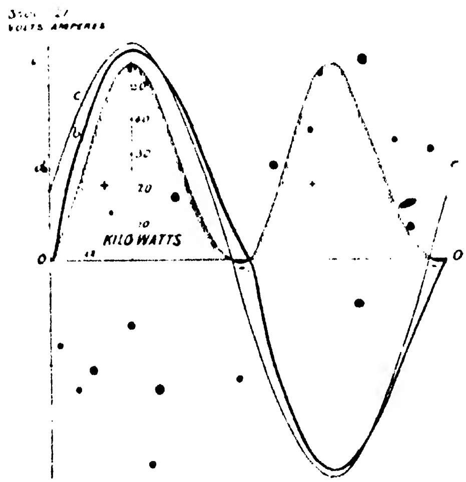


FIG. 40

to the product of the ordinates ab ac, which represent the amperes flowing at that instant and the impressed volts. By thus multiplying together a number of simultaneous values of the impressed E.M.F. and current, and plotting their products along the same horizontal axis, a third curve (shown with a shaded fringe in Fig. 40) is obtained, representing the instantaneous rate of development of energy in watts throughout an entire period, and the area which it encloses, being the product of power and time, represents work done. In so doing we must pay attention to the algebraic signs of the E.M.F. and current, all ordinates above the horizontal line being reckoned as + and all below as −; and further, if their product be positive, it must be plotted above the horizontal line as positive work; if negative, below it, as negative work. Since the product of two quantities, one + and the other −, is negative, the product of two

ordinates is negative unless both are above or both are below the horizontal line, *i.e.* positive work is done only when current and impressed E.M.F. are in the same direction.

The positive work of an alternator is the sum of two separate portions. One part is expended in heating resistances or in forcing a current against a back E.M.F. due to causes external to itself, as in transformers, both being useful work so far as the alternator is concerned. It is developed at the rate i^2R watts, where R is the effective resistance, as defined in § 4. The other part is the work done in the creation or re-establishment of the magnetic field of the circuit as described in § 5. The negative work is due to the liberation of the energy stored in the magnetic field which is expended partly in heating the ohmic resistance of the circuit and partly in driving the machine as a motor.

In each half-period, in virtue of the growth of the magnetic field, energy first passes outwards from the circuit into the surrounding medium where it is temporarily stored, and then at a later stage in the same half-period this stored energy is returned by the surrounding medium into the conductive circuit. The final result is that when a periodic alternating current has been established, so far as the magnetic stress upon the ether is concerned, the net energy which has passed in either one of the two directions, *i.e.* either into the surrounding medium or out of the medium into the circuit, is at the end of a half-period or any whole number of half-periods zero. If the duration of a period considerably exceeded the time of one revolution, say, of a steam-engine, the latter would be called upon to develop first more and then less than its average power; but since in alternating circuits as commercially used many complete periods occur in the time of one revolution, the prime mover does not show on its indicator diagram any sign of difference in its rate of doing work, so far as this is dependent upon the reaction of the magnetic field. The rhythmical fluctuation of the power, as energy is alternately stored in or released from the field of the generator, is in practice obscured from our view by the mechanical inertia of the fly-wheel and other moving parts, which suffices to absorb or give out the necessary energy as required by its electrical analogue of inductance.

Thus iR (when R is not merely the ohmic resistance but is interpreted as explained above) represents the energy component of the E.M.F., and its product with the current corresponds to the net expenditure of energy, while the product of e_s with i corresponds to the energy which surges to and fro between the conductive circuit and the medium which surrounds it. Consequently the total effective work done by the alternating current and E.M.F. in one complete period, or the total energy transformed from electrical energy into heat or other useful work, is measured by the net area

enclosed by the fringed curve in Fig. 40, when the two negative areas (shown black and marked –) are added together, and their sum subtracted from the two positive areas. It will be seen that the undulating form of the curve is due to the fact that current and E.M.F. are continually varying, but the appearance of the negative (shown black) power, or the double frequency of the power transference is solely due to the "lag" of the current curve behind the impressed E.M.F. curve, by reason of which the impressed E.M.F. and current can be in opposite directions. The physical explanation of the negative work in the alternator is that the direction of the armature current displaced in phase becomes periodically so related to the field poles that it assists in driving the machine as a motor. The greater the lag of the current curve, the smaller the net amount of work done. If the lag were to amount to as much as a quarter of a period, the entire current curve being retarded in phase by an angle of 90° as compared with the E.M.F. curve, so that the current value is zero when the impressed E.M.F. is a maximum, and *vice versâ*, then in each half-period the negative area is exactly balanced by an equal positive area, and the net work done in a half or any number of half-periods would be nil, the explanation being that the magnetic field would then be giving back as much energy in one quarter of a period as was previously stored in the preceding quarter by the source of the impressed E.M.F. On the other hand, if there is no inductance and no lag, there is no negative work done, since the phases of E.M.F. and current coincide. Both cases are ideal, but serve to indicate the theoretical limits to which practical cases approximate, and as a circuit realizes one or other ideal more or less closely, it is classed as either an ***inductive*** or a ***non-inductive*** circuit.

Given, therefore, the two curves of impressed E.M.F. and current, a curve can be deduced whose area represents the work done, and whose ordinates represent the power developed at any instant. The mean power or mean rate of development of energy will be the mean of all the values of the product of current and impressed E.M.F. taken over a sufficiently long time; since the positive and negative half-waves of E.M.F. and current are in each case alike, it will actually suffice to take the mean of all the values during one half-period. The mean value of the power during one period will evidently be represented by the mean ordinate to the curve of power, *i.e.* an ordinate *Od* (Fig. 40) of such a height that when multiplied by the length *OO′* the area of the rectangle so formed *OdeO′*, is equal to the net work done, or the difference between the areas of positive and negative work.

The whole of the above is applicable not only to any portion of a circuit, but also to the circuit as a whole; in the case of an alternator supplying energy to the external circuit its output or rate of

development of energy in the external circuit is equal to the mean ordinate of a curve formed by multiplying together simultaneous values of the P.D. impressed on the external circuit from its terminals and of the current flowing in the circuit. Thus in Fig. 40 if the curves give simultaneous readings of the volts and amperes in the external circuit of a single-phase alternator, its output, as shown by the mean ordinate to the power curve, is 26 kilowatts, and would be so recorded on a wattmeter.

§ 11. The power factor of an alternating circuit.—In the theoretical case of an alternating circuit having no inductance or capacity, but simply ohmic resistance, the product of the *effective* or *virtual* or *root-mean-square* values of the volts and amperes will measure directly the mean power therein developed. But in the more general case of a circuit having inductance and capacity, the product of the $\sqrt{\text{mean square}}$ or R.M.S. values of the E.M.F. and current or EI is the *apparent power* of the circuit, and will be greater than the true power if there be any phase difference between impressed E.M.F. and current. The ratio of the true power or watts to the apparent power or watts is called the *power factor* of the circuit. In the case of a circuit which is either non-inductive or in which the E.M.F. and the current are in phase by reason of the capacity effect exactly balancing the inductance effect, the power factor is unity, but in all other cases it is some fraction less than 1. Its value may be obtained from the ratio of the wattmeter reading to the product of the ammeter and voltmeter readings for the same circuit. Thus the apparent power of the alternator of Fig. 40 is 2255 volts × 12·1 amperes = 27·3 kilovolt-amperes, while the true power is 26 kilowatts, whence the power factor is 26/27·3 = 0·95. A calculation of the power of an inductive circuit traversed by an alternating current can, however, readily be made, if it is permissible to assume some simple law to govern the periodic variations of the E.M.F. and current.

§ 12. Simplification by assumption of a sine-law hypothesis.—In § 8 the effect of self-induction has been expressed in general terms independent of the exact shape of the curves drawn and assumed, but for the purpose of the simpler process now required the curves are assumed to obey a *sine law*. From the mathematical nature of a sine curve, it follows that the curve derived from it and expressing its rate of change will also be a sine curve of the same periodic time, but differing by 90° in its phase. If, therefore, the current curve in any piece of alternating machinery follows a sine law and if its inductance be strictly constant throughout the whole periodic time, so that the flux-lines due to the current are strictly proportional to the current and follow simultaneously upon its variations, the curve of self-induced linkages (Fig. 38) will be a sine curve; consequently the dotted curve derived from it, which,

when inverted, represents the self-induced E.M.F., will also be a sine curve of the same periodic time but differing by 90° in its phase. Further, if two sine curves of the same periodic time be compounded together, as was done in § 8, the third curve so deduced is a sine curve again of the same periodic time, but differing in phase

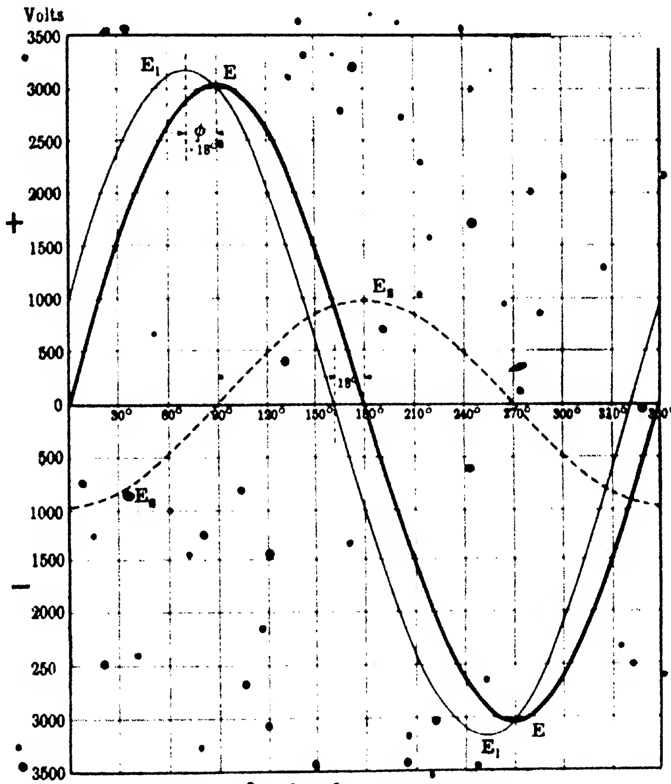


FIG. 41.—Equivalent sinusoidal E.M.F. curves.
Angle of lag of current, φ, = 18°.

from either of the two components, and therefore the curve of impressed E.M.F. (in Fig. 39) will be such a curve. Fig. 41 shows the combination of two sine curves to form a third, and it is therefore the counterpart of Fig. 39 on the supposition that both current and E.M.F. are sinusoidal. From a comparison of the two figures it will be seen that the chief difference is that in Fig. 41 there is a definite angle of lag (18° in the diagram) between the current and

impressed E.M.F. curves. Not only does the current pass through its zero at a certain angle behind the zero of the impressed E.M.F., but it passes through its maximum with the same angle of lag. Further, the maximum ordinate of the resultant E.M.F. coincides with its point of intersection with the impressed E.M.F., and lastly the maximum value of the resultant E.M.F. is necessarily less than the maximum ordinate of the impressed E.M.F. The effect, therefore, of constant inductance in any portion of a circuit traversed by an alternating current is to shift the entire current curve backwards by a constant angle of lag, and to reduce the height of all the ordinates of the resultant E.M.F. curve as compared with those of the impressed E.M.F. curve. We thus obtain three sine curves, representing respectively impressed E.M.F., self-induced E.M.F., and resultant E.M.F. or current, mutually related to one another by simple mathematical laws, so that from any two the third is easily derived, and from the mathematical nature of sine curves we can go on to deduce other facts respecting the three related factors. Now the assumption that in any given alternator the three curves are sine curves is not by any means necessarily true, but experience has shown that in modern machines no great error arises in theory or practice from assuming that they *are* sine curves—at least for a first approximation to enable practical problems to be readily treated mathematically and graphically. The curves of impressed E.M.F. of modern alternators do closely resemble sine curves in their general nature, and so also but to a less degree do the curves of current under normal conditions of working.

§ 13. Power factor given by the cosine of angle of lag on sine hypothesis. If simultaneous values of the ordinates of two simple sine curves of equal period, but of different amplitude and phase, be multiplied together, it can be shown mathematically that the mean of all the products so obtained is equal to half the product of the maximum values of the two sets of ordinates multiplied by the cosine of the angle which expresses their difference of phase. Hence if the two curves of E.M.F. and current be sine curves as in Fig. 41, the mean ordinate to the power curve can be thus directly obtained; if e_i and i be the impressed E.M.F. and alternating current at any instant, and $\mathbf{E}_i$ and $\mathbf{I}$ be their maximum values, the mean value of $e_i i$ or the mean rate of expenditure of energy on the circuit during a half-period or any whole number of half-periods is $\frac{1}{2}\mathbf{E}_i\mathbf{I}\cos\phi$, where ϕ is the angle of lag or difference of phase between the impressed E_iM.F. and current.

For any function that obeys a sine law, the effective or virtual stands to the maximum value as $1/\sqrt{2}$ to 1, or is 0·707 of the maximum value. The product of two virtual values is thus always equal to half the product of the maximum values. If, therefore, V_e and I_e be the virtual values of the terminal voltage and external

current of the single-phase alternator, the maximum values being $\mathbf{V}_e$ and $\mathbf{I}_e$,

$$\frac{\mathbf{V}_e\mathbf{I}_e}{2}\cos\phi = \frac{\sqrt{2}.V_e \times \sqrt{2}I_e}{2}\cos\phi = V_eI_e\cos\phi.$$

Hence the output of the alternator is equal to *the product of the virtual voltage and virtual amperes multiplied by the cosine of the angle of lag*, or $V_eI_e \cos\phi$, and in this form it is usually expressed, $\cos\phi$ being the power-factor or the ratio of the true to the apparent watts. It will be seen later that the value of the power-factor varies with the nature of the load, *e.g.* whether the alternator is lighting lamps directly or through transformers, or is driving motors. So small an angle of lag as that which has been illustrated above, or so high a power-factor as 0·95, would in actual working correspond to the case of a machine lighting lamps directly or connected to fully loaded transformers working on non-inductive loads.

§ 14. The mechanical torque in an alternator. In Chapter IV § 6 it was assumed that the alternating current was in phase with the impressed E.M.F., but it is now evident that this will in general not be the case, so that for the average torque the expressions given in Chapter IV § 6 (*b*) on the sinusoidal hypothesis will require to be multiplied by the cosine of the angle of lag of the current behind the curve of an assumed sinusoidal resultant field.[1]

The effect of armature reaction in the alternator has been treated above on the basis of two superposed fields, the one of which is the main field due to the field excitation, while the other is self-induced by the armature current. If now the external circuit had no inductance, and if the two component fields of the alternator were strictly confined to one and the same magnetic circuit or system of circuits, the resultant curve of the field-density would correspond precisely in phase with the current curve. Under these circumstances the direction of the tangential drag on the armature would be always opposed to the direction of rotation, and would merely fluctuate greatly in amount during the period in a single-phase alternator. But as a matter of fact the magnetic circuit of the self-induced flux is not entirely the same as that of the main flux. So far as it is separate, the case then becomes analogous to Fig. 39, and the curve of flux-density in the main field corresponds to an impressed curve of E.M.F. E_e; the product of the instantaneous

[1] Also in more complicated cases the distribution factor of the winding enters into the question, as foreshadowed in Chap. IV, § 6, so that the average torque is more readily obtained through the division of the output by the speed, *i.e.* from the watts per rev. per min. Both expressions have, in fact, to be used by the designer.

values of this with the current values for the same instants when plotted will now give certain areas of negative work, as in Fig. 40. The current in the armature conductors lags behind the induced E.M.F. and behind the B_g curve; the result is that the direction of the pull on an active conductor changes four times in each period, and twice for short intervals during each period it drives the alternator forwards as a motor, or gives out mechanical energy instead of absorbing it. If the external circuit itself has inductance, an entirely separate magnetic circuit (or circuits) is evidently added, and there is a still greater difference between the phase of the main field and that of the current. In other words, the angle of lag is increased, and the forward impulses on the machine become more powerful and last longer as compared with the backward drags, although they must always be less in amount so long as the machine continues to act as a generator. Thus the mechanical forces of a single-phase alternator, owing to its own inductance or still more to any inductance in its external circuit, are of a "racking" nature, which subjects it to strains of greater severity than are found in the equivalent continuous-current dynamo.

§ 15. Vector diagram for inductive circuit. Although the methods of § 10 or 13 have given the net rate at which irreversible work is done, they have not definitely determined at each instant how far the impressed E.M.F. is consumed in overcoming the effective resistance of the circuit and how far it is consumed in work upon the elastic medium surrounding the circuit. But when once it is assumed that L is constant, and that there is no hysteresis (Chapter XIV § 6) so that the magnetic flux follows the current changes instantly without any lag in point of time and further in strict proportion, then if any one of the three E.M.F. curves is sinusoidal, all are, including the current curve which is proportional to the e_r curve. The E.M.F.'s and current being simple harmonic functions having the same periodic time, they may be graphically represented by rotating radii or vectors combined in a single clock or time diagram. Thence upon the sine-law hypothesis we are enabled to deduce immediately not only the net rate at which irreversible work is developed over any considerable period of time, but also the rates at which each of the different kinds of work is done at any particular instant.

Let time be reckoned from the instant when the current is passing through zero, so that its value at time t is $i = I \sin \omega t$. Here and throughout in such expressions based on the sinusoidal hypothesis it must always be clearly borne in mind that ω is a purely electrical quantity; it is the electrical angular velocity in radians per sec. at which a vector must rotate in order to complete f revolutions in a second, where f is the number of cycles or complete waves which an alternating function passes through in a second. Thus

if T_p = the periodic time in seconds, $f = 1/T_p$ and $\omega = 2\pi f$ radians per sec. The latter only becomes identical with the mechanical angular velocity in the case of a rotating 2-pole machine in which one revolution corresponds to one complete cycle of E.M.F. or current. When mechanical and electrical angular velocities occur in the same expression (as in Chapter IX § 2) and are liable to confusion, the latter may be distinguished by a subscript as $\omega_e = p$ times the mechanical angular velocity, ω.

Corresponding then to $i = \mathbf{I}\sin\omega t$, in Fig. 42 the time or angle moved through in terms of a bipolar machine is reckoned from the instant when the radius OR, whose length represents $\mathbf{E}_r = R\mathbf{I}$, the maximum value of the "active" or resultant E.M.F., coincides with the horizontal axis; its projection on the vertical axis is then zero, corresponding to zero current.

The instantaneous value of the the self-induced E.M.F. is thus—

$$e_s = -\mathcal{L}\frac{di}{dt} = -\mathcal{L}\mathbf{I}\frac{d(\sin\omega t)}{dt} = -\omega\mathcal{L}\mathbf{I}\cos\omega t$$

or since $\cos\omega t = -\sin\left(\omega t - \frac{\pi}{2}\right)$,

$$e_s = \omega\mathcal{L}\mathbf{I}\sin\left(\omega t - \frac{\pi}{2}\right)$$

Thus the curve of self-induced E.M.F. passes through zero or a maximum or any particular percentage of the maximum at an actual time $= \pi/2\omega$ seconds after the current has passed through zero or its maximum, or the same percentage value of its maximum; *i.e.* relatively to the current it lags behind by a quarter of a complete period, or 90°. This *relative* phase difference is of chief interest, and is given by the term $-\pi/2$, the negative sign showing that it is a case of "lag." The self-induced E.M.F. is therefore a sine function of which the maximum value is $\mathbf{E}_s = \omega\mathcal{L}\mathbf{I}$, and which is at its negative maximum when $t = 0$. Hence if another rotating radius be drawn of length $OS = \mathbf{E}_s$ at an angle of 90° behind $\mathbf{E}_r$, or $\pi/2 - \omega t$ behind the horizontal OX at the instant t, the projection of this vector upon the vertical axis will at any moment represent the instantaneous value of the self-induced E.M.F. The maximum value $\mathbf{E}'_s$ of the E.M.F. consumed by the self-induction has the same value, but leads by a quarter of a period before the current, and is therefore represented by OS'. The impressed E.M.F. e_t must then at each instant not only counterbalance the E.M.F. of self-induction by its component e'_s, but also yield the resultant E.M.F. $e_r = Ri$, *i.e.* algebraically $\mathcal{L}di/dt + Ri = e_t$. We have then only to complete the parallelogram, having for its side OS and for its diagonal OR, to find the direction and length of $OE = \mathbf{E}_t$, the maximum value of the impressed E.M.F. Or we may produce OS to give the

maximum value OS' of the E.M.F. consumed by self-induction, and compound this with OR.

The complete time or clock vector diagram is given to the left of Fig. 42, and by its side are plotted in wave form the instantaneous values of e_t, e_r, e'_s and i, so that the correspondence of the projections of the rotating radii upon the vertical axes to the instantaneous values given by the curves when plotted by the method of rectangular co-ordinates may be seen. The position of the rotating radii relatively to the horizontal axis which has been chosen for illustration is shown by the dotted projections to correspond to 20° later than zero time. The maximum value $\mathbf{E}_t$ is assumed to be 1,000 volts, and of $\mathbf{E}_r$ to be 500 volts. To the right of the diagram is plotted the instantaneous value of the rates at which energy is either being stored magnetically in the medium surrounding the circuit or is being released therefrom, or is being expended in heating the circuit, or partly it may be in useful mechanical or electrical work, since R is the effective resistance as defined in § 4.

§ 16. Active and reactive components of impressed E.M.F. or current.—Either, therefore, the active E.M.F. may be regarded as the resultant of the impressed and self-induced E.M.F.'s or the impressed E.M.F. may be resolved into the two components (1) the E.M.F. consumed by self-induction, and (2) the active E.M.F. which drives the current against the resistance of the circuit. The vectors of these two component E.M.F.'s being at right angles to and along the vector of the current, the phase difference between the impressed E.M.F. and the current is some angle less than 90°; a certain amount of effective work is spent usefully in the circuit in each half-period. The instantaneous values of the two components of the impressed E.M.F., namely, e_r and e'_s, are shown dotted in Fig. 42 as obtained by projection of their maximum values on the vertical. When the projection upon the vertical of the maximum *active* component $\mathbf{E}_r = R\mathbf{I}$ is multiplied by the projection of the current $\mathbf{I}$ at the same instant, the product or Ri^2 is the rate in watts at which energy is developed in the circuit in a useful form or as heat, or both. When the maximum *reactive* component $\omega \mathcal{L}\mathbf{I}$, which is in quadrature with the current, is projected upon the vertical axis and multiplied by the simultaneous projection of the current, the product or $i\,\mathcal{L}di/dt$ is the rate at which energy is being stored in or released from the medium surrounding the circuit. While the latter has twice the frequency of the alternating E.M.F., and in any half-period is first positive and then negative, the former or the effective work is always positive or above the horizontal line, and simply varies from zero to a maximum and back again to zero in a half-period. Further, it is seen from Fig. 42 that in a half-period the gross power of the generator is at first spent partly in storing energy in the magnetic field and partly effectively in its

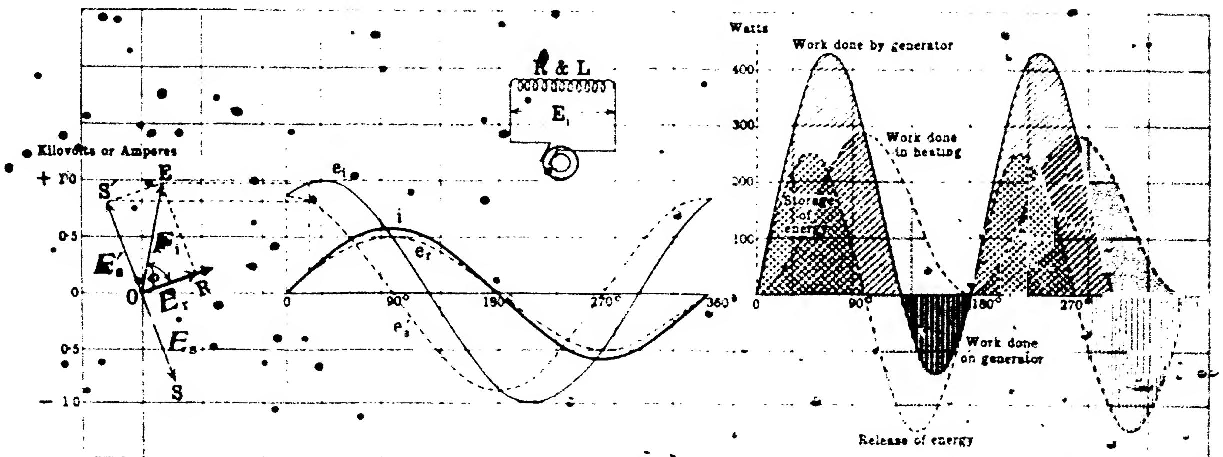


FIG. 42.—Circuit with inductance and resistance. Angle of lag of current [illegible] = 60°.

useful work; after a quarter of a period some of the effective work is obtained from release of the stored energy, and finally this not only supplies all the effective work but also drives the generator and returns energy to it mechanically. In the next half-period the same processes are repeated, so that *the power curve of the generator alternates with twice the frequency of the E.M.F. or current.*

The two components of the impressed E.M.F. have also been called respectively the *energy E.M.F.* and the *wattless E.M.F.*, but the terms are not to be recommended, since the instantaneous reactive watts are just as real as the energy watts. The only difference between the two is that the time integral of the latter can be used and therefore sold by the central station manager; the reactive watts, on the other hand, although not directly a source of loss or gain, since the corresponding energy is recovered immediately after it has been expended, yet are indirectly disadvantageous, since by reason of them a greater current-carrying capacity is called for in the machine and cables for a given amount of useful work developed, and regulation for constant volts is rendered more difficult.

The complete solution of the equation for the instantaneous value of the current,

$$i = \frac{e_i - \mathcal{L}\dfrac{di}{dt}}{R}$$

is usually given in a form which reckons time from the instant when the impressed E.M.F. is zero, so that

$$Ri + \mathcal{L}\frac{di}{dt} = e_i = \mathbf{E}_i \sin \omega t$$

When a steady periodic state has been reached, it is

$$i = \frac{\mathbf{E}_i \sin(\omega t - \phi)}{\sqrt{R^2 + \omega^2 \mathcal{L}^2}} = \mathbf{I} \sin(\omega t - \phi) \qquad (16)$$

so that the current is a sine function of the time with a relative phase difference of ϕ, or lagging behind the impressed E.M.F. by an actual time $= \phi/\omega$. The E.M.F. of self-induction $-\mathcal{L}\,di/dt$ from the present starting point is $-\omega\mathcal{L}\mathbf{I}\cos(\omega t - \phi) = \omega\mathcal{L}\mathbf{I}\sin(\omega t - \phi - \pi/2)$; since it lags behind the current by the time $\pi/2\omega$, it lags behind the impressed E.M.F. by the time

$$\frac{\phi + \dfrac{\pi}{2}}{\omega}$$

If each of the maximum values of the E.M.F.'s are divided by $\sqrt{2}$, the complete clock diagram of rotating radii is converted into a relative phase diagram of virtual values. Although the one

is based upon the other, the diagram showing the relative phases of the E.M.F.'s has in itself no reference to instantaneous temporal values, so that there is no need to add the vertical and horizontal axes which are required in order that the clock diagram may yield results from the point of view of actual time. The relative phases are, however, of chief importance as giving the average results of a half or whole period. The projection of the virtual impressed E.M.F. on to the current line when multiplied by the virtual current, or the product of the virtual value of the active E.M.F. $E_i \cos\phi$ with the current I, gives the average active watts or the value over a period of the effective power, while the projection of the virtual value of the reactive E.M.F. $E_i \sin\phi$ on to the current line is zero, and correspondingly the reactive watts over a period are zero.

Just as the impressed E.M.F., whether maximum or virtual, has above been resolved into its two components, it is also possible mentally to resolve the current into two components along and at right angles to the impressed E.M.F., the former being a so-called energy or working current and the latter a wattless or reactive current. Such a mental resolution of the current vector when applied to its maximum value has not, however, the same physical truth as the analogous resolution of the maximum impressed E.M.F., since when the components are projected on to the vertical and multiplied by the simultaneous projection of the impressed E.M.F. the instantaneous rates for the heating and magnetic work respectively are made to appear in reversed order in time. The reason is that while the impressed E.M.F. can truly be resolved into two components of which the one may appear below the horizontal axis with its projection opposing the projection of the other above the horizontal axis, the resolution of the current leads to one component current at times opposing the other component; the latter components do not then have any real physical existence, since in nature the current can only flow in one direction. The resolution of the current is therefore unreal as applied to the instantaneous values or the diagram of maximum rotating radii, but the method is perfectly valid when applied to the phase diagram of virtual values; it then yields the same value for the average rate at which effective work is done in the circuit, as the equivalent resolution of the E.M.F. Thus the product of the virtual working current $I \cos\phi$ with the virtual impressed E.M.F. gives the active watts over a period, while the projection of $I \sin\phi$ on to the impressed E.M.F. is zero.

§ 17. Magnetic reactance and impedance.—On the sine-law assumptions of a sinusoidally alternating E.M.F. or current and of constant $\mathscr{L}$, the quantity $\omega\mathscr{L} = 2\pi f\mathscr{L}$ is seen to express a fundamental characteristic of the circuit or conductor. It is known as

its ***magnetic reactance***; it is also symbolized by X, and is expressed in ohms, since its dimensions in terms of length and time (the true dimensions of μ, the permeability, being unknown) are those of a resistance, to which, in fact, it is analogous since its product with the current gives the reactive volts.

The vectors of either the maximum or the virtual values of the active and reactive volts being at right angles to one another, with the impressed volts as the hypotenuse of the right-angled triangle formed by them, we have *e.g.* in the case of the virtual values

$$E_t = \sqrt{E_r^2 + E_x^2} = I\sqrt{R^2 + \omega^2 l^2} = I\sqrt{R^2 + 4\pi^2 f^2 l^2} \quad (17)$$

The combination of the resistance and of the magnetic reactance in the special form $\sqrt{R^2 + \omega^2 l^2}$, being the square root of the sum of their squares, is known as the *impedance*, Z, of the portion of an inductive circuit which is under consideration, and is also measured in ohms. The relative angle of lag ϕ of the current behind the impressed E.M.F. is thus defined by any of the three relations—

$$\tan \phi = \frac{\omega l}{R} = \frac{X}{R} = \frac{\text{reactance}}{\text{resistance}} . \quad (18a)$$

$$\text{Power factor}, \cos \phi = \frac{R}{\sqrt{R^2 + \omega^2 l^2}} = \frac{R}{Z} = \frac{\text{resistance}}{\text{impedance}} . \quad (18b)$$

$$\sin \phi = \frac{X}{\sqrt{R^2 + X^2}} = \frac{X}{Z} = \frac{\text{reactance}}{\text{impedance}} . \quad (18c)$$

Ohm's law for a steady current, $I = E/R$ is therefore a special form of the more general expression which covers sinusoidal alternating currents, $I = E/Z$. In Fig. 42 R is assumed to be 866 ohms, and ωl to be 1,500 ohms; the impedance is therefore 1,732 ohms, and $\tan \phi = 1{,}500/866 = 1{\cdot}732$, whence $\phi = 60°$. The maximum value of the current is thus $1{,}000/1{,}732 = 0{\cdot}577$ ampere, and its virtual value $707/1{,}732 = 0{\cdot}407$, while the maximum $\mathbf{E}_r = 866$, and $\mathbf{E}_x = 500$ volts.

Given any one of the three E.M.F.'s with a knowledge of the R and X of the circuit and of the frequency, it is easy to determine the values of the other two. A semicircle (Fig. 43) described on the impressed E.M.F. gives the locus of the ends of the vectors, of the E.M.F. consumed by the reactance (OS') and of the active E.M.F. consumed over the effective resistance ($S'E = OR$) for all the possible combinations that the impressed E.M.F. admits, so that a knowledge of either R or X defines the case. Or from a knowledge of the current and of R and X, the three E.M.F.'s can be determined.

§ 18. The effect of capacity in the circuit.—The possibility of capacity being also present as a property of the circuit or conductor

has been incidentally alluded to above, and its effect must also be considered, at first in a general way. The work passing from the conductive circuit into the medium surrounding it has so far been assumed to be magnetically stored; the ether can however be stressed not only magnetically, but also electrically, as is the case with the dielectric film dividing the plates of a condenser. Any circuit traversed by an alternating current acts in some degree as a condenser and exhibits more or less marked capacity effects which may be reproduced by an imaginary condenser or condensers, either in series or in parallel with the circuit (the latter being itself then supposed to have no capacity).

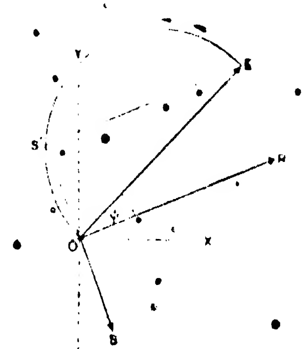


FIG. 43.

When a condenser of capacity C farads is connected to a source of E.M.F., its charge of q coulombs at any instant will vary proportionately to the difference of potential in volts at its terminals, or $q = C \cdot e$. The current which flows into or out of the condenser, charging or discharging it, is at any moment identical with the rate of change of q, or $i = dq/dt = C de/dt$. With an alternating difference of potential at its terminals, when once a steady periodic state has been established, the flow of current into the condenser is a maximum when the impressed E.M.F. is zero, and as the E.M.F. rises, the rate of charging diminishes until, when the E.M.F. is a maximum, the current is zero, and the condenser is fully charged. At any instant the charge upon the condenser will, if the impressed E.M.F. be imagined to be instantaneously withdrawn, set up a current round the circuit in the opposite direction to the impressed E.M.F. at the moment of withdrawal; in relation to the rest of the circuit, therefore, the condenser may be credited with an *E.M.F. due to capacity* $= -q/C$, which is exactly balanced by a component of the impressed E.M.F., just as the self-induced E.M.F. is balanced by the E.M.F. consumed by self-induction.

§ 19. Capacity reactance on the sine-law hypothesis.—If now it be assumed that the alternating difference of potential impressed upon the condenser follows a sine law, and reckoning time from the instant when the impressed E.M.F. is zero, $e = \mathbf{E}_i \sin \omega t$. The current at any instant is

$$i = C \cdot \frac{de}{dt} = C\mathbf{E}_i \frac{d(\sin \omega t)}{dt} = C\mathbf{E}_i\omega \cos \omega t$$

$$= C\omega\mathbf{E}_i \sin\left(\omega t + \frac{\pi}{2}\right) = \omega\mathbf{Q} \sin\left(\omega t + \frac{\pi}{2}\right) = \mathbf{I} \sin\left(\omega t + \frac{\pi}{2}\right)$$

The capacity current flowing into or out of the condenser is therefore a sine function of the time, and may be represented by a vector of maximum length $C\omega\mathbf{E}_t$ preceding the vector of E.M.F. $\mathbf{E}_t$ by 90°. Of course, this must not be interpreted to mean that the current actually precedes the E.M.F. which causes it to flow, but merely that their phases differ, and that when the steady periodic state has been reached a + maximum of current precedes a + maximum of E.M.F. by a quarter of a complete period. At any instant there is a charge upon the condenser $q = \mathbf{Q}\sin\omega t$, so that $\frac{q}{C} = \frac{\mathbf{Q}}{C}\sin\omega t = \frac{\mathbf{Q}}{C}\sin\left(\omega t + \pi\right) = \frac{\mathbf{I}}{\omega C}\sin(\omega t + \pi)$. The E.M.F. due to capacity is therefore represented by a vector of length $= \mathbf{I}/\omega C$, and the impressed E.M.F. must contain its counterpart in the opposite direction.

The quantity $1/\omega C$ is called the *capacity* or *condensive reactance* of the condenser, since when multiplied by $\mathbf{I}$ it determines the maximum E.M.F. due to capacity, or when multiplied by the virtual value of the current it determines the virtual E.M.F. due to capacity. It is therefore to be measured in ohms, and is exactly analogous to the magnetic reactance $\omega\mathcal{L}$, which similarly determines the counter E.M.F. of self-induction.

The exactly opposite nature of the effects from self-induction and from capacity are evident from the foregoing. Whereas self-induction causes the current to lag behind the impressed E.M.F., capacity causes it to lead before the impressed E.M.F. Whereas the E.M.F. of self-induction lags a quarter of a period behind the current, the E.M.F. due to capacity leads a quarter of a period before the current. If, then, capacity and inductance are combined in the same circuit, the two E.M.F.'s due to self-inductance and capacity respectively will oppose one another, and this possibility leads us to the complete case of a series circuit containing resistance and also capacity and inductance.

§ 20. Vector diagram for circuit containing resistance, inductance and capacity in series.—In the above combination the impressed E.M.F. must at one and the same time supply the active E.M.F. which drives the current against the effective resistance of the circuit R, and also balance the E.M.F.'s due to self-induction and capacity, the two latter acting against one another. Hence algebraically,

$$Ri + \mathcal{L}\frac{di}{dt} + \frac{q}{C} = e_t$$

The relative phase diagram of vectors will therefore be given by Fig. 44 (i), where $RI = E_r$ represents the active E.M.F. in phase with the current; $OA = \omega\mathcal{L}I$ ahead of the current by 90° represents the E.M.F. consumed by self-induction, and $I/\omega C$ lagging behind the current by 90° represents the E.M.F. required to balance the

E.M.F. due to capacity. The two components $\omega\mathcal{L}I$ and $I/\omega C$ are directly opposed to each other; they may or may not be equal, and in our case give OB as their difference. The relative phases and values are thus again given by a right-angled triangle, which, *e.g.* in the case of the virtual values, will have sides RI, $I(\omega\mathcal{L}-1/\omega C)$, and E_i.

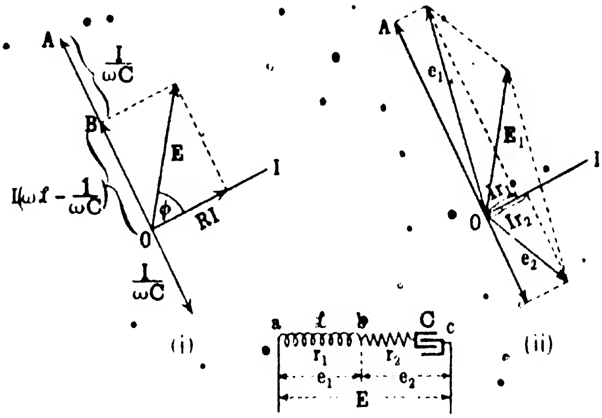


FIG. 44.—Circuit with resistance, capacity and inductance in series.

The solution of the algebraic equation for the instantaneous value of the current, time being reckoned from the instant when the impressed E.M.F. is zero, namely,

$$\mathcal{L}\frac{di}{dt}+Ri+\frac{q}{C}=\mathbf{E}_i\sin\omega t \quad . \quad . \quad . \quad (19)$$

gives

$$i=\frac{\mathbf{E}_i\sin(\omega t-\phi)}{\sqrt{R^2+\left(\omega\mathcal{L}-\frac{1}{\omega C}\right)^2}} \quad . \quad . \quad . \quad (19a)$$

where $\tan\phi=\dfrac{\omega\mathcal{L}-1/\omega C}{R}$. The virtual value of the current is

$$I=\frac{E_i}{\sqrt{R^2+\left(\omega\mathcal{L}-\frac{1}{\omega C}\right)^2}}.$$

When $\tan\phi$ is positive, $-\phi$ in the above expression is negative, *i.e.* if $\omega\mathcal{L}>1/\omega C$, the current lags behind the impressed E.M.F. When $\tan\phi$ is negative, $-\phi$ becomes positive, *i.e.* if $\omega\mathcal{L}<1/\omega C$, the current leads before the impressed E.M.F. As the quantity $\omega\mathcal{L}-1/\omega C$ is made larger in comparison with R, the phases of the E.M.F. and current diverge, and according as it is the magnetic or the capacity

reactance which decisively predominates, so will the angle ϕ in the above equation approach its limiting values of $+90°$ or $-90°$.

§ 21. The total reactance of a circuit, and the conditions of resonance. Thus the quantity $\omega \mathcal{L} - 1/\omega C$ is *the reactance of the circuit*, X, in the fullest sense of the term, and $\sqrt{R^2 + (\omega \mathcal{L} - 1/\omega C)^2}$ is the *total impedance*, Z. Its reciprocal $\dfrac{1}{\sqrt{R^2 + (\omega \mathcal{L} - 1/\omega C)^2}}$ is known as the *admittance*,[1] $Y = \sqrt{G^2 + B^2}$, where G is the *conductance* $= R/Z^2$ and B is the *susceptance* $= X/Z^2$. The total reactance may be either positive or negative according as the magnetic or the capacity reactance is the larger. Further, whenever magnetic and capacity reactance are both present, the possibility at once arises that within the outer terminals, a, c, of the circuit under consideration there may exist voltages much higher than the impressed E.M.F., *i.e.* such an increased voltage may exist either between the terminals bc of the condenser, or between the terminals ab of the inductive resistance, or within both of these portions. This possibility, which holds equally for either virtual or maxima values, may be followed from Fig. 44 (ii). The vectors of the self-induced E.M.F. and of the E.M.F. due to capacity being opposed to one another, either or both may be much greater than E_t. The voltage at the terminals of the inductive coil, if it had no resistance, would be $\omega \mathcal{L} I$, and at the terminals of the condenser, on the assumption that this also has no effective resistance (owing to dielectric hysteresis), would be $I/\omega C$, and both of these are determined by the value of the current and not directly by the impressed E.M.F. Or if the portions of the circuit respectively have resistances r_1 and r_2 (Fig. 44, ii), the difference of potential at the terminals of the inductive resistance is the resultant of $I\omega \mathcal{L}$ combined with the loss of volts Ir_1, *i.e.* e_1, while the difference of potential at the terminals of the condenser is similarly the resultant of $I/\omega C$ combined with the loss of volts Ir_2, i.e. e_2. It is evident that according to the relative values of $\mathcal{L}$ and C with a given frequency and impressed E.M.F., either e_1 or e_2 or both may be much higher than E_t, and there is a danger of the insulation being punctured.

The magnitude of the total reactance $\omega \mathcal{L} - 1/\omega C$ for given values of $\mathcal{L}$ and C evidently turns entirely upon the frequency. If $\mathcal{L}$ and C are in any way comparable, for a very low frequency the second term will tend to predominate, and the difference of potential at the terminals of the condenser will be the higher, while for a very high frequency the first term will tend to predominate, and the difference at the ends of the inductive portion will be the higher. But in either case the current will be largely checked down from its maximum value of E/R as fixed by the effective resistance alone.

[1] See Steinmetz, *Alternating-Current Phenomena* (5th edit.), chap. 8.

But when the frequency and constants are of such an order that $\omega\mathcal{L}$ and $1/\omega C$ approach one another, the current very rapidly rises until it reaches its greatest virtual value $I = E/R$ when $\omega\mathcal{L} = 1/\omega C$, or very rapidly falls off after this critical frequency is passed, and $\omega\mathcal{L}$ and $1/\omega C$ again begin to diverge. At the particular frequency when $\omega\mathcal{L} = 1/\omega C$, the self-inductance and capacity effects exactly annul one another, and the total reactance is zero. The angle ϕ is then zero, or the current is in phase with the impressed E.M.F., and the circuit acts as if it were a simple ohmic resistance, R,—so far at least as the impressed E.M.F. and value of the current is concerned. Complete electrical *resonance* is thus present at the critical frequency when $\omega\mathcal{L} = 1/\omega C$, or $\omega = \sqrt{\frac{1}{\mathcal{L}C}}$ *i.e.* since, $\omega = 2\pi f$, when

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{\mathcal{L}C}} \qquad . \quad . \quad . \quad . \quad . \quad (19b)$$

The reciprocal of this frequency, or $T_p = 2\pi\sqrt{\mathcal{L}C}$, is the natural period of discharge of the circuit if it was first charged up as a condenser. Hence an electrical circuit is said to be resonant to an impressed pressure when its natural period coincides with the period of the impressed E.M.F. The physical explanation of the fact that when the magnetic and capacity reactances exactly neutralize one another the circuit acts to an alternating E.M.F. as a pure ohmic resistance, and the virtual current reaches its greatest value for given values of E, R, $\mathcal{L}$, and C, is simple; in such a case the rate at which energy is being stored or released at any instant in the magnetic field is precisely the same as the rate at which it is being released or stored in the charge of the condenser. Just as the ether is being released from the magnetic stress, it becomes stressed electrically and at the same rate, so that externally, as it were, the circuit appears to act as a simple resistance, and the whole of the impressed E.M.F. is spent in overcoming the effective resistance in the development of heat or useful work. The energy which is being transferred backwards and forwards between the inductive resistance and the condenser may be much greater than the energy imparted to the circuit by the generator, and indeed bears no relation thereto, but is correlative to the development of increased voltages between the terminals of the two divisions of the circuit. The frequency of complete resonance gives the maximum possible rise of voltage within the circuit, but at any frequency approaching this critical value surprisingly high voltages may arise even in a low voltage circuit. As a general rule in alternating circuits the ohmic resistance is very low as compared with the impressed E.M.F., and, in the absence of a high useful back E.M.F., the current is **kept within bounds by the inductance; yet if, as may very well happen, the capacity by itself checks down the current to somewhat**

the same value, as that to which the inductance by itself reduces it, so that the critical state of resonance is approached, a large current and very high differences of potential may arise, with consequent danger of the insulation becoming pierced.[1]

In the extreme case when the resistance both of the inductive portion of the circuit and also of the condenser is negligible, and when resonance occurs, it can be seen from Fig 44 that, so far as the rest of the circuit from *c* to *a* through the generator is concerned, the combination of pure inductance and of the condenser acts as a perfect conductor; a current $I = E/R$ flows through it where R is the resistance of the rest of the circuit, but there is no *P.D.* between the terminals *a*, *c*. The virtual voltage between *a*, *b* is equal to that between *b* and *c*, and each is equal to $I\sqrt{\mathcal{L}/C}$, giving the maximum rise possible with the given R. The combination of equal magnetic and capacity reactances would thus act as a short-circuit, were it not for the resistance of the rest of the circuit, and if this were zero we should have an infinitely large current I, and infinitely high potential, but no energy absorbed.

§ 22. Vector diagram for an inductive circuit in parallel with a capacity, each with resistance.—The currents in the two branches are algebraically

$$i_1 = \frac{\mathbf{E}_i \sin(\omega t - \phi_1)}{\sqrt{R_1^2 + \omega^2 \mathcal{L}_1^2}} \quad \text{where } \tan\phi_1 = \frac{\omega\mathcal{L}_1}{R_1}$$

$$i_2 = \frac{\mathbf{E}_i \sin(\omega t + \phi_2)}{\sqrt{R_2^2 + \frac{1}{\omega^2 C^2}}} \quad \text{where } \tan\phi_2 = \frac{1}{\omega C R_2}$$

or in virtual values

$$I_1 = \frac{E_i}{\sqrt{R_1^2 + \omega^2 \mathcal{L}_1^2}}$$

$$I_2 = \frac{E_i}{\sqrt{R_2^2 + \frac{1}{\omega^2 C^2}}}$$

As the inductance or the positive reactance is increased relatively to the resistance, the current I_1 in the inductive branch lags more and more behind the impressed E.M.F.; as the capacity or negative reactance is increased relatively to the resistance, I_2 in the condenser branch leads more and more ahead of the E.M.F. The two currents thus become more and more opposed to one another (Fig. 45), and each may be very much greater than the resultant current I in the circuit as a whole. The leads joining the inductance

[1] *Cp.* La Cour and Bragstad, *Theory and Calculation of Electrical Currents*, translated by Dr. S. P. Smith (Longmans), p. 213.

and condenser in parallel might therefore become overheated, even though the resultant current through the combination is but small.

Thus, if a certain inductive resistance gives a lagging current I_1 and a condenser is placed in parallel with it, the new joint current can be made smaller than I_1, and at the same time be brought more

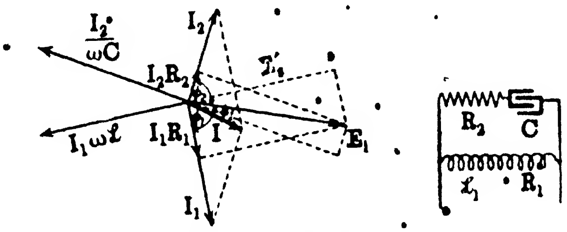


FIG. 45.—Inductance and capacity, each with resistance, in parallel.

nearly into phase with the E.M.F. impressed at the terminals of the combination. Such an arrangement, or a similar one in which an over-excited synchronous motor takes the place of the condenser, is used in practice with induction motors of low power factor in

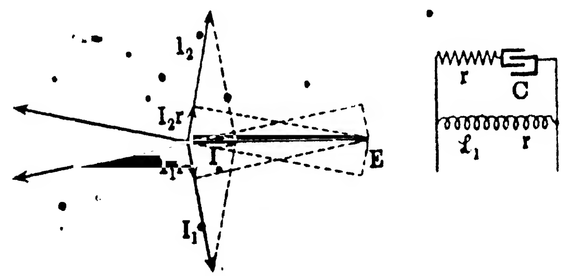


FIG. 46.—Inductance and capacity, each with equal resistance, in parallel.

order to reduce the line current. The reactive component of the lagging current I_1 being $\frac{E\omega\mathfrak{L}}{R_1^2+\omega^2\mathfrak{L}^2}$ and of the leading current I_2 being

$$\frac{E}{\left(R_2^2+\frac{1}{\omega^2C^2}\right)\omega C}$$

it is only necessary to make these two equal to bring the resultant line current into exact phase with the E.M.F.; one such case is

when $\omega\mathcal{L} = 1/\omega C$, and $R_1 = R_2 = r$ (Fig. 46), but since R_2 is usually small, a negative reactance

$$\frac{1}{\omega C} = \frac{R_1^2 + \omega^2\mathcal{L}^2}{\omega\mathcal{L}}$$

will give nearly exact coincidence of phase when R_1 is much greater than R_2. The condenser then, as it were, supplies the reactive component which is called for by the motor or system of motors.

If the resistance in both branches is negligible, the two component currents are strictly opposite to one another, and the resultant current is their numerical difference, or

$$I = I_1 - I_2 = E(1/\omega\mathcal{L} - \omega C).$$

When resonance also occurs and $\omega\mathcal{L} = 1/\omega C$, $I_1 = I_2$ and the current through the circuit as a whole is zero; the combination then acts as a pure insulator with regard to the impressed E.M.F., yet a large current $E/\omega\mathcal{L}$, or $E\omega C$, is circulating round the inductance and condenser as closed upon themselves. The multiplication of the current when condenser and inductance are in parallel is thus correlative to the multiplication of E.M.F. when they are in series.[1]

§ 23. The analogous linear oscillations of a mechanical system.— The real analogy which holds between an electric circuit containing inductance, capacity and resistance under the action of an impressed sinusoidal E.M.F. as in §§ 20, 21 and a mechanical system with inertia, an elastic controlling force and damping friction, when set in oscillation by an external applied force, is well-known, and need not be elaborated. But since mechanical oscillations come into question in the centrifugal whirling of shafts and in the parallel working of alternators, something more may here be introduced on the analogous mechanical problem, in regard to linear oscillations which are simple harmonic motions, leaving its conversion to the case of rotary oscillations to a later Chapter.

In order to comply with the requirements of a simple harmonic motion (of which one instance is given by the projection of the movement of a point rotating at uniform velocity in a circle on a diameter of the circle), a mass M when linearly displaced from a central position of rest must be under the action of a controlling or retarding force which is proportional to the displacement and always directed towards the centre, and of a damping force which is strictly proportional to the instantaneous velocity. The general equation for a linear forced oscillation is then

$$M\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = y$$

where x is the displacement, and d^2x/dt^2 is the acceleration; b is

[1] For a short account of pressure rises on high-tension transmission lines, which illustrate the importance of the effects of inductance and capacity, see E. Hudson, *Electr. Eng.*, vol. 39, p. 406. A fuller treatment of the effects of capacity will be found in the 5th edition of *The Dynamo*.

the damping factor giving the proportionality between the damping force and the velocity dx/dt, or the damping force per unit velocity, c is the factor giving the proportionality between the controlling force and the displacement which gives rise to it, or the controlling force per unit displacement, and y is the applied force as a function of the time. This being given as $\mathbf{F} \sin pt$, when $p = 2\pi/T$, and T is the periodic time of the external applied force varying sinusoidally, the equation becomes

$$M\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = \mathbf{F} \sin pt \quad . \quad . \quad . \quad (20)$$

The full solution is the sum of the natural and forced oscillations, but when the natural oscillation dies away under the action of friction, the motion is solely due to the applied force and retains the periodic time of the latter. The motion must then be such as to give $x = A \sin pt + B \cos pt$, where A and B are certain constants. It is shown in text-books that

$$A = \mathbf{F}\frac{(c - Mp^2)}{b^2p^2 + (c - Mp^2)^2}$$

$$= \frac{\mathbf{F}}{\sqrt{b^2p^2 + (c - Mp^2)^2}} \times \frac{c - Mp^2}{\sqrt{b^2p^2 + (c - Mp^2)^2}}$$

$$B = -\mathbf{F}\frac{bp}{b^2p^2 + (c - Mp^2)^2}$$

$$= \frac{\mathbf{F}}{\sqrt{b^2p^2 + (c - Mp^2)^2}} \times \frac{-bp}{\sqrt{b^2p^2 + (c - Mp^2)^2}}$$

In the second part of each expression the numerators and denominator are equivalent to the sides and hypotenuse of a right-angled triangle. The solution, therefore, is

$$x = \frac{\mathbf{F}}{\sqrt{b^2p^2 + (c - Mp^2)^2}}(\sin pt \cdot \cos \eta - \cos pt \cdot \sin \eta)$$

$$= \frac{\mathbf{F}}{\sqrt{b^2p^2 + (c - Mp^2)^2}} \sin (pt - \eta) \quad . \quad . \quad . \quad . \quad (20a)$$

where

$$\tan \eta = \frac{bp}{c - Mp^2} = \frac{b}{c/p - Mp}$$

Theoretically the final displacement

$$x = \frac{\mathbf{F}}{\sqrt{b^2p^2 + (c - Mp^2)^2}} = \frac{\mathbf{F}}{p\sqrt{b^2 + (c/p - Mp)^2}}$$

is only reached after an infinite number of cycles, but except under special conditions the final value as given above is closely attained after a few oscillations, and may, therefore, in practice at once be adopted.

The equation of motion analogous to (19) has now been solved for the displacement x which corresponds to quantity Q in the electrical case. To solve it for $v = dx/dt$ which corresponds to i, the current, as is more usual in the electrical case, we have—

$$v = pA \cos pt - pB \sin pt$$

$$= \frac{p\mathbf{F}}{\sqrt{b^2p^2 + (c - Mp^2)^2}} (\cos pt . \cos \eta + \sin pt . \sin \eta)$$

$$= \frac{p\mathbf{F}}{\sqrt{b^2p^2 + (c - Mp^2)^2}} \cos (pt - \eta)$$

$$= \frac{p\mathbf{F}}{\sqrt{b^2p^2 + (c - Mp^2)^2}} \sin \left(pt - \eta + \frac{\pi}{2}\right) \quad . \quad . \quad . \quad (20b)$$

i.e. the velocity vector is p times as great as the vector of the displacement and precedes it by 90°.

Equation (20*b*) may also be written

$$v = \frac{\mathbf{F}}{\sqrt{b^2 + (c/p - Mp)^2}} \sin \left(pt - \eta + \frac{\pi}{2}\right)$$

$$= \frac{\mathbf{F}}{\sqrt{b^2 + (Mp - c/p)^2}} \sin (pt - \phi)$$

where $\phi = \eta - \frac{\pi}{2}$, or $\tan \phi = \frac{Mp - c/p}{b} = \frac{Mp^2 - c}{bp}$.

The expression for v is then exactly analogous to (19*a*) and in this form it is usually given in the electrical case. Here ϕ is the angle of lag of the velocity behind the applied force when Mp exceeds c/p.

The *natural oscillation* and *natural periodic time* of the damped system is given if the applied force is instantaneously removed, *i.e.* when the three terms on the left-hand side of equation (20) are equated to zero instead of to $\mathbf{F} \sin pt$, or

$$M\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0.$$

The nature of the oscillations then resulting is dependent upon the value of b. Usually $4cM > b^2$, and in this case the solution of the differential equation is

$$x = e^{-\frac{b}{2M}t}(C \sin p_1 t + D \cos p_1 t)$$

where C and D are constants depending upon the initial conditions and

$$p_1 = \frac{\sqrt{4cM - b^2}}{2M} \quad . \quad . \quad . \quad (20c)$$

It follows that if b is negative,[1] however small its value, the amplitude

[1] The possibility and importance of a negative value for b will be found in other cases to be mentioned later, *e.g.* in the centrifugal whirling of shafts.

of the oscillations continually increases. If b is positive, the amplitude of the oscillations continually diminishes until they are finally damped out. If the damping is very small, we approach the ideal case of $b = 0$, *i.e.* $Md^2x/dt^2 = -cx$, or the acceleration and controlling force are proportional. The amplitude of an oscillation before the removal of the applied force is then continually repeated unchanged after it is removed, since there is nothing to increase it and nothing to reduce it.

(*a*) In the *ideal undamped case*, with $b = 0$, it will be seen from (20*a*) and (20*b*) that the value of η depends also upon the relation between c/p and Mp, for these may be equal. At the particular frequency p_0 when this occurs, an infinitesimal divergence of the frequency on either side must be considered in combination with an infinitesimal amount of damping. Thence when c/p exceeds Mp by an infinitely small amount $\tan \eta$ is zero, but is positive, so that $\eta = 0°$ and remains at this value for all lesser values of p. But when Mp exceeds c/p by an infinitely small amount, $\tan \eta$ is zero but is negative, so that $\eta = 180°$, and remains at this value for all larger values of p. When this latter case is represented by a clock diagram of vectors rotating at an angular speed of $2\pi/T$, radians per second, the phase of the velocity should lag 90° behind the phase of the applied force, and the displacement should lag 90° behind the velocity or 180° behind the force. It is then usually stated in explanation that when a cyclic state of affairs has been established, during the half period when the force, and consequently also the acceleration, is directed, say, towards the right, its first effect for a quarter period is a slowing down of the instantaneous velocity towards the left, which at the beginning of the half period is at its maximum: when the acceleration reaches its maximum to the right, the body has just been brought to rest at the end of its swing to the left. During the next quarter period, the instantaneous velocity towards the right grows up to a maximum; the direction of the acceleration then changes, and the second half period follows with the same sequence of events.

The case is, however, entirely unreal, and the account given of it, although mathematically true, is physically misleading in two respects. It is not merely that in nature there will always be some small damping force, but that when there is some damping present, even to an infinitesimal degree, the angle by which the vector of displacement lags behind the vector of applied force diverges slightly from 0° at low frequencies, passes rapidly to the value of 90° at the resonant frequency (Fig. 47, ii), and at higher frequencies rapidly becomes nearly 180° but only reaches this latter value at an infinite frequency of the applied force. This angle of lag for various degrees of damping is plotted in Fig. 48, and it will be seen that as the damping decreases, the curve becomes more

and more square-cornered, but even in the limiting case of zero damping, it continues to pass through 90° at the resonant

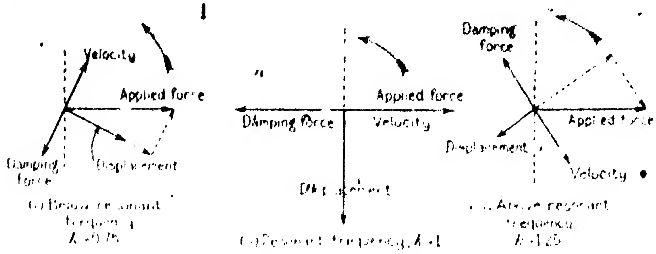


FIG. 47. Vector diagrams for damped oscillating system ($a = 0{\cdot}15$).

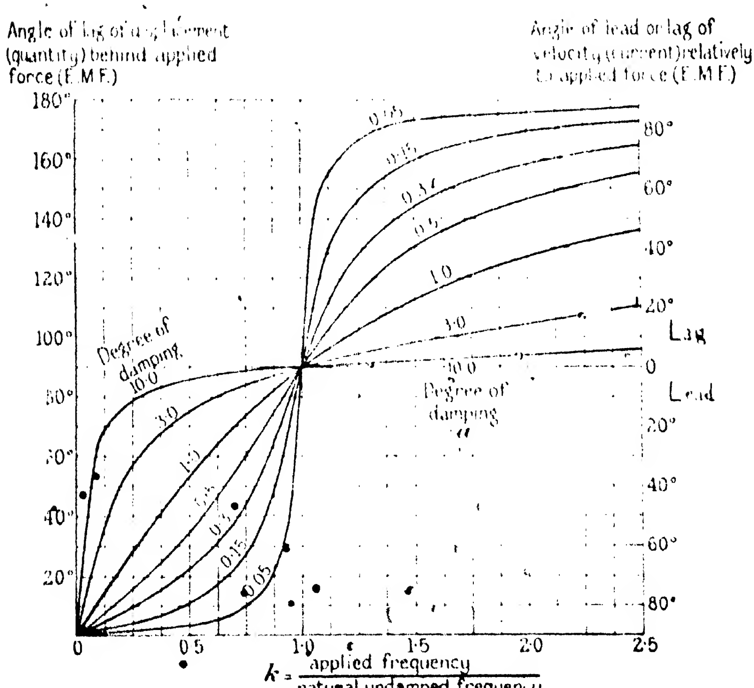


FIG. 48.—Phase angles of velocity and displacement relatively to applied force for different degrees of damping ($a = b/b_{crit}$).

frequency; below this frequency, the angle approaches 0° and it is only at higher values of the frequency that it approaches 180°.

The undamped system, however, possesses practical interest owing to the fact that the natural frequency of damped systems

is most conveniently expressed in terms of the natural frequency of the undamped system as a standard of reference. The undamped natural periodic time of an oscillation at once follows from equation (20c) since when $b = 0$, p_1 takes the value $p_o = \sqrt{c/M}$; therefore,

$$T_p = \frac{2\pi}{p_o} = 2\pi\sqrt{\frac{M}{c}}$$

From the importance of this quantity, the undamped case deserves a little further consideration from first principles. A body of mass M having been taken to one extreme limit of its swing by the external force which is then removed and damping being assumed to be absent, its velocity as it passes the central position is a maximum $\mathbf{V}$, and at any distance x from the centre is v. The difference in its kinetic energy when it passes the centre and at point x, being equal to the potential energy at that point, is also equal to the integral effect of the controlling force c acting through the displacement x. Therefore,

$$\tfrac{1}{2} M (\mathbf{V}^2 - v^2) = \int_0^x cx \, dx = \frac{c \cdot x^2}{2}$$

whence

$$v = \sqrt{\mathbf{V}^2 - \frac{c \cdot x^2}{M}}$$

When the body is at its maximum displacement, *i.e.* at $\mathbf{x}$, $v = 0$, so that $\mathbf{V} = \mathbf{x}\sqrt{c/M}$.

But $\mathbf{V}$ is also equal to the uniform velocity of a rotating body describing a circle, the radius of which is the amplitude, in the time of a complete to-and-fro swing, *i.e.* in the natural periodic time T_p. Hence

$$\mathbf{V} = \mathbf{x}\sqrt{\frac{c}{M}} = \frac{2\pi \mathbf{x}}{T_p}$$

or

$$T_p = 2\pi\sqrt{\frac{M}{c}} \qquad (20d)$$

exactly analogous to equation (19b).

This is often expressed as $T_p = 2\pi\sqrt{\mu}$, where μ is the constant ratio which the acceleration towards the centre bears to the displacement. Thus in words,

$$T_p = 2\pi\sqrt{\frac{\text{mass}}{\text{force per unit displacement}}},$$

or

$$= 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}.$$

since the actual force F = mass × acceleration. Apart from damping by air friction, such is the case of the simple pendulum, provided its swings are small. The tangential force acting upon its bob is $Mg . \sin \theta$, where θ is the angle of displacement from the vertical; the actual displacement is the arc moved through. So long, then, as the chord in $\sin \theta$ = chord/radius may be regarded as equal to the arc, the force is proportional to the arc or to the displacement, $\mu = g/l$ and its natural periodic time becomes $2\pi \sqrt{l/g}$.

Since $p_o = 2\pi/T_o = \sqrt{c/M}$, it follows that when $p = p_o$, $Mp^2 = c$ or $Mp = c/p$, so that the statement that this equality holds is only another way of stating that the periodic time of the forced oscillation is equal to the periodic time of the undamped natural oscillation. It is thus convenient to express all frequencies in terms of the natural undamped frequency, *i.e.* $p = k\ p_o$ where k is the ratio of the forced to the natural undamped frequency and = 1 at resonance (*cp.* Figs. 48-50).

(*b*) *The damped case.* Let there now be added the condition that the oscillating velocity of the body gives rise to or is accompanied by a force proportional but opposed to it; *i.e.* a damping force $b\,dx/dt$ is present. To overcome this there must then be present in the vector of the applied oscillating force a component in exact opposition to the damping force, and therefore in phase with the velocity and by its product therewith accounting for a certain expenditure of energy in every cycle. The result is that only the remaining component of the applied force vector, *i.e.* the component at right angles to that which balances the damping force, is effective in accelerating or in overcoming the controlling force. The amplitude of the component in **F** which opposes the damping force is

$$b\mathbf{v} = \mathbf{F} \sin \eta = \mathbf{F} \frac{b}{\sqrt{b^2 + (Mp - c/p)^2}}$$

and the amplitude of the component which is at right angles to the damping force and which is therefore unbalanced by it is

$$\mathbf{F} \cos \eta = \mathbf{F} \frac{c/p - Mp}{\sqrt{b^2 + (Mp - c/p)^2}}$$

Below the resonant frequency the latter component is in phase with the displacement which lags less than 90° behind the applied force, while the velocity vector leads the applied force (Fig. 47, i). At the resonant frequency the vectors of velocity and applied force are in phase, and the displacement lags 90° behind either; the applied force then has no component in phase with or in opposition to the displacement (Fig. 47, ii), but is entirely spent in overcoming the damping force, and the interaction between controlling force and displacement keeps the mass swinging. Above

the resonant frequency the velocity and displacement lag 90° and 180° behind the component of the applied force which is not expended over the frictional or damping force (Fig. 47, iii), just as in the imaginary case of zero damping they appear to lag behind the total force by these angles; in reality their lag behind the total force gradually increases towards 90° and 180°. To represent truly the whole process, a three-dimensional model would be required from which it would be seen how the vector of the displacement remaining finite passes through the angle of 90° at resonance when the vectors of applied force and velocity coincide.

The periodic time of the damped natural oscillation when the applied force is removed has already been given in (20c). It differs, of course, from the periodic time of the undamped natural oscillation, *i.e.* p_1 differs from p_0 according to the value of b. As the value of b (still assumed to be positive) is increased in proportion to cM, the damped periodic time lengthens or the frequency diminishes. As soon as b reaches the value $2\sqrt{cM}$, or $b_{crit}^2 = 4cM$, we then for the first time reach a stage when the curve of displacement no longer crosses the horizontal axis of time. At this point the system is just aperiodic, and the damping is said to be *critical.* With a further increase of b relatively to cM, the system remains aperiodic, but for any given initial conditions, the body more quickly reaches its maximum displacement and more quickly returns towards its central position of rest, although theoretically it never reaches it.

The special interest of the *critical damping* point lies in the fact that the effect of any other value of the damping factor and of any frequency of applied force can be conveniently expressed in terms of b_{crit} and of the resonant frequency. A particular combination of values which would give $b^2_{crit} = 4cM$ would be

$$b_{crit} = 2Mp_o = 2c/p_o$$

and from the two last expressions would be determined the displacement or speed at the resonant frequency. We are thus able to compare the value of the displacement and its lag behind the applied force for critical damping with their values at the resonant frequency, and also for any other values of damping and frequency in terms of the results for critical damping. Thus let $a = b/b_{crit}$, and k = applied frequency/resonant frequency. Then $b = 2aMp_o = 2ac/p_o$ and $p = kp_o$. Hence by substitution in (20*a*) the angle of lag between the vectors of displacement and applied force is given by

$$\tan \eta = \frac{b}{b/2ak - kb/2a} = \frac{2a}{1/k - k} \qquad (21a)$$

The values of η for different degrees of damping are plotted in relation to k in Fig. 48.

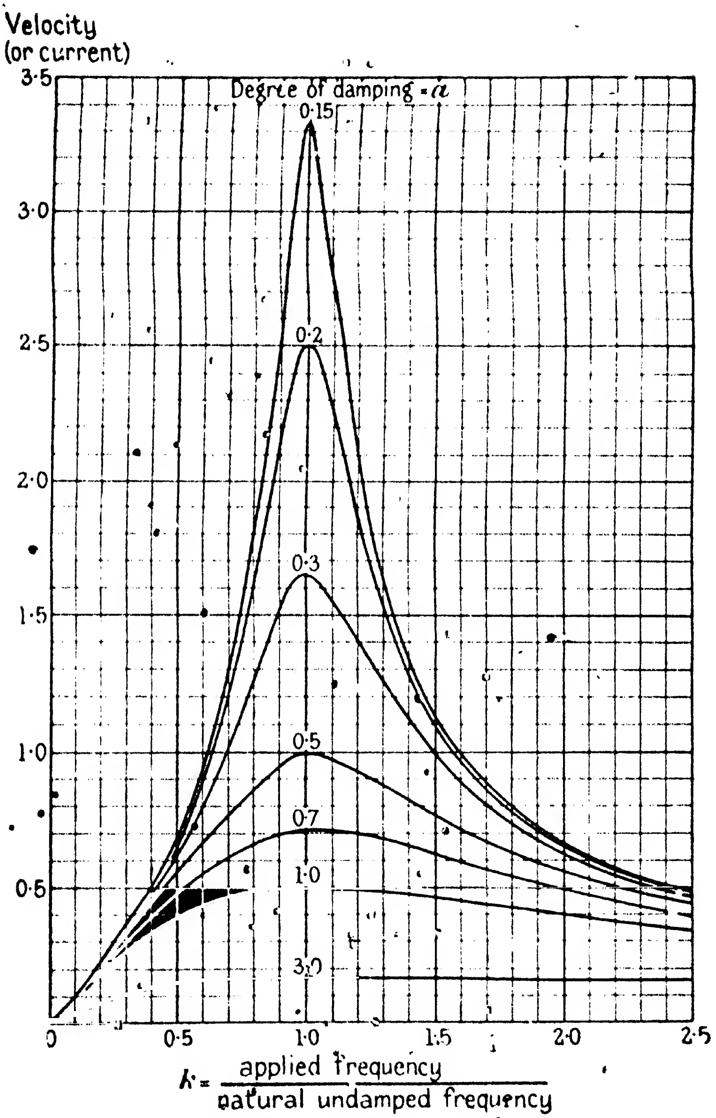


FIG. 49.—Amplitude of velocity (or current).

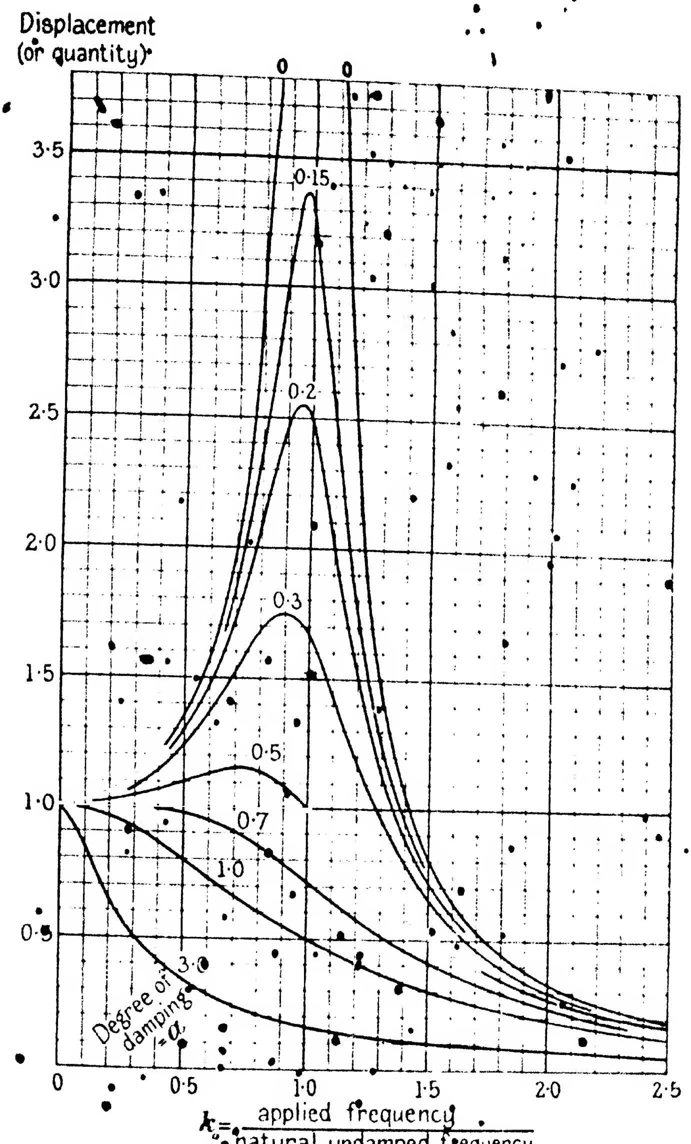


FIG. 50.—Amplitude of displacement (or quantity).

Now, whether there be damping or not, the maximum speed or the maximum current in the electrical case is always reached when the applied frequency is equal to the natural undamped frequency, *i.e.* at the resonant frequency (Fig. 49). Expressed in terms of p_0, equation (20*b*) becomes

$$\mathbf{v} = \frac{p_0\mathbf{F}}{c} \times \frac{1}{\sqrt{4a^2 + \left(\frac{k^2-1}{k}\right)^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21b)$$

and for any given degree of damping or value of a, the denominator is a minimum when $k = 1$. Yet this does not imply that the displacement or (electrically) the quantity reaches its maximum at the resonant frequency. The corresponding expression for this in the same terms is

$$\mathbf{x} = \frac{\mathbf{F}}{c} \times \frac{1}{\sqrt{4a^2k^2 + (k^2-1)^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21c)$$

The values of

$$\frac{1}{\sqrt{4a^2k^2 + (k^2-1)^2}}$$

are plotted in Fig. 50 from which it will be seen that as a is increased, the maximum displacement which without damping coincides with the resonant frequency occurs earlier and earlier.[1] The unit of the vertical scale of Fig. 50, viz., $\mathbf{F}/c$, is simply the displacement that would arise at zero frequency from the maximum value of the applied force acting against the controlling force which has the value c per unit distance. The unit of the vertical scale of Fig. 49 is p_0 times that of Fig. 50, *i.e.* it is the maximum velocity of a system oscillating with the displacement which is the unit of Fig. 50 but with the undamped natural frequency.

[1] For a similar treatment of the electrical case in its bearing on the truthfulness of oscillograms, see G. W. O. Howe, "The Amplitude and Phase of the Higher Harmonics in Oscillograms," *Journ. I.E.E.*, vol. 54, p. 19. *Cp.* also D. Robertson, p. 24, in the same volume.

CHAPTER VII

HOMOPOLAR CONTINUOUS-CURRENT DYNAMOS

§ 1. Difficulties in the homopolar continuous-current machine.— If the terminal voltage of a dynamo at a constant speed of rotation is uni-directed during passage of the armature past a double pole-pitch and also practically constant, it will cause an equally constant and steady or continuous flow of current through a circuit of fixed resistance. A dynamo yielding a terminal voltage which is at once uni-directed, continuous and constant in value is called somewhat inadequately a *continuous-current* or *direct-current* machine.[1] It has already been shown that the homopolar dynamo is fundamentally a continuous-current machine, giving in itself an uni-directed E.M.F. of constant value, without fluctuation or pulsation. It might therefore be expected that our first group of machines, namely, homopolar dynamos with field uniform in the path of movement (Class I, i), would be almost universally used for continuous-current work. In contrast to the heteropolar continuous-current machine, the current does not need to be stopped and reversed in direction in each armature coil as it passes the brushes; it does not therefore require any commutator with its attendant troubles of sparking at the brushes. To the homopolar type, too, belong the earliest forms of motor and dynamo that were made, since the first motor and dynamo, invented by Barlow and Faraday in 1823 and 1831 respectively, were disc homopolars.

The reverse is, however, the case; the heteropolar continuous-current machine has been and still is in possession of almost the entire field of continuous-current work, and practical experience shows that there are inherent disadvantages in the homopolar dynamo which have prevented its wide adoption.[2] All the ingenuity of inventors and designers has not succeeded up to the present in removing these disadvantages which arise from the fact that the simple machine gives but a small E.M.F. even at a very high speed of rotation. It has been already shown that in its simplest form it has only one active element, and therefore yields at best only a very low voltage. It remains to describe the methods that can be employed to lessen this objection.

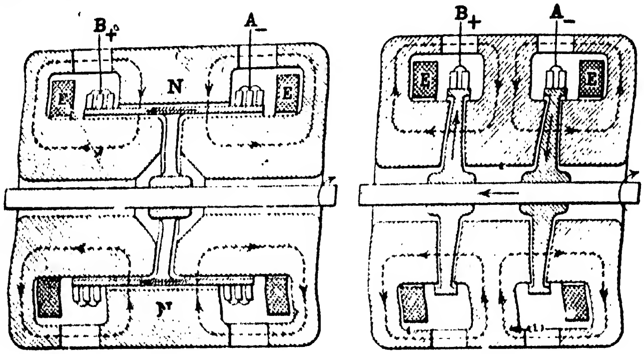
[1] English lacks an exact equivalent to the more expressive German term "level-current" (*Gleich-strom*).

[2] For an interesting summary of the various stages in the historical development of the homopolar continuous-current dynamo, see "Die unipolare Gleichstrommaschine," by Boris von Ugrimoff, in *Arbeiten aus dem Elektrotechnischen Institut zu Karlsruhe*, vol. 2, p. 132 (1911: Julius Springer, Berlin).

§ 2. Homopolar continuous-current [illegible] circuit.—To increase the E.M.F., the first step will be to [illegible] magnetic circuit, and so to double the effective cutting length of [illegible] element. Thus Fig. 51 (*a*) shows how from Fig. 8 (*a*) can be [illegible] an inducing cylinder of double the axial length and twice the flux; the latter after passing through the cylinder face bifurcates and half is taken out of each and instead of the whole out of one end of the armature. The E.M.F. is therefore double that of Fig. 8 (*a*) for the same diameter, and correspondingly the current must now be collected from the periphery of the cylinder at both ends, since half of the flux must pass through each end of the cylinder.

Similarly Fig. 51 (*b*) shows the use of two of the radial-type discs of Fig. 8 (*b*) in series; the current passes radially inwards in one disc, along the shaft, and radially outwards in the other disc. The brushes must therefore press on the periphery of each disc, and again half of the total flux must pass *through* each brush ring.

It is this necessity for the flux to pass *through* the brush collecting circle which really limits the possible voltage from a single active element rather



(*a*) (*b*)

FIG. 51.—Homopolar continuous-current machines with double magnetic circuit.

than the permissible peripheral speed as fixed solely by considerations of mechanical strength. It is evident, for example, in the axial type that it is useless to increase the length of the armature with a view to using a greater flux if this flux cannot be passed through the area of the iron or steel at the ends of the cylindrical rotor. But if the diameter be increased, then the collection of large currents at very high peripheral speeds presents formidable practical difficulties due to friction, heating from mechanical and electrical causes, and wear of the brush paths.

§ 3. The cylindrical and disc types compared.—The type of machine shown in Fig. 51 (*a*) with a plain cylindrical rotor of copper or brass as the single inducing element, running at, say, 1,500 revs. per min., has been successfully employed for outputs of a few volts and several thousands of amperes for electroplating or meter testing.[1] The cylindrical or axial type has in fact been most used in commercial engineering, although the disc or radial type was the first invented. This preference for the former type has been partly due to the difficulties that arise with the disc type from axial thrust. If the magnetic flux that passes into one face of the disc does not all issue out of the other face but part leaks out of the periphery, the disc becomes magnetically unbalanced and is drawn to one or other side, producing a powerful end-thrust. A thrust bearing and means for adjusting the disc to a great degree of nicety are, therefore, advisable.[1]

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[1] *Cp.* Boris v. Ugrimoff, *loc. cit.*, p. 209.

But the preference for the cylindrical type is still more due to the fact that in it the next step by which the E.M.F. can be increased is more easily made.

§ 4. The necessity of slip-rings.—When once the permissible collecting speed has been reached in combination with a double magnetic circuit, the only method of increasing the E.M.F. (short of duplicating the entire machine) is to increase the number of active conductors and to connect them so as to add up their E.M.F.'s.

If in Fig. 7, with rotating armatures, a second active conductor be added at the side of the first, and the far or outer end of the first be joined by a connecting wire to the near or inner end of the second active wire, this connecting wire if fixed to the active wires and rotating with them must [illegible] through a neutral space in the magnet circle as in the methods described in Chap. III, § 6, or must pass through the same air-gap [illegible] traversed by the active conductors or must cross the magnetic circuit [illegible] similar air-gap at some other point in its length. In the first case, as [illegible] shown in Chap. III, § 6, the connecting wires themselves become active as [illegible] rotation proceeds, and the result is that the E.M.F. round the loop so formed alternates, so that the continuous-current feature which is proper to the homopolar type is entirely lost. In either of the two latter cases the connecting wires must necessarily cut the lines of flux at exactly the same rate and in the same direction as the active conductors; they are thus continuously active themselves, producing an E.M.F. in the reverse direction, and the total E.M.F. of the combination is *nil*. Hence if the armature rotates, the connecting wires must be led back through the face of the iron *field-magnet or* through the iron and round outside the field-magnet, and so must be *stationary*. At once some kind of sliding contact becomes necessary in order to maintain connection between the fixed and the rotating conductors. If, on the other hand, the armature is stationary, the connecting wires must rotate with the field-magnet, and the same necessity exists for sliding contacts or *slip-rings*.

Thus there is no method of winding, properly so called, which is applicable to the homopolar armature, so long as it has to give a uni-directed E.M.F. All that can be done is to join one active conductor or element on to another by means of slip-rings, and the multiplication of the collecting surfaces and of the brush contacts, each carrying the full-load current is a necessary disadvantage.

§ 5. The E.M.F. from a single element with double magnetic circuit.—In the axial type the first stage will be again to divide the complete inducing cylinder of Fig. 8 (*a*) into a number of separate insulated sectors, each connected at *each* end to a slip-ring upon which may press the brushes necessary to lead the current out of the slip-ring at the back end of one element through a stationary connection into the front end of the next element. Thus the total number of slip-rings is always double that of the active elements.

If a peripheral collecting speed of 100 metres per sec. (nearly 20,000 ft. per min.) be taken as the maximum permissible, and a steam turbine running at 3,000 revs. per min. be assumed as the prime mover, the diameter of the collecting surface must not be more than 62·6 cm. (nearly 25 inches), and the net cross-sectional area of iron through which the flux must pass within the rings after all deductions are made cannot be more than about 2,000 sq. cm. at one end or 4,000 sq. cm. at both ends. With a density $B_c = 18{,}000$, the flux Φ_a cannot be more than 72×10^6 C.G.S. lines. Since this is cut in $60/N$ seconds, the E.M.F. from one bar is

$$e = \Phi_a \times \frac{N}{60} \times 10^{-8} \qquad (22)$$

$$= 72{,}000{,}000 \times 50 \times 10^{-8} = 36 \text{ volts.}$$

Three elements must therefore be connected in series to give about 105 volts at the terminals, and six slip-rings are required. This alone indicates one of the disadvantages inherent in the design.[1]

§ 6. The cylindrical homopolar continuous-current machine.—With the use of several elements it becomes necessary to make part of the iron magnetic

[1] The limitations of the homopolar dynamo are further discussed in Chap. XXII.

circuit into a rotating armature core, to which the bars may be attached. A second air-gap of small length in each magnetic circuit, analogous to that in Fig. 9, thus becomes necessary, and we reach the type illustrated in Fig. 52. This shows the essential parts of a machine with eight rings at each end, and a solid rotor core carrying eight sectors. Fig. 53 shows diagrammatically a developed plan of the winding with the stationary conductors marked by

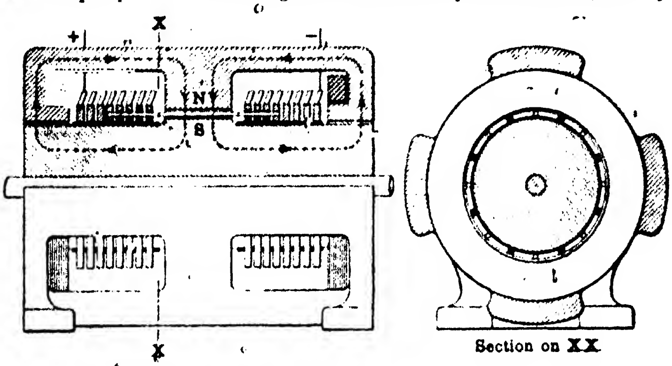


FIG. 52.—Continuous-current homopolar dynamo (axial type) with slip rings.

dotted lines. In the 300 kw. homopolar giving 500 volts described by Mr. J. E. Noeggerath (*Trans. Amer. I.E.E.*, vol. 24, p. 1) twelve flat sheets of copper bent to the radius of the armature and driven by lugs projecting from the core were bound to the solid cast-steel armature by steel binding wire.

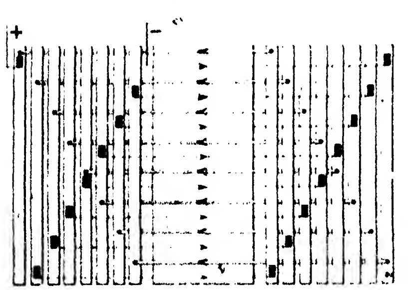


FIG. 53.—Diagram of connexions of armature of Fig. 52.

Instead of sectors, it is more usual to employ a number of equally spaced solid copper bars sunk into insulated slots or passing through tunnels in the iron body of the rotor. The stationary conductors which pass the current in the reverse direction from a ring at one end to a ring at the other end may similarly be formed of solid bars sunk in equally spaced slots in the pole-faces of the magnet: they then also serve the additional purpose of a compensating winding which neutralizes the armature reaction, as will be described in Chap. XIX.

A valuable and interesting account of a 2,000 kw., 260-volt, 7,700-ampere homopolar running at 1,200 revs. per min., and of the difficulties that had to be overcome to secure and maintain its satisfactory working has been given by Mr. B. G. Lamme.[1] This machine with a peripheral collecting speed of 13,200 ft. per min. had eight collector rings at each end of the rotor, and each of the eight active elements giving 32.5 volts was subdivided into six one-inch round solid conductors in insulating tubes threaded through nearly closed holes in the solid steel forging forming the rotor.

[1] "Development of a Successful Direct-current 2,000-kw. Unipolar Generator," *Trans. Amer. I.E.E.*, vol. 31, part II, (1912), p. 1811.

CHAPTER VIII

HOMOPOLAR ALTERNATORS

§ 1. Homopolar alternators.—Although of great historical interest and at one time much in favour, homopolar alternators which form the second group of machines need only brief consideration, since they have been practically superseded by the heteropolar type. As already stated, the homopolar machine naturally gives a uni-directed E.M.F., but is made to give an alternating E.M.F. through the device of interrupting the circle of the magnetic field by one or more neutral spaces through which very few lines of flux pass. Connecting wires can then be brought through these spaces and it can be *wound*, which is a necessity for an alternator that is to give a high E.M.F. The drum and disc methods of winding as described in Chapter III, § 6, will therefore be our starting-point.

§ 2. Concentrated, grouped, and uniformly distributed windings.—With the formation of a coil of many loops the question arises how far the increase in the E.M.F. will correspond to the number of active conductors which have been added and connected in series.

In the case of the smooth-surface armature, if the several loops of the drum coil are wound on the top of one another, each will at every instant produce exactly the same E.M.F. as the first original loop, provided the density of the field be kept the same; the winding may then be called *concentrated*, as it will become piled up into two or more layers of small width. But this must result in an increase of the length of the air-gap, and therefore with a given exciting power the density of the flux is greatly reduced. With a large number of loops it becomes necessary to wind them by the side of each other as in Fig. 18, so as to permit of the retention of the same air-gap as for a single loop. The winding thus becomes *uniformly distributed* over some appreciable extent of the armature core, and as the coil-side has a certain width, all the loops cannot occupy identically the same position relatively to the polar surfaces at any point of time. It follows that the E.M.F.'s induced in the different loops will not at the same moment be always alike either in direction or amount; in other words, they differ in phase. The effect of the coil is then to compound together a large number of phases, each of which differs a little from that of its neighbour as we pass from one side of the coil to the other.

Next, if the armature be slotted and the additional loops are wound within the same slot as the original loop, the winding may

again be called *concentrated*, and the E.M.F. of each new loop so added will very nearly be the same in magnitude and phase as that induced in the first loop. If, however, this statement is to hold true in practice, the dimensions of the slot in order to receive the numerous conductors must not be increased above certain normal limits. Should the slots become too deep or too wide, it will be necessary to divide the wires between two or more slots placed at some distance apart, and in this case the joint E.M.F. produced by the conductors in one slot will not be in phase with that of the conductors in the second or other slots. The slots being spaced out cover, in fact, a certain proportion of the field, and a ***grouped distribution*** is obtained which falls midway between a concentrated winding and the uniform distribution of the smooth-surface armature. In the grouped winding of the slotted armature,[1] a small number of sharply distinct phases are compounded, but as the slots between which the side of a coil is divided are increased in number, the distinction between the uniform and grouped distributions gradually diminishes. In all cases the effect of adding in series two or more E.M.F.'s differing in phase must be a reduction of the total as compared with the product of the E.M.F. of one active conductor multiplied by the number in series, and it becomes necessary to consider the effect of the width of a coil-side, whether on a smooth or toothed armature.

§ 3. The pitch-line and the pole-pitch of the homopolar alternator.—In order to investigate the best proportioning of the widths of pole and coil, so as to make the most advantageous use of an armature of given dimensions running at a given speed, the three widths of the coil, pole, and gap between the poles must be measured along the mean circular path traversed by the rotating coil or magnetic field, as the case may be.

This circle is called the *mean pitch-line*, and in the case of the homopolar machines now under consideration which have one or more fields separated by a neutral space or spaces of very weak flux-density, the *pitch* is to be defined as the distance measured along the pitch-line between the centre of a field or polar projection and the centre of the adjacent interpolar gap (Fig. 54). As shown in Chap. III, § 6, the maximum permissible width of field, even with only one loop, is not more than the pitch, since otherwise differential action reduces the time during which any resultant E.M.F. is produced. The field can, however, be as wide as the pitch, since in any one circle of polar projections all are of one sign, and no magnetic flux leaks across from one into the other without entering the armature; on this account the ratio $\dfrac{\text{width of pole}}{\text{pitch}}$ usually approaches unity more nearly than in the corresponding heteropolar machine. When there are several loops wound side by side, or several slots between which each side of the coil is distributed so that it has appreciable width, there will be considerable differential action unless the

[1] It is here assumed that the slots are parallel to the axis of the shaft. If skewed through a slot-pitch, uniform distribution is reached.

width of the field be reduced to equality with the inner loop and the width of the outer loop of the undivided coil be not greater than the width of the gap. Various proportions of coil and pole are therefore possible, and their relative advantages can be decided by mathematical analysis, if the flux-density curve along the pitch-line is known.

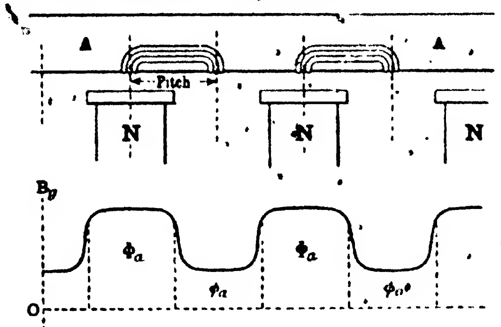


FIG. 54.—Flux-density along pitch line in homopolar alternator

§ 4. Types of homopolar drum alternators. Usually in the single phase alternator with undivided coils as shown in Fig. 54, the width of the inner loop is equal to half the pitch, and the width of the outer loop equal to one and a half times the pitch, so that half the armature is covered with winding. But any such single coil may be more conveniently divided into two, so that the end-connecting portions have less depth, as in the heteropolar drum alternator (Chap. IX, § 5).

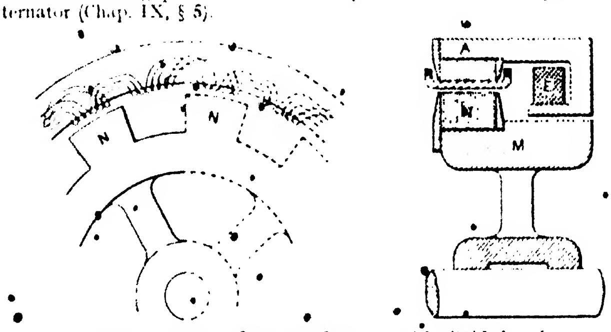


FIG. 55.—Drum inductor alternator with divided coils.

Divided coils are therefore illustrated in the simple drum alternator of Fig. 55, which also shows by dotted lines how the whole may be repeated with a second field. We thus arrive at *twice* as many coils as there are fields, the coils being virtually arranged in pairs, and each pair corresponding to one field instead of two fields as in the heteropolar machine. The width of the outer loop of each of the two coils is now equal to the pitch, but the coil-ratio is best still expressed in terms of the total width of the winding or the width of one belt of active wires forming the adjacent sides of two coils. The several fields are produced by polar projections from one central revolving iron mass. The exciting coil E is not rotated, but is supported from the stationary armature ring, so that there is no revolving copper and no need for any collecting rings either for armature or exciting

current. The rotating iron system may be likened to a number of "keepers," and is sometimes known as the "inductor," whence homopolar drum alternators of this class have received the name of "*inductor*" *generators*.

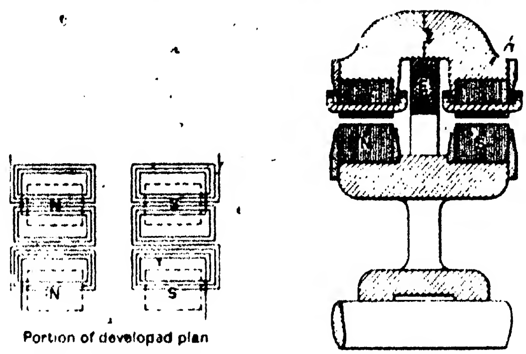


FIG. 56. Drum inductor alternator with double armature.

From Fig. 55 is easily derived what is known as a "double-armature" inductor alternator. Instead of the simple continuous air-gap shown on the right-hand side of Fig. 55, the second gap in the magnetic circuit may be broken up into separate poles and utilized by a second armature similar to the first (Fig. 56). If the poles on one side of the machine are abreast of those on the other side, and the coils on the two sides are staggered relatively to one another, the machine will give two phases. Or the poles may be themselves staggered, when we may again consider that there is only one pitch-line, a pole being in each case opposite to a neutral interpolar gap; and from this another arrangement, shown in Fig. 57, may be obtained which is suitable for a single-phase machine. The armature conductors are now taken straight through both halves of the armature core in tunnels or slots, and each straight wire consists of an active portion and a connector end to end. Finally, the two armatures may be arranged one inside and one outside the circle of revolving "inductors" or "keepers" (Fig. 58); these latter are bolted to the side of the fly-wheel of the driving engine. In all cases of double armatures the windings are preferably connected in series so as to avoid any liability to slight differences in the curve of E.M.F. or of current on the two sides.

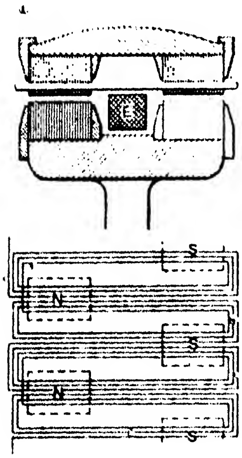


FIG. 57.—Drum inductor alternator with staggered poles.

§ 5. Frequency of homopolar drum alternator.—Since the passage of one field past a coil gives one complete double wave of E.M.F., the number of periods per second, or the frequency of a homopolar drum or disc alternator, is $pN/60$, where p is the number of fields or polar projections on one side of the armature. Thus in the drum or disc homopolar machine p is not, as in the heteropolar machine, the number of pairs of fields, and this difference in the definitions is the counterpart of the difference in the definition of the pitch for the heteropolar drum or disc machine as given in Chap. IX, § 1. Since p is also equal to the number of undivided coils on the armature or to

the number of pairs of divided coils, the frequency of the homopolar drum alternator is the same as that of the heteropolar with an equal number of armature coils, although in the latter there are twice as many fields as in the homopolar machine.

A characteristic feature of all homopolar alternators is that the poles form branches of one and the same magnetic circuit, and therefore only require one exciting coil wound on that portion of the circuit where all the lines unite to flow in a common stream. Each active wire thus cuts the

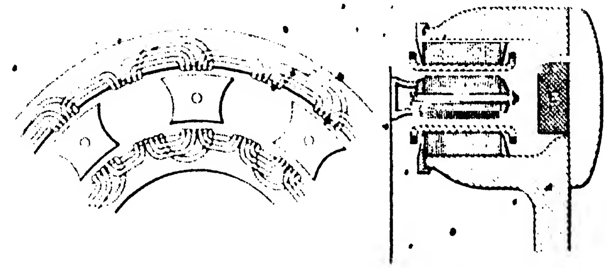


Fig. 58.—Drum inductor alternator with double armature.

lines from the numerous subdivisions of one common pole, as the term "homopolar" emphasizes when applied to an alternator with many fields, and it is on this account that we have spoken of the fields rather than of poles. Although not confined to homopolar alternators, the single exciting coil is always employed in them.

§ 6. Useless lines entering the armature.—In all homopolar alternators if a line VL be drawn symmetrically between two neighbouring poles of like sign, any flux which enters (or leaves) the armature in the interpolar gap up to the limit of half the pitch on either side of the symmetrical line, *i.e.* between BD (Fig. 59), is not only useless but is positively detrimental, since it causes a back E.M.F. by cutting the wires, which should act purely as connectors. In the homopolar machine this differential action is always present, whatever the width of the winding. The harmful lines are often called "leakage," but the name is not strictly appropriate, since they enter the armature, although not in a useful manner. Their number must therefore be deducted from the useful lines which enter immediately from a pole-face. Thus in Fig. 59 if Φ'_a be the total number that pass into the armature from the N. pole within the limits of the pitch, *i.e.* between either EB or FD, and ϕ_a be the number of harmful lines between BD, the total number which must be reckoned as forming one field is $\Phi_a = \Phi'_a - \phi_a$ (*cp.* Fig. 54, at the foot of which is given a curve indicating the flux distribution in the air gap). After thus discounting the effect of the direct differential action peculiar to the homopolar alternator, we are left with the differential action due to the coil having width; this can be taken into account by a winding factor k_d, and the only difference between the homopolar and heteropolar machines with similar coil and pole-ratios will lie in the slight difference of the densities of the fringes between the edges of the poles and the lines AG, CH (Fig. 59), when the width of the pole is not equal to the pitch. Thus with but little modification the values of K as given later for different ratios of pole- and

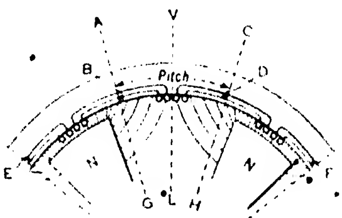


Fig. 59.—Air gap fluxes in inductor alternator.

coil-width to the pitch in the heteropolar alternator are equally applicable to the analogous homopolar machine.

§ 7. The E.M.F. equation of the homopolar drum alternator.—In the drum homopolar machine with single armature, p has already been defined as the number of fields, or of polar projections. If, then, Z be reckoned as the total number of conductors which at different times actively induce an E.M.F., and has therefore the same meaning as in the corresponding heteropolar machines, each wire in one revolution cuts $p\ (\Phi'_a - \phi_a)$ lines. The average E.M.F. is therefore, with the same pole width and the same net flux in the air gap, only half that of the similarly wound heteropolar armature, and so also is the virtual E.M.F. The equation for the virtual E.M.F. in one phase of the drum homopolar is then

$$E_a = \frac{K}{N_{ph}q} \cdot (\Phi'_a - \phi_a) \cdot \frac{pN}{60} \cdot Z \times 10^{-8} \text{ volts} \quad . \quad . \quad (23)$$

where N_{ph} is the number of phases, q is the number of parallel paths in the winding, and K has the meaning given on p. 168. But as already mentioned, the pole-face is usually of greater width as compared with the pitch in the homopolar than in the heteropolar form, and also owing to the field not being reversed in direction in the armature core a higher flux-density is permissible; hence $\Phi'_a - \phi_a$ of the homopolar may not be so very different from the $2\Phi_a$ of the heteropolar alternator with equal numbers and dimensions of armature coils in both cases, *i.e.* the virtual E.M.F.'s are more nearly equal for the same values of p, Z, and N. In the case of the combined double armature of Fig. 57, if Z be reckoned as the number of straight wires or bars passing from end to end across both armatures and $\Phi'_a - \phi_a$ be reckoned on one side only, the equation becomes

$$E_a = \frac{K}{N_{ph}q} \cdot 2\ (\Phi'_a - \phi_a)\ \frac{pN}{60} \cdot Z \times 10^{-8} \text{ volts} \quad . \quad . \quad (24)$$

and the same holds for the two separate armatures of Figs. 56 and 58 if these are in series and Z be the number of active conductors in one armature.

CHAPTER IX

HETEROPOLAR ALTERNATORS

§ 1. The E.M.F. equation of a drum loop in a heteropolar field.—Having thus briefly dismissed the subject of homopolar machines both continuous- and alternating-current, we return to heteropolar machines (Class II), which, as being by far the most common type in practical use, must be the main subject of the present book. Resuming from Chapter III § 12, the E.M.F. of a single drum loop in a heteropolar field will first be considered. The fundamental E.M.F. equation of the dynamo may here be recalled in its two forms

$$e = d\Phi/dt \times 10^{-8} \quad \text{or} \quad dN_s/dt \times 10^{-8}$$

where Φ is the number of C.G.S. lines cut in time t seconds, or N_s is the number of linkages of lines with the electric circuit. By means of these equations, any case can be solved, however complicated may be the closed circuit, if it be mentally split up into small elemental portions, single conductors or loops as the case may be, and their E.M.F.'s be summed up throughout the whole of the circuit, due regard being paid to their several directions (one direction round the circuit being taken as positive) and to the question of whether any or all of the elements are in parallel.

Given a single drum loop of any span mounted on the surface of a smooth armature, whether in a bi- or multipolar field of heteropolar type, let the flux-distribution in the air-gap under the poles be plotted to give a curve, as in Fig. 12, and let B_x and $B_{x'}$, each having its own appropriate sign, be the values of the flux-density at the positions x and x' (reckoned spacially from an interpolar line where the field changes direction relatively to the armature), which the two sides of the loop occupy at some instant (*cp.* Fig. 61). Then the instantaneous values of the E.M.F. induced in the two sides respectively for a constant linear velocity V cm. per sec. are $B_xLV \times 10^{-8}$ and $B_{x'}LV \times 10^{-8}$; for the length of the active side is in each case at right angles to the direction of the field and cuts through it at right angles. If the span of the loop is such that B_x and $B_{x'}$ have the same algebraic sign, the E.M.F. in each loop-side is in the same direction along it, but when the loop as a whole is considered, one direction *round* it must be taken as positive, and in this relation the resultant instantaneous E.M.F. is the difference

$$(B_x - B_{x'})\, LV \times 10^{-8} \text{ volts.}$$

For the linear velocity V may be substituted ωR, where R is the

6—(5065)

radius of the conductors and ω is the constant angular velocity $= 2\pi N/60 = 2\pi n$; so that alternatively the E.M.F. of the loop is

$$(B_x - B_{x'}) L\omega R \times 10^{-8}.$$

In the heteropolar machine (in contrast to the homopolar case of Chapter VIII, § 3) the *pole-pitch* is the distance measured along the pitch-line between the centres of a pair of neighbouring poles of opposite sign, just as in a toothed wheel the pitch is the distance measured along the pitch-line from the centre of one tooth to the centre of the next. If a drum armature such as that of Fig. 18 be imagined to be cut across along the line marked X and flattened out, as in Fig. 60, AA is the pitch-line and $p\,p$ is the pole-pitch Y.

Now if the span of the loop or the distance between x and x' is exactly equal to the pole-pitch, Y, *i.e.* to the total width of a

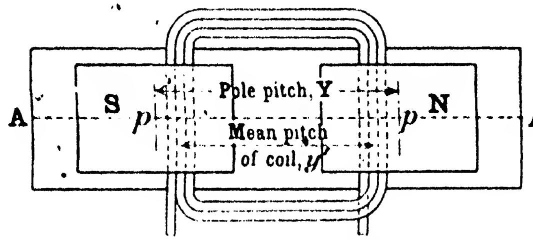


FIG. 60. Development of armature showing pitch line of heteropolar alternator.

field, all fields being assumed to be alike in their total flux and flux-distribution, the two sides of the loop at any instant are symmetrically situated in fields of equal density but of opposite sign, or $-B_{x'} = B_x$. The E.M.F.'s of the two sides are then additive round the loop, and the total E.M.F. is

$$2B_x LV \times 10^{-8} \quad \text{or} \quad 2B_x L\omega R \times 10^{-8}$$

and it is this condition which in practice would be aimed at.

Thus a single conductor on a smooth-armature at constant speed gives a time-wave of E.M.F., the shape of which is an exact replica of the spacial curve of flux-density round the armature, and so also does a single loop equal in span to the pole-pitch if the field-distribution under each pole is similar; but if the distribution under each pole is not alike the E.M.F. curve of the loop would be freed from the even harmonics which would then be found in the flux-curve. In practice, however, such a case seldom needs to be considered. The same results may also be applied, at least as a first approximation, to the case of a conductor or loop on a toothed armature, or combined with a toothed pole-face, but in this case any ripples superposed on the smooth wave of E.M.F. and due to the presence of teeth and slots in stator or rotor or both are not taken into account.

§ 8. The above on the supposition of a sinusoidally distributed field.—If the flux-density curve is sinusoidal as shown in Fig. 61, it is expressible spacially as $B_{gs\,max} \sin \alpha$, where $B_{gs\,max}$ is the amplitude and α is the angular distance reckoned (in terms of a bipolar machine for which 180° or π radians correspond to the pole-pitch Y) from the point where the flux changes its direction between two poles. $B_{gs\,max}$ does not itself have any sign, but the value to be assigned to the angle α for any given point x follows at once from the fact that α stands to x as π to Y; hence, $\alpha = \pi x/Y$. The instantaneous E.M.F. of the loop is then

$$B_{gs\,max} LV \left(\sin \frac{\pi x}{Y} - \sin \frac{\pi x'}{Y}\right) \times 10^{-8}$$

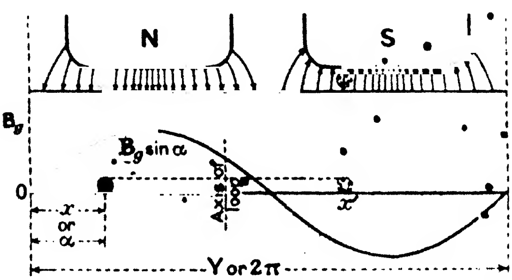


FIG. 61.—Loop in field with sinusoidal flux distribution.

When the span of the loop is Y, or $x' = x + Y$ and $\sin \pi x'/Y = \sin (180° + \pi x/Y)$, this becomes

$$2B_{gs\,max} LV \sin \pi x/Y \times 10^{-8}$$

The above may also be expressed in terms of time t instead of space x. Reckoning time from the instant when a single conductor crosses the neutral line between two fields, its instantaneous E.M.F. is $B_{gs\,max}\ LV \sin \omega_s t \times 10^{-8}$ where $\omega_s = p\omega$ (Chapter VI, § 15). But in the case of a loop of span more or less than the pole-pitch, time is best reckoned from the instant when the *axis* of the loop is at right angles to the neutral line or zero space-position from which x was measured. The E.M.F. of the loop is then

$$B_{gs\,max} LV \left[\sin (\alpha + \omega_s t) - \sin \{(\pi - \alpha) + \omega_s t\}\right] \times 10^{-8}$$
$$= 2B_{gs\,max} LV \sin \omega_s t \cos \alpha \times 10^{-8}$$

and when the span of the loop is equal to Y, so that $x = 0$,

$$= 2B_{gs\,max} LV \sin \omega_s t \times 10^{-8} = 2B_{gs\,max} LR\omega \sin \omega_s t \times 10^{-8} \quad (25)$$

If T_s = the periodic time in secs. taken by the loop to pass a double pole-pitch, giving a complete wave, $\omega_s = 2\pi/T_s$; for $\sin \omega_s t$ may then also be substituted the equivalent expression $\sin 2\pi t/T_s$ which

is analogous to the previous expression sin $2\pi x/2Y$, the position of the loop of full span Y being identical for the zeros chosen for both space and time.

The flux-distribution being here assumed to be sinusoidal, the maximum density $B_{gs\,max} = \frac{\pi}{2} B_{g\,av}$. Substituting this expression and remembering that $\pi R/p = Y$, the E.M.F. is

$$B_{gs\,av} L Y \omega_e \sin \omega_e t \times 10^{-8}$$

But LY is the total area of one field, so that $B_{gs\,av} LY = \Phi_{gs}$, the total flux of the field sinusoidally distributed over a pole-pitch. Hence when the span of the loop is equal to Y and when, in consequence, it is linked in its zero position with all the flux of a field, its E.M.F. may also be expressed as

$$e = \Phi_{gs} \omega_e \sin \omega_e t \times 10^{-8} \quad . \quad . \quad . \quad (25a)$$

The virtual value of a sine function being $\frac{1}{\sqrt{2}}$ of the maximum and the maximum value being $\frac{\pi}{2}$ times the average value of a half wave, the virtual value in the above cases is

$$E = \frac{1}{\sqrt{2}} \times \frac{\pi}{2} \times \text{average value}.$$

The ratio $\frac{\text{virtual value}}{\text{average value}}$ of an alternating E.M.F. or current is known as the *form factor*, and its value for a sine function is thus $\frac{\pi}{2\sqrt{2}} = 1{\cdot}11$.

The preceding results may usefully be checked on the linkage hypothesis as follows. The flux linked with the loop at any moment is

$$\Phi_t = \int_x^{x'} B_x L \, . \, dx$$

so that

$$-\frac{d\Phi_t}{dt} \times 10^{-8} = -d\int_x^{x'} B_x L \frac{dx}{dt} \times 10^{-8}$$

But $\frac{dx}{dt} = V$, the constant velocity in cm. per second, so that the above expression becomes

$$= -LV\int_x^{x'} dB_x \times 10^{-8} = (B_x - B_{x'}) LV \times 10^{-8}$$

When the flux-distribution is sinusoidal, and the coil has a span = Y,

$$\Phi_t = \int_x^{x+Y} B_{gs\,max} \sin \frac{\pi x}{Y} L \,.\, dx = 2B_{gs\,max} L \frac{Y}{\pi} \cos \frac{\pi x}{Y}$$

and

$$-\frac{d\Phi_t}{dt} \times 10^{-8} = -2B_{gs\,max} L \frac{Y}{\pi} \frac{d\left(\cos \frac{\pi x}{Y}\right)}{dt} \times 10^{-8}$$

$$= -2B_{gs\,max} L \frac{Y}{\pi} \frac{d\left(\cos \frac{\pi x}{Y}\right)}{dx} \times \frac{dx}{dt} \times 10^{-8}$$

$$= 2B_{gs\,max} LV \sin \frac{\pi x}{Y} \times 10^{-8}.$$

Or

$$-2B_{gs\,max} L \frac{Y}{\pi} \frac{d\left(\cos \frac{2\pi t}{T_p}\right)}{dt} \times 10^{-8}$$

$$= 2B_{gs\,max} L \frac{R}{p} \times \frac{2\pi}{T_p} \sin \frac{2\pi t}{T_p} \cdot 10^{-8}$$

$$= 2B_{gs\,max} LR\,\omega \sin \omega_e t \times 10^{-8}$$

$$= \Phi_{gs}\,\omega_e \sin \omega_e t \times 10^{-8}$$

§ 3. The ratio of pole-width to pole-pitch in heteropolar machines.—As in the case of homopolar machines, when we pass to coils instead of simple loops, it will be necessary to determine the best widths of pole and of winding; the same principles must be our guide, and the same distinction already drawn (Chapter VIII, § 2 *q.v.*) between concentrated, grouped and distributed windings still holds. Since the N. and S. poles of the magnet system are in the same line round the armature, a first condition is that the leakage of flux across from the edge of a pole into the adjoining edge of the next pole should not be excessive. In the modern alternator since the magnet system is usually internal to the armature, a second condition is that room must be found for the necessary exciting coils with sufficient copper in them to avoid overheating. In a machine with radial poles of the salient type as it is termed (*cp.* Fig. 20), the first condition might be partially met by thinning off the pole-shoe edges, but since the useful flux which they transmit to the surrounding armature is also reduced thereby, it results in practice that the width of the pole-face is usually made less than the pole-pitch by at least 30 per cent. On the other hand, if the ratio of the pole-width to the pitch be made too small, the flux of each field must also be small, and the exciting coils on the poles will be comparatively uneconomical in weight of wire or in exciting watts. *Ceteris paribus,*

the larger the area of a magnet-core the more cheaply is the flux obtained, since, *e.g.*, with a round core the area is proportional to the square of the diameter, while the length of an exciting turn is only proportional to the diameter.

In the non-salient-pole type which is now generally adopted for turbo-alternators (Figs. 234-C), the pole-face width is nominally equal to the pole-pitch, but here also the two conditions are met by the fact that the exciting coil is distributed in slots over more than half the pole-pitch, so that between teeth near its centre there is but a small M.M.F. acting to cause leakage, while the neighbouring edges of the unslotted pole-centres forming the virtual poles are removed to an appreciable distance apart. A greater concentration of the exciting winding in fewer slots would for the reason given above be magnetically advantageous, but is forbidden by the second of the two conditions.

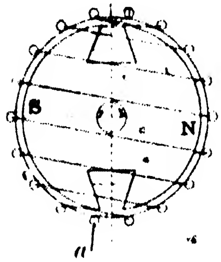


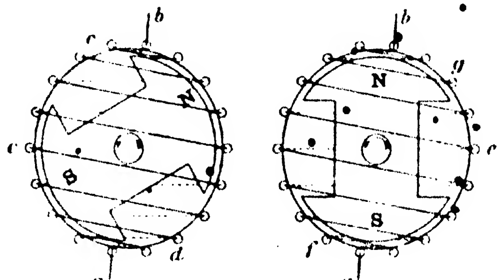
FIG. 62—Coil and pole both of maximum width.

It thus results that in all types of the heteropolar alternator, the field of flux from one pole is mainly concentrated within some 50 to 70 per cent. of the pole-pitch, and the shape of the curve of distribution of the flux over the pole-pitch is made to resemble more or less closely a sine curve—usually with a somewhat flattened top.

§ 4. The ratio of coil-width to pole-pitch. Avoidance of direct differential action.—Considering a bipolar machine with internal revolving magnet, it will be seen that if a drum winding spread over the whole internal periphery of the armature, as in Fig. 62, is combined with the maximum pole-width equal to the pitch, as soon as the magnet is rotated away from the symmetrical position of maximum E.M.F. shown in Fig. 62, each side of the coil is at the same time under the action of two poles of opposite sign. The E.M.F.'s generated in the active conductors at the outer edges of each coil-side will then tend to neutralize one another, and the net E.M.F. from each coil-side will be only the difference between the sums of the E.M.F.'s generated in the two portions which are respectively under a N. and a S. pole. In this extreme case, when not only the coil-side has its maximum width equal to the pitch, but also the pole-width is a maximum, direct *differential action* lasts throughout the whole period. But the time during which such differential action lasts can be reduced by decreasing the width of the winding; its serious amount can also be reduced by limiting the effective width of the pole-face, or both remedies can be applied. As explained in the preceding section, the flux is for other reasons more or less concentrated over a certain proportion of the pole-pitch,

but the diagram serves to show that if the extreme coil-width be retained, however much the pole-width is reduced, there must still remain some direct differential action.

That it is specially the loops at each edge of the coil-side which are ineffective as compared with the central turns will be better seen if the pole-width is reduced as in Fig. 63. Starting from the position there shown, the E.M.F.'s of the wires at *c* and *d* begin to oppose that of their respective coil-sides, and this action increases until the axis of the coil coincides with the general direction of the field (Fig. 64), when the net E.M.F. of either side and of the coil as a whole is zero. As rotation proceeds, the E.M.F.'s of the wires at *f* and *g* oppose the rise of the main E.M.F., but to a gradually diminishing



FIGS. 63 and 64.—Pole width reduced, coil of maximum width.

extent. But the central turns at *e* are only subjected to differential action in a minor degree when the symmetrical line dividing the interpolar gap passes across them. Hence it is chiefly when a coil-side is wider than the interpolar gap that all the loops are not equally active; and there will be comparatively little reduction in the net E.M.F. if some of the loops at each edge of the coil are removed and the width of coil-side reduced. Inspection of the developed plan of Fig. 65 shows that if direct differential action is to be avoided, (1) the inner loop *a* must exceed the width of pole-face, so that its opposite sides are never moving under the same pole; and further, (2) the outer loop *b* must be of such width that when deducted from twice the pole-pitch the difference is not less than the width of the pole-face. The outer *b* loop will thus in a bipolar machine bear the same relation to one pole as the inner *a* loop does to the other pole. In other words, the width of a coil-side must not exceed the width of the gap between two poles, so that one coil-side may never be under two poles of opposite sign. There will then only be left the small amount of differential action due to the spreading outwards of the lines from the edges of the poles;

this effect can, strictly speaking, only vanish when the coil is concentrated into a single line, but in any case is practically unimportant by comparison with the direct differential action when the E.M.F.'s of active conductors immediately under a pole are in opposition. The narrow coil to which we are thus led if the pole-width be great must appreciably limit the number of active conductors; and further, the greater portion of the available space round the armature core on which loops could be wound will not be utilized. A compromise must thus be struck between the extreme cases of complete concentration and distribution over the entire pole-pitch. Hence in all alternators the coil-sides are confined within certain proportions of the pitch, although it may be to different degrees, and in single-phase alternators the spread of the coil-side does not much exceed one-half or at most two-thirds of the pitch. A half only of the armature is shown diagrammatically in Fig. 65 as covered with winding.

§ 5. Undivided and divided coils.—When the inactive end-connectors at both ends of the single coil on a bipolar armature (Fig. 65) are arranged entirely on one side of the shaft, then if the armature is cut across to the centre along the line X and opened out, the plan developed on the flat shows one single undivided coil containing two groups of active conductors with E.M.F.'s in opposite directions (Fig. 65). But this single coil may equally well be divided into two halves by taking one-half of the end-connectors round on either side of the shaft, as in Fig. 66. The same active groups are now divided between two separate coils in which the current circulates in opposite directions at each moment without in any way affecting the total E.M.F. if they are connected in series. When so divided, the space taken up by the connecting ends of the loops in an axial direction is halved, as is seen by comparing Figs. 65 and 66. This economy of space is specially advantageous in the case of multipolar alternators, and the arrangement is therefore common. There are then two coils for a bipolar, or as many coils as there are poles in the multipolar, alternator; the outer loop of the divided coils has a width equal to the pitch, and if the inner loop has a width equal to the polar arc, the width of each coil-side is equal to half the gap between the poles. It must be borne in mind that the question whether the coil is divided or undivided[1] is quite distinct from the question whether the distribution be concentrated, grouped, or uniform, and what we have described as a "coil" may actually be composed of two or more coils in separate slots; *e.g.* Fig. 68 shows an undivided coil which

[1] Windings with undivided coils are called by Dr. S. P. Thompson "hemitropic," and by Mr. H. M. Hobart "half-coiled," with the opposite term "whole-coiled" for divided coils.

is nevertheless spaced out between two slots, one-quarter of the pole-pitch apart.

§ 4. Multipolar alternators. Frequency.—In most commercial alternators it is desired to produce a large number of alternations per second or a high "frequency" of 25 or 50 complete periods per second. The reversals of the direction of the current are thus very rapid, the flow in one direction only lasting a small

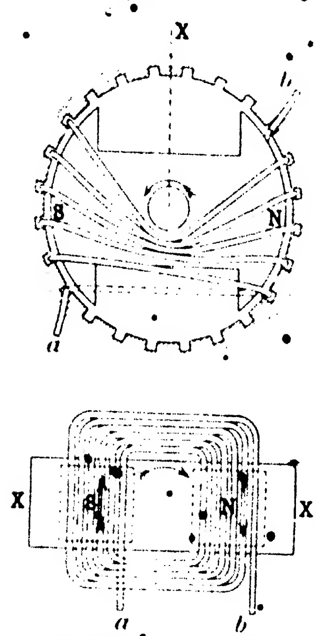


Fig. 65.—Drum bipolar alternator armature with undivided coil.

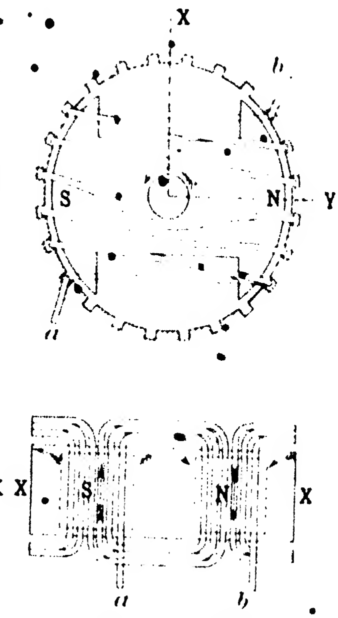


Fig. 66.—Drum bipolar armature with coil divided into two halves.

fraction of a second (from $\frac{1}{50}$th to $\frac{1}{100}$th). If the speed at which a magnet of sufficient size to give the required output can be safely or conveniently driven is not sufficiently high to give the desired frequency, it is necessary, while retaining the same width of winding on the armature relatively to the width of pole, to make more than one pair of fields pass the coils in each revolution. In other words, the alternator must be multipolar. If the bipolar alternator is imagined to be cut through, and then to be opened out, it is evident that the pair of poles can be repeated indefinitely any number of times, if they are arranged symmetrically round the enlarged armature core. The full-line portion of Fig. 67 shows the alternator of Fig. 66 cut through on the Y line when the magnet has turned

through a quarter of a revolution; the dotted portion to the right-hand side shows a pair of poles thus repeated, and similar N. and S. poles will succeed each other all round until they again form a complete circle. By thus multiplying the numbers of pairs of fields which pass across the coils in each revolution, the number of complete waves of the E.M.F. of the coils shown in full lines is increased correspondingly. If p = the number of pairs of poles, each pair passes any given point once every revolution, so that the number of periods in one revolution is p, and if N = the number

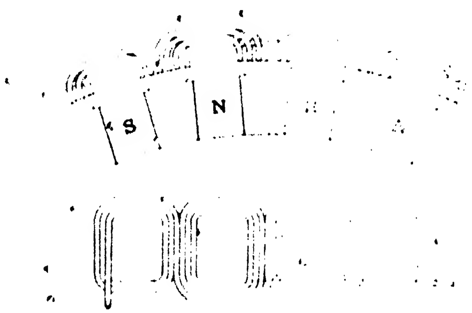


FIG. 67.—Portion of drum armature of multipolar single-phase alternator with divided coils.

of revolutions per minute, the periodicity or frequency of the multipolar alternator is $f = \frac{pN}{60}$.

Hence, by increasing the number of pairs of poles the frequency of the ***multipolar alternator*** for a given number of revolutions per minute can be raised to any required figure. The most common frequency is 50, or, as it is symbolically expressed, 50 ~, although in the United States of America 60 is commonly employed. The larger the output and size of the alternator, the slower the speed at which it is desirable to drive it; consequently the number of poles required to give a frequency of 50, will range from 2 poles with 3,000 revolutions per minute to 90 poles with 66⅔ revolutions per minute.

But it is not only the pairs of poles which can be multiplied: the coils can also be repeated for every pair of fields, and Fig. 67 shows by dotted lines a second pair repeated and connected in series with the original pair. We thus obtain in the multipolar alternator several pairs of poles, and in a single phase as many coils as there are poles, or with undivided coils, as many as there are pairs of poles, and each coil-side may, if desired, be distributed between two or more slots, as in Fig. 68. All the complete coils will in this case occupy at any moment exactly the same position relatively to a

magnetic field, and the phase and magnitude of the alternating E.M.F. induced in each will be identical. Owing to the latter fact they can legitimately be divided into two or more paths in parallel, if a large current is to be carried. Or they can be coupled together in series, so as to form a continuous winding if we desire a high E.M.F. with a small current; and the total E.M.F. of the machine at any moment is equal to the E.M.F. induced by any one coil multiplied by the number of coils that are in series. By increasing the number of coils that are in series, the E.M.F. of the

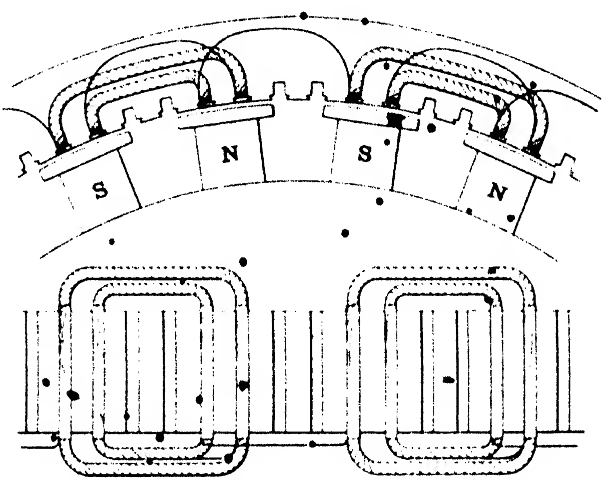


FIG. 68. -Portion of toothed drum armature of multipolar single phase alternator with undivided coils.

alternator is multiplied while the frequency is multiplied by increasing the number of pairs of poles, but whether the coils are in series or in two or more parallels, the machine remains a *single-phase alternator.* The presence, therefore, of several pairs of poles and coils when thus repeated introduces nothing new in the theory of the action, and the multipolar alternator so obtained may be simply regarded as made up of several bipolar alternators. Thus in single-phase alternators the active conductors are confined within a comparatively small number of sharply defined coils, separated off from one another by intervening spaces of core on which there is no winding. Although less marked in the case of polyphase machines next to be described, this characteristic is still present in all cases, and by it the alternator is to a great extent distinguished not only in action but also in appearance from dynamos which give a unidirected current.

§ 7. Polyphase alternators.—Since in the single-phase alternator the winding only covers a part of the armature core, it is evident that an entirely distinct set of coils forming a second armature circuit might be wound in the vacant spaces between the coils of the first circuit. Thus in Fig. 68, in order that the reluctance of the magnetic circuit may be kept approximately constant, a pair of empty slots occurs alternately between the full slots, and could

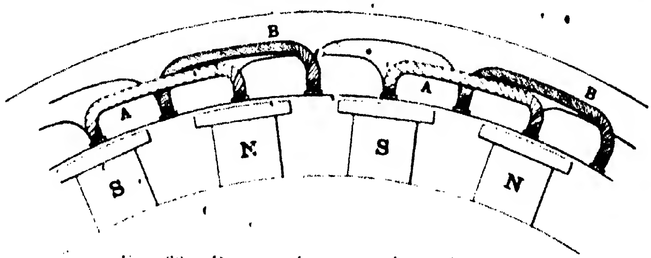


FIG. 69. Portion of quarter phase alternator.

be filled by a second winding (Fig. 69). The second set of coils may be used to feed an entirely separate external circuit, possibly at a different pressure; or if the original winding only covers one-half of the armature, and the second winding is a duplicate of the first, as regards number and size of the wires, they may be interconnected to form a *quarter-phase alternator*. The geometrical degrees of displacement of the two windings will of course depend

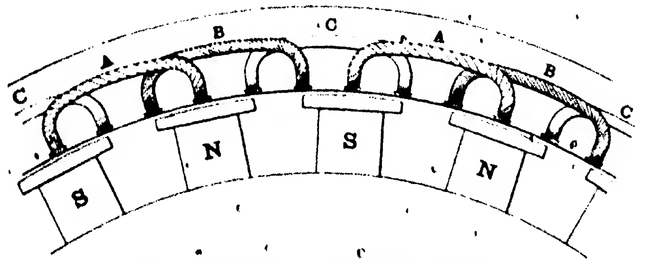


FIG. 70.—Portion of 3-phase alternator.

upon the number of poles, but since twice the pole-pitch corresponds to 360 electrical degrees, and the coils of the *A* winding are displaced from the corresponding coils of the *B* winding by half the pitch, the phase of the E.M.F.'s in the two circuits will differ by a quarter of a period, or 90°, the one reaching its maximum when the other is at zero. This process may be carried still further if the slots between which each coil-side is divided occupy less than half the pitch; *e.g.* with undivided coils if they cover not more than one-third, or with divided coils, not more than one-sixth of the pitch,

three distinct armature windings can be wound on the same core, giving curves of E.M.F. differing in phase by 120°, and a 3-*phase alternator* is obtained (Fig. 70). Since 1890 such alternators have

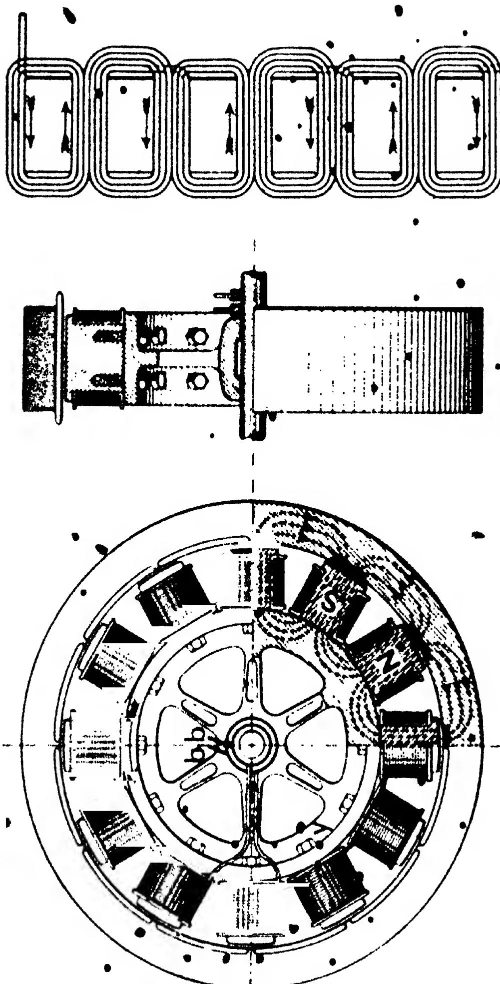


FIG. 71.—Diagram of single-phase drum multipolar alternator.

been largely used, and are called in general *polyphase* machines. Their construction and design will again be dealt with in Chapter XXIV and subsequent Chapters.

§ 8. General description of single-phase drum heteropolar alternator.—Whether the armature or field magnet system rotates is decided by practical considerations of mechanical or electrical convenience. As it is easier to insulate stationary coils for high pressures, the armature is usually the *stator*. The field-magnet or *rotor* is then internal, and carries two *collecting rings*, connected to the field winding; the exciting current is led into and out of the field winding by means of brushes or rubbing contacts, while the ends of the armature windings are attached to stationary terminals. Fig. 71 shows diagrammatically a single-phase drum-wound stationary armature, with twelve internal poles, pointing radially outwards, and each wound with an exciting coil; one-half of the machine is shown in section in the side view, and one-quarter in the end view, the paths of the magnetic flux being there marked by dotted lines. It will be seen that the total flux of lines forming one field passes, entire through each magnet core, and bifurcates on reaching the armature core or the yoke-ring. In practice the

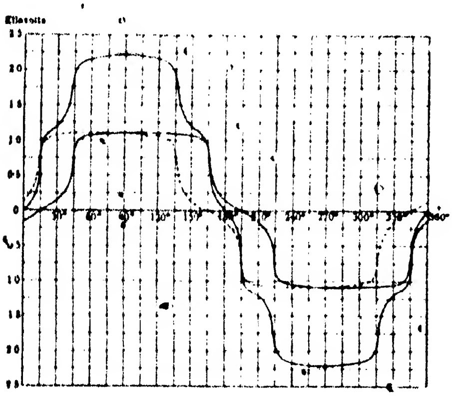


FIG. 72.—Two groups of coils 30° = one-sixth of pole-pitch apart, on smooth armature.

system of magnets would be supported from the bed-plate, which is omitted in the diagram for the sake of clearness. The twelve armature coils are connected to form a single series; the active wires of each coil are shown as disposed in one layer only of six turns, although in most cases there would be a large number of turns, possibly in two or more slots. Current is supplied to the exciting coils by the brushes *bb* and collecting rings, while the alternating E.M.F. and current is obtained from the stationary terminals of the armature. At the right hand is shown an internal view of one-half of the armature winding after removal of the magnet-system.

§ 9. Shape of the E.M.F. curve.—The exact shape of the E.M.F. curve of an alternator admits of great variations with different dispositions of iron and copper in field-magnet and armature (*see* Chapter XXVI), but the general principles bearing on the subject may be shortly traced.

Owing to the gradual shading off of the flux at the edges of the pole-pieces the rise and fall of the E.M.F. always follows a continuous curve and can never show abrupt changes. Starting with a single coil with side of narrow width on a smooth armature, which gives a nearly flat-topped wave of E.M.F. with a normal ratio of pole-width to pole-pitch of 0·66, Figs. 72 and 73 show the effect of a grouped distribution of two and three such coils, spaced respectively one-sixth and one-ninth of the pole-pitch apart, as they would be in a three-phase alternator with two and with three slots per pole per phase. Proceeding still further, Fig. 74 shows to a different scale the effect of compounding the E.M.F.'s of a 6-turn coil; the turns are spaced out at intervals of 15°, so that each coil-side actually covers 75°, and a near approach is made to practically uniform distribution over half the pole-pitch. As the number of groups, or the spread of the coil-side, is increased, the flat-topped portion decreases in width, until in the last case as the active conductors of a coil-side one by one come into or pass out of full action, the resultant curve is smoothed out, and its shape closely resembles a sine curve. Any undulatory effect left in the case of Fig. 74 and seen more markedly in Figs. 72 and 73 is a true "spacing ripple," due to the spacing out of the groups.

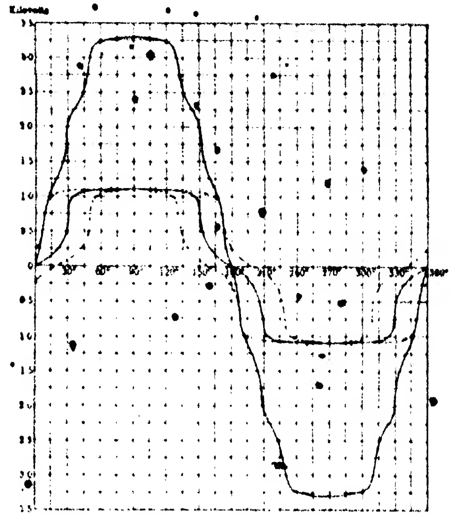


FIG. 73.—Three groups of coils 20° or one-ninth of pole-pitch apart, on smooth armature.

But a marked distinction exists between the smooth armature even when the distribution is grouped and the toothed armature of practice in which the groups are embedded in open or semi-closed slots. In the former case, it will be seen from the preceding figures that when the E.M.F. is rising, though its rate of rise alters, the E.M.F. never decreases, and *vice versâ*, when it begins to fall, it never again increases during its fall. But in the toothed armature there may be and frequently are such marked ripples superimposed

upon the main wave of the curve that its rise or fall is no longer progressive. So far as the winding in the toothed armature is grouped in the slots, there is the same spacing effect as already described. But in addition there is a further effect, owing to the presence of the slots, which must be taken into account, namely, the pulsation of the flux in magnitude or its oscillation to and fro if the slots affect the reluctance of the magnetic circuit, or alter the spacial distribution of the lines.[1] When the slots are few in number and of large dimensions, and especially if they are open at the top, the iron and air which are presented at any moment to the polar face by the armature core may vary appreciably either in their proportionate amounts or in their position relatively to the pole-face. Further, the wave-shape alters with the degree of excitation which alters the permeability of the iron circuit (Fig. 75).

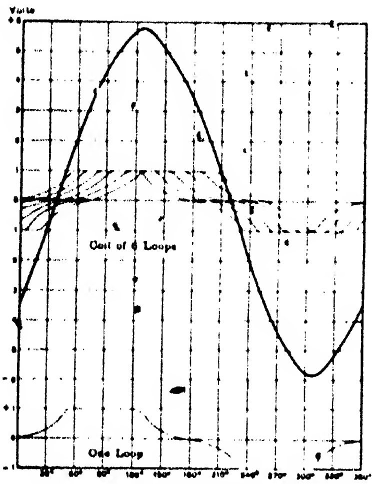


FIG. 74.—Combined E.M.F. of coil on smooth armature.

§ 10. Analysis of E.M.F. wave into its fundamental and harmonics.—By Fourier's theorem, as is well known, any periodic function, however complex or distorted may be the form of its curve, so long as it is single-valued and finite, may be represented by the sum of a series of sine curves of different frequency, phase, and amplitude, the number of such component curves increasing with the complexity of the function.[2] Or expressed analytically, if y is any periodic function of frequency f periods per second,

$$y = C + C_1 \sin(\omega t + \theta_1) + C_2 \sin(2\omega t + \theta_2) + C_3 \sin(3\omega t + \theta_3) + \text{etc}$$

where C_1, C_2, C_3, etc., are the amplitudes of the different constituent curves. The angular velocity ω being a constant $= 2\pi f$, ωt is an angle varying with the time, and θ_1, θ_2, θ_3, etc., are the phases of the different constituents. Or the same expression may also be written as

$$y = C + C_1 \sin 2\pi f(t + t_1) + C_2 \sin 4\pi f(t + t_2) + C_3 \sin 6\pi f(t + t_3) +$$

[1] Cp. Chap. XXVI.

[2] See Fleming, *The Alternate-Current Transformer*, vol. 1, chap. iii, §§ 2–6, and especially S. P. Thompson, *Dynamo-Electric Machinery*, vol. 2, chap. ii.

where t_1, t_2, t_3 . . . are the times when each component wave first passes through zero. If the X axis along which time is measured divides the curve of a single period into a positive and a negative half of equal area, the constant C vanishes, and this is normally the case with alternating E.M.F.'s or currents. Of the remaining terms the first containing $\sin(\omega t + \theta_1)$ obviously passes through a cycle of values in the same time as the complex periodic function, and so has the same frequency; the second containing $\sin(2\omega t + \theta_2)$ has double the frequency, and so on. Hence the first is called the *fundamental*, and the rest from the analogy of "overtones" in acoustics are called the *harmonics* of the first term. The second term of double frequency is strictly called the first harmonic, the third term of triple frequency the second harmonic, and so on. The first, third, etc., harmonics, the frequencies of which are even multiples of that of the fundamental, are known as the even harmonics, and similarly the second, fourth, etc., are known as the odd

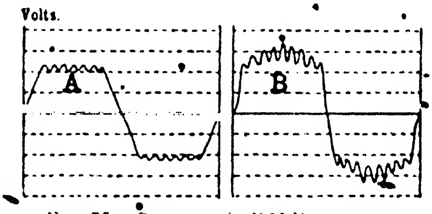


FIG. 75.—Open-circuit E.M.F. curves of alternator, (A) with moderate, and (B) with strong excitation.

harmonics, their frequencies being uneven multiples. By some writers the fundamental term is called the first harmonic, and the terms of double, triple, etc., frequency are called the second, third, etc., harmonic, but with either terminology the odd and even harmonics have reference to the multiple which their frequency is of that of the fundamental.

If even harmonics are present, either the two half-waves of the complex periodic function are entirely dissimilar, or, if they have the same shape, they are passed through in opposite order in point of time, the rise *e.g.* of the negative half-wave repeating the decline of the positive half-wave, and *vice versâ*. Since in alternators successive poles are as far as possible made of equal strength and of similar distribution of flux, even harmonics very rarely occur and may be regarded as exceptional. In the absence, then, of even harmonics the general expression for an alternating E.M.F. or current becomes

$$y = C_1 \sin(\omega t + \theta_1) + C_3 \sin(3\omega t + \theta_3) + C_5 \sin(5\omega t + \theta_5) + \dots \text{ etc.}$$

If $\theta = \tan^{-1}\frac{B}{A}$, so that $\cos\theta = \frac{A}{\sqrt{A^2+B^2}}$ and $\sin\theta = \frac{B}{\sqrt{A^2+B^2}}$, it follows by trigonometry that

$$\sqrt{A^2+B^2}\sin(\beta+\theta) = A\sin\beta + B\cos\beta$$

Hence if for C_1 above is substituted $\sqrt{A_1^2+B_1^2}$, where $\frac{B_1}{A_1} = \tan\theta_1$, the first term may also be resolved into the sum of two terms, $A_1\sin\omega t + B_1\cos\omega t$; analogously, the second term becomes $A_3\sin 3\omega t + B_3\cos 3\omega t$, where $\frac{B_3}{A_3} = \tan\theta_3$, and similarly for all the other harmonics. This simply amounts to a statement of the fact that if any radius vector as C_1 is resolved into two components at right angles to each other, and one of these components is so chosen that it lags by the angle θ_1 behind C_1, the instantaneous value $C_1\sin(\omega t + \theta_1)$, or the projection of C_1 on the datum axis, is equal to the sum of the instantaneous values $A_1\sin\omega t + B_1\cos\omega t$ or the projections of A_1 and B_1 on the same axis; or, when plotted by rectangular co-ordinates, the curve of C_1 is at any moment equal to the algebraic sum of the ordinates of the A_1 and B_1 curves. Thus each constituent sine curve of the complex periodic function may itself be resolved into a pair of curves, a sine and a cosine curve of the same frequency, and the general expression for the alternating function becomes—

$$y = A_1\sin\omega t + A_3\sin 3\omega t + A_5\sin 5\omega t + \ldots$$
$$+ B_1\cos\omega t + B_3\cos 3\omega t + B_5\cos 5\omega t + \ldots$$

It will now be seen that all the sine components pass through zero, and all the cosine curves reach their maximum at one and the same time, and the reason for making this transformation is that in this form the sine and cosine amplitudes A and B for any particular harmonic are readily obtainable either by analytic or graphical methods. One of the quickest methods is that given by Dr. S. P. Thompson (*loc. cit.*, or *Electrician*, vol. 55, p. 78), based on Runge's method, and a convenient schedule for practical use on a similar principle has been given by A. E. Clayton[1]; after determining by either of these methods the values of A and B for any harmonic, they are easily recombined, and the phase relation of lag or lead determined for the harmonic from the typical relation

$$\sqrt{A_1^2+B_1^2}\sin\left(\omega t + \tan^{-1}\frac{B_1}{A_1}\right) = C_1\sin(\omega t + \theta_1).$$

[1] *Journ. I.E.E.*, vol. 59, p. 491, where the degree of accuracy is also investigated. A set of tables on a different method is given in *Electrical World and Engineer* (vol. 43, p. 1034); cp. also R. Beattie (*Electr.*, vol. 67, p. 326), La Cour and Bragstad, *Theory and Calculation of Electric Currents*, pp. 195 ff., and P. Kemp, Supplement to vol. 57, *Journ. I.E.E.*, p. 85.

When an alternating E.M.F. (or current) with only odd harmonics is resolved into a Fourier series, its virtual or R.M.S. value is

$$\sqrt{\tfrac{1}{2}(\mathbf{C}_1^2 + \mathbf{C}_3^2 + \mathbf{C}_5^2 + \ldots)} = \sqrt{\tfrac{1}{2}(\mathbf{A}_1^2 + \mathbf{B}_1^2 + \mathbf{A}_3^2 + \mathbf{B}_3^2 + \ldots)}$$

i.e. the square root of half the sum of the squares of the amplitudes.

$$\text{If } e = \mathbf{C}_1 \sin(\omega t + \theta_1) + \mathbf{C}_3 \sin(3\omega t + \theta_3) + \ldots\ldots$$
$$\text{and } i = \mathbf{D}_1 \sin(\omega t + \phi_1) + \mathbf{D}_3 \sin(3\omega t + \phi_3) + \ldots\ldots$$

the average power or mean product of simultaneous values of e and i over a half-period is the algebraic sum of the powers resulting from the fundamental E.M.F. and fundamental current and from each of the harmonic E.M.F.'s and currents, *i.e.* half the sum of the products of the amplitudes of similar order with due regard to their relative phase-angles $(\theta_1 - \phi_1)$, etc., as in Chapter VI, § 13; *i.e.* the true watts are

$$\frac{\mathbf{C}_1\mathbf{D}_1}{2}\cos(\theta_1 - \phi_1) + \frac{\mathbf{C}_3\mathbf{D}_3}{2}\cos(\theta_3 - \phi_3) + \ldots$$

When further resolved into sine and cosine terms, with the typical relations $\sqrt{\mathbf{A}_1^2 + \mathbf{B}_1^2} = \mathbf{C}_1$, and $\sqrt{\mathbf{J}_1^2 + \mathbf{K}_1^2} = \mathbf{D}_1$, these several sine and cosine curves are in phase, and the mean power is half the sum of the products of the amplitudes of similar sine or cosine terms taken in pairs from each expansion, *i.e.*

$$\frac{\mathbf{A}_1\mathbf{J}_1}{2} + \frac{\mathbf{B}_1\mathbf{K}_1}{2} + \frac{\mathbf{A}_3\mathbf{J}_3}{2} + \frac{\mathbf{B}_3\mathbf{K}_3}{2} + \ldots\ldots \text{etc.}$$

In order to analyse with considerable accuracy the E.M.F. curves of alternators, it is often sufficient to determine only the fundamental and the harmonics of triple and quintuple frequency, those of higher frequency forming but a small residuum. The importance of such analysis lies in the fact that the determination of the frequency of the principal components whose amplitudes bear an appreciable percentage ratio to that of the fundamental throws light on the causes to which the distortion of the wave-shape is due. Further, for exhaustive study of the conditions in an alternating circuit it is necessary to consider not merely the equivalent sine waves for E.M.F. and current, but also the several effects and powers from the fundamentals and harmonics,[1] especially if capacity is present and resonance may occur.

The distribution of the flux over the pole-pitch is seldom itself so exactly sinusoidal that on the assumption of perfectly constant speed, which is very nearly attained in practice, it will give a sinusoidal curve of E.M.F. Harmonic terms are therefore required

[1] Cp. Steinmetz, *Alternating Current Phenomena*, chaps. xxv–xxvii (5th edit.).

in order to express the actual curve of E.M.F.; *e.g.* a flat-topped curve requires a triple and quintuple harmonic in combination with the fundamental in order to express it even approximately.

§ 11. The E.M.F. of the heteropolar alternator in terms of the fundamental and harmonics of the field.—But not merely can the fundamental and harmonics of the E.M.F. wave be found: it can itself be expressed in terms of the fundamental and harmonics of the flux-curve of the field.

Since a single conductor gives with a sinusoidal field[1] an instantaneous E.M.F. of $LVB_{g1\,max} \sin \omega_e t \times 10^{-8}$ volts, a second conductor displaced from the first, say, in another slot, by an electrical angle γ_e (also reckoned on a basis of 180° = one pole-pitch) will similarly give an instantaneous E.M.F. $= LVB_{g1\,max} \sin (\omega_e t + \gamma_e) \times 10^{-8}$. The algebraic sum of two sine curves of equal periodic time and amplitude, but displaced from one another, is shown by simple trigonometry to be another sine curve of the same periodic time, displaced in phase from either of its components and having an amplitude which is necessarily less than twice that of either component. Although there may be no direct differential action under a pole-face, this reduction in amplitude is due to the two components not coinciding in phase, and may be expressed by a *differential factor.*

Now since the component E.M.F.'s of a number of conductors or coils traversing a sinusoidal field are all sinusoidal, by an extension of the same process of compounding sine curves the resultant amplitude as compared with that due to one conductor multiplied by the number of conductors in series in a phase can be readily calculated mathematically, as will be shown in the next section, and later in Chapter XXV. They can be added as vectors graphically or by trigonometry, and in relation to a sinusoidal field there is no difficulty in determining the value of the resultant differential factor for any number of slots or coils or for any kind of winding.

Considering the total number of active conductors which form a path in series and yield one combined phase of E.M.F., they may be divided into a number of belts or sheafs, each one of which corresponds to one pole. Let t be the number of active conductors in one such belt; *i.e.* in the drum coil if undivided, t is the number of conductors in one coil-side or the number of turns in the large coil, but if the coil is divided into two halves, t is equal to twice the number of turns in the half-coil. Thus in Fig. 65 or Fig. 66 the belt corresponding to one pole consists of six active conductors, and this is formed either by the single coil-side or by the adjacent

[1] The field is here provisionally regarded as "steady," or of constant shape in time, although strictly this would only be true for a smooth armature. But the conclusions deduced from the steady field of the smooth armature apply in the main to the toothed armature of practice, so that for the present elementary treatment the relative displacement of the conductors is spoken of as due to their being placed in different slots.

sides of two divided coils. In the drum alternator which from its practical importance is alone being considered here, the duplicate belt of active conductors forming the opposite coil-side has also to be considered at the same time. In addition, therefore, to the differential action within the belt forming a coil-side, whether the conductors are grouped in two or more slots or are uniformly distributed over some fraction of the pole-pitch, there will be a difference of phase between the E.M.F.'s of the two coil-sides if the mean pitch y' between the two belts is less than the full pole-pitch Y (Fig. 60). Thence there results a further reduction in the amplitude, which must be taken into account in the differential factor.

The instantaneous E.M.F. of two belts corresponding to one pole-pair may, therefore, be expressed as $2t LV\, k_{ds}\, B_{gs\,max} \sin \omega_e t \times 10^{-8}$, where k_{ds} is the factor expressing all the effects of differential action and the subscript letter s serves to keep before the mind that a sinusoidal field is alone under consideration. Further in the whole machine if there are p pairs of poles, there are p such large undivided coils or p pairs of divided coils, situated similarly to one another, and therefore in phase with one another; these again may be divided into q parallel paths, where q may be one or two or any whole number of which p is a multiple, so that the E.M.F. of a phase is due to $\frac{p}{q}$ coils. Finally if N_{ph} be the number of separate phases (§ 7), and Z be the total number of active conductors counted all round the armature periphery, the number of conductors in one belt corresponding to a pole is $t = \frac{Z}{N_{ph} 2p}$. The total instantaneous E.M.F. of one phase as due to a simple sinusoidal field is therefore

$$e_a = \frac{p}{q} \times \frac{2Z}{N_{ph} \cdot 2p} \times LV\, k_{ds} B_{gs\,max} \sin \omega_e t \times 10^{-8},$$

time being reckoned from the instant when the axes of the coils are in line with the centres of the sinusoidal fields. Since $V = \frac{\pi DN}{60}$, where D is the diameter in cm.,

$$e_a = \left(\pi DL \frac{N}{60} \times \frac{Z}{N_{ph} \cdot q} \times 10^{-8}\right) k_{ds} B_{gs\,max} \sin \omega_e t \quad . \quad . \qquad (26)$$

$$= c \, . \, k_{ds} B_{gs\,max} \sin \omega_e t \text{ volts}$$

where c is the constant within the bracket for a given machine.[1]

[1] Although it has above been supposed in deducing the formula that there is a whole number of slots per pole, so that each succeeding pole-pitch is a repetition of the first, this assumption is not essential when k_{ds} is adjusted to suit a fractional number of slots per pole.

As already stated, the flux-distribution under the pole of an alternator is seldom strictly sinusoidal, but whatever the shape of the flux-density curve, since it is single-valued, finite and cyclic, it can always be resolved into a Fourier series of sinusoidal waves, the expression for the flux-density being thus in general

$$B_{g1} \sin (\alpha + \theta_1) + B_{g3} \sin (3\alpha + \theta_3) + B_{g5} \sin (5\alpha + \theta_5) + \ldots \quad (27)$$

No even harmonics are present in normal cases, and the subscript numerals indicate respectively the fundamental and higher uneven harmonics.

Now all that has been said of the E.M.F. due to a simple sinusoidal field is equally true of each separate sinusoidal component of the actual complex field, provided only that k_{d} is given its proper value appropriate to the nth harmonic, and this will now be symbolized as k_{dn}. The total instantaneous E.M.F. of a phase is therefore

$$e_a = c \left\{ k_{d1} B_{g1} \sin (\omega_1 t + \theta_1) + k_{d3} B_{g3} \sin (3\omega_1 t + \theta_3) + k_{d5} B_{g5} \sin (5\omega_1 t + \theta_5) + \ldots \right\} \quad (28)$$

Thus the E.M.F. in a single coil may be expressed as the sum of a number of components due to the several sine waves of the flux-density, and in a phase as the sum of the sinusoidal components of the E.M.F.'s of the coils. In either case any component may be represented by a vector or rotating radius of length proportional to the amplitude, whose projection on a time axis will measure its value at any instant.

§ 12. The addition of vectors and general expression for the differential factor. The addition of a number of vectors, each of the same length and displaced successively from one another by the same angle ψ enters into so many problems in connection with the design of both continuous-current and alternating-current machinery that the subject will first be introduced in a general manner.

If q_1 vectors, each of length e and displaced consecutively by the same angle ψ, are drawn either end to end or radiating from a common centre after the fashion of Fig. 76, the sum of their projections on any two rectangular co-ordinates or axes at right angles to one another and each passing through the starting point of the first vector or through the centre O from which they radiate, will yield respectively the two series

$$e\left[\sin \alpha + \sin (\alpha + \psi) + \sin (\alpha + 2\psi) + \ldots + \sin \left\{\alpha + (q_1 - 1)\psi\right\}\right]$$

$$= e\,\frac{\sin \left\{\alpha + (q_1 - 1)\dfrac{\psi}{2}\right\} \sin q_1 \dfrac{\psi}{2}}{\sin \dfrac{\psi}{2}} \quad (29)$$

and $\{\cos\alpha+\cos(\alpha+\psi)+\cos(\alpha+2\psi)+\ldots+\cos\{\alpha+(q_1-1)\psi\}\}$

$$= \frac{\cos\left\{\alpha+(q_1-1)\frac{\psi}{2}\right\}\sin q_1\frac{\psi}{2}}{\sin\frac{\psi}{2}} \quad . \quad . \quad (30)$$

Here α is the angle of displacement of the first vector from the axis upon which the cosine series is measured, the sine series giving the projections on the second axis at right angles thereto, and ψ may be positive or negative. The summations are stated at once as they are to be found in standard works on trigonometry, and by inserting various values for α or ψ, the special forms required in practice are quickly found.

Usually the maximum value of the vectorial sum in point of time is required, as given by the projections of the component rotating vectors or of the closing vector of the polygon upon an axis of time. Just as in a clock diagram the projections of the rotating radii upon a vertical axis of time give their instantaneous values, so also the projection of the closing side of the polygon if rotated about O gives the instantaneous value of the sum. For this to be a maximum, it is evident, when q_1 is uneven, that the central vector must coincide with or be parallel to the vertical axis by reference to which time is measured, the remaining vectors being paired on either side of the central vector (Fig. 76, i), while if q_1 is even, they must be placed symmetrically on either side of the vertical axis (Fig. 76, ii). In these circumstances, making use of the sine series, the angle α by which the first vector is displaced from the horizontal axis becomes $\frac{\pi}{2}-(q_1-1)\frac{\psi}{2}$. Since $\sin q_1\frac{\psi}{2}$ and $\sin\frac{\psi}{2}$ are constant, it is obvious from (29) that the sum is a maximum when $\sin\left\{\alpha+(q_1-1)\frac{\psi}{2}\right\}$ is a maximum, and that this is a maximum when $\alpha=\frac{\pi}{2}-(q_1-1)\frac{\psi}{2}$. Thus, at the chosen instant when the vectorial sum is a maximum, its value is

$$= \frac{\sin q_1\frac{\psi}{2}}{\sin\frac{\psi}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Or, making use of the cosine series on the same vertical axis of time, the angle α by which the first vector is displaced from it is $-(q_1-1)\frac{\psi}{2}$, and $\cos\left\{\alpha+(q_1-1)\frac{\psi}{2}\right\}=\cos 0°=1$, so that for the same instant the same result may also be reached from expression (30).

If the vectors are paired as mentioned above, with q_1 uneven and a central vector giving its full effect as shown in Fig. 76, i, then at the chosen instant the vector sum is

$$OA + 2AB + 2BC + \ldots$$

$$= e\left\{1 + 2\cos\psi + 2\cos 2\psi + 2\cos 3\psi + \ldots + 2\cos(q_1 - 1)\frac{\psi}{2}\right\}$$

With q_1 even, there is no central vector, but when they are paired, their sum falls on the line bisecting the angle between the central pair and is

$$2OA + 2AB + \ldots$$

$$= e\left\{2\cos\frac{\psi}{2} + 2\cos\frac{3\psi}{2} + \ldots + 2\cos\frac{q_1 - 1}{2}\psi\right\}$$

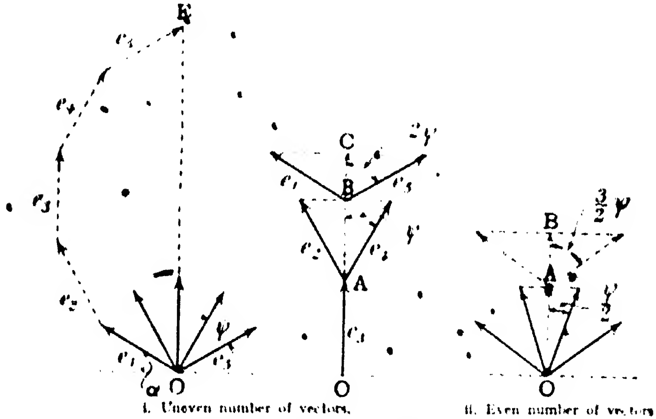


FIG. 76. Addition of vectors.

Either of the above two series is therefore the equivalent of the preceding series and can be shown to be equal to

$$\frac{\sin q_1 \frac{\psi}{2}}{\sin \frac{\psi}{2}}$$

which is the ratio of the vectorial sum to the magnitude of a single vector of amplitude e. But the arithmetical sum of the amplitudes is $q_1 e$, so that the ratio

$$\frac{\text{vectorial sum of the amplitudes}}{\text{arithmetical sum of the amplitudes}} = \frac{\sin q_1 \frac{\psi}{2}}{q_1 \sin \frac{\psi}{2}} \quad . \quad (32)$$

The virtual values, as for an alternating E.M.F., being in each case proportional to the amplitudes, the same ratio is the *differential factor* for sinusoidal E.M.F.'s differing in phase, expressed in its most general form.

The same expression is also reached by another method which is often employed in considering the subject. As it brings out additional features, it is also here given. In Fig. 77 if θ be half the

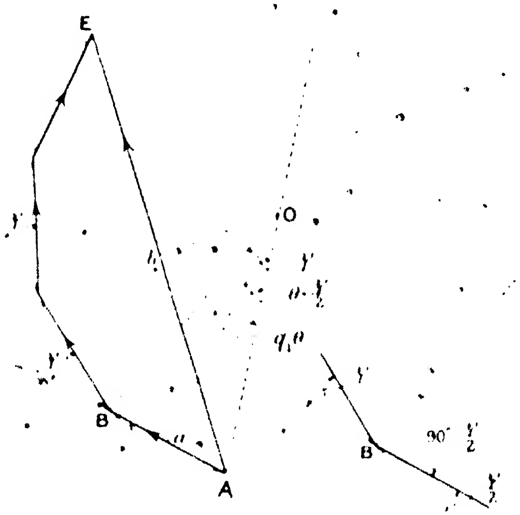


Fig. 77 Deduction of differential factor

angle subtended at the centre of the circle of radius R by the vector of length AB, and there are q_1 vectors, the angle $AOb = q_1\theta$. Then

$$\frac{Ab}{AO} = \frac{AE}{2R} = \sin q_1\theta$$

$$\frac{Aa}{AO} = \frac{AB}{2R} = \sin\theta$$

$$\therefore \frac{AE}{AB} = \frac{\sin q_1\theta}{\sin\theta}$$

But the arithmetical sum of the vectors is q_1 times AB. Therefore the ratio

$$\frac{\text{vectorial sum}}{\text{arithmetical sum}} = \frac{\sin q_1\theta}{q_1 \sin\theta}$$

But, as shown at the side of Fig. 77, the angles OBA and OAB are each equal to $\pi/2 - \varphi/2$; the remaining angle BOA of the triangle is therefore equal to φ, and $\theta = \varphi/2$, so that

$$\frac{\text{vectorial sum}}{\text{arithmetical sum}} = \frac{\sin q_1 \frac{\varphi}{2}}{q_1 \sin \frac{\varphi}{2}}$$

When q_1 is very large, and φ correspondingly small, for $q_1 \sin \theta$ may be written $q_1\theta$. In the limiting case of an infinite number of vectors subtending very small angles, which corresponds to a winding perfectly uniformly distributed over some arc of the bore, *i.e.* over an angle χ, the ratio becomes

$$\frac{\text{vectorial sum}}{\text{arithmetical sum}} = \frac{\sin \frac{\chi}{2}}{\frac{\chi}{2}}$$

where χ is now the angular width in radians of the belt over which the winding is uniformly distributed.[1] This ratio is also equal to $\frac{\text{chord}}{\text{arc}}$, for $\sin\frac{\chi}{2} = \frac{Ab}{R} = \frac{\text{chord of } \chi}{2R}$ and $\frac{\chi}{2}$ in radians $= \frac{\text{arc of } \chi}{2R}$. Therefore in this special case

$$\text{differential factor} = \frac{\sin \chi/2}{\chi/2} = \frac{\text{chord}}{\text{arc}} \quad . \quad . \quad . \quad (32a)$$

The general expression (32) will be applied later (in Chapter XXV) to determine k_{en} in definite cases.[2] One further case, elementary in nature, but of fundamental importance, must however here be added. The E.M.F.'s in the two sides of a drum coil are in phase, so far as their action *round the coil as a whole* is concerned, when, as separate vectors, they are exactly opposed to one another at an electrical angle of $180° = \pi$ radians. If, therefore, the mean pitch of the coil-sides or the coil-span when expressed as a proper or improper fraction of the pole-pitch is λ_c, the relative displacement of the two vectors of the coil sides for

[1] For the functions $\frac{\sin \alpha}{\alpha}$ and $\frac{\sin q_1 \alpha}{q_1 \sin \alpha}$ the late Prof. S. P. Thompson (*Journ. I.E.E.*, vol. 53, p. 240) proposed the names "cursine" and "persine," and that they should be written "cursin α" and "pers$_{q_1}$ α."

[2] Much valuable information on the vectorial treatment of E.M.F.'s in both alternators and continuous current machines is given in *The Shape of the Pressure Wave in Electrical Machinery*, by S. P. Smith and R. S. H. Boulding, *J.I.E.E.*, vol. 53, p. 228, and *Theory of Armature Windings*, by S. P. Smith, *J.I.E.E.*, vol. 55, p. 18.

insertion in expression (32) as ψ is $\pi(1-\lambda_c)$; whence the pitch or coil-span differential factor is for the fundamental

$$k'_{s1} = \frac{\sin \pi (1-\lambda_c)}{2 \sin \frac{\pi}{2} (1-\lambda_c)} = \cos \frac{\pi}{2} (1-\lambda_c)$$

or in general for any order n of harmonic

$$k'_{sn} = \cos n \frac{\pi}{2} (1-\lambda_c) = \cos n \frac{\varepsilon}{2} \quad . \quad . \quad . \quad (33)$$

where ε is the excess or deficiency by which the coil-pitch differs from the pole-pitch, and the E.M.F.'s of the two sides depart from synchronism in phase. As Fig. 78 shows, the two sides OA, OB

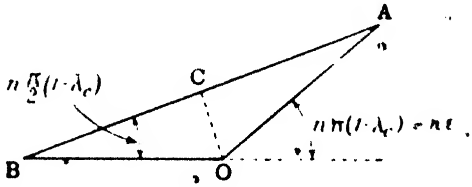


FIG. 78.—Pitch differential factor of drum coil

being equal, and the two angles OAB, OBA each of value $n\frac{\pi}{2}(1-\lambda_c)$, $AB = 2BC = 2OB \cos n \frac{\pi}{2} (1-\lambda_c)$.

§ 13. Resolution of the field density into fundamental and harmonics.—At no load, the curves of flux for each pole-pitch being not only similar but symmetrical within themselves about their middle ordinate, no cosine or B terms are required, all harmonics having a phase of 0° or 180°. The no-load flux-curve[1] will therefore be expressible as

$$B_{g1} \sin a + B_{g3} \sin 3a + B_{g5} \sin 5a + \ldots \quad . \quad (34)$$

Fig. 79 shows how closely a flat-topped curve is reproduced with harmonics of only triple and quintuple frequency, the equation to the curve being in percentages

$$y = 100 \sin a + 20 \sin 3a + 3{\cdot}5 \sin 5a \ldots \quad . \quad (35)$$

Under load the resultant flux-curve becomes distorted in shape and cannot be so simply expressed. An approximate curve composed of only a few terms is shown in Fig. 80, its equation being

$$y = \sin a + \frac{1}{3^2} \sin (3a + 45°)$$
$$- \frac{1}{5^2} \sin (5a + 45°) - \frac{1}{7^2} \sin (7a + 90°).$$

[1] Any one or more of the harmonic amplitudes B_{g3}, B_{g5}, etc., may itself be negative.

or in percentages

$$y = 100 \sin \alpha + 11{\cdot}1 \sin (3\alpha + 45^\circ) - 4 \sin (5\alpha + 45^\circ) - 2{\cdot}04 \sin (7\alpha + 90^\circ)$$

Here α is measured from the origin of the fundamental, but the resultant curve is found to pass through zero at $-1{\cdot}6^\circ$, so that when

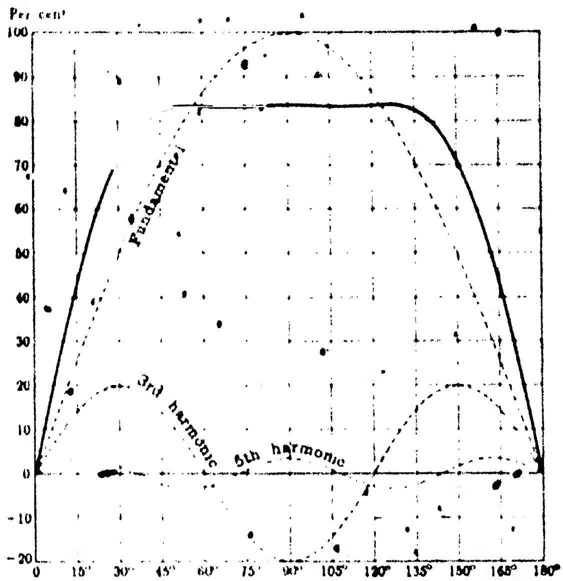


FIG. 79. No load flux curve in terms of fundamental and harmonics

α' is measured from the origin of the resultant curve, the same equation becomes

$$y = C_1 \sin (\alpha + \theta_1) + C_3 \sin (3\alpha + \theta_3) + \ldots$$

$$= 100 \sin (\alpha - 1{\cdot}6^\circ) + 11{\cdot}1 \sin (3\alpha + 45^\circ - 4{\cdot}8^\circ) - 4 \sin (5\alpha + 45^\circ - 8^\circ) - 2{\cdot}04 \sin (7\alpha + 90^\circ - 11{\cdot}2^\circ)$$

$$= 100 \sin (\alpha - 1{\cdot}6^\circ) + 11{\cdot}1 \sin (3\alpha + 40{\cdot}2^\circ) - 4 \sin (5\alpha + 37^\circ) - 2{\cdot}04 \sin (7\alpha + 78{\cdot}8^\circ)$$

The amplitudes C_1, C_3 . . . can then be resolved into A and B terms, by the relation $A_1 = C_1 \cos \theta_1$, $B_1 = C_1 \sin \theta_1$, $A_3 = C_3 \cos \theta_3$ . . . . and the equation becomes

$$y = 99{\cdot}96 \sin \alpha + 8{\cdot}46 \sin 3\alpha - 3{\cdot}19 \sin 5\alpha - 0{\cdot}386 \sin 7\alpha - 2{\cdot}79 \cos \alpha + 7{\cdot}165 \cos 3\alpha - 2{\cdot}4 \cos 5\alpha - 2{\cdot}00 \cos 7\alpha \quad (36)$$

in which form it is more readily applied to practical calculations.

§ 14. The virtual E.M.F. in terms of the fundamental of the flux-density curve.—The full purpose and advantage of such a division of the E.M.F. into its sinusoidal components, and its extension to the case of an alternator under load, must be left for

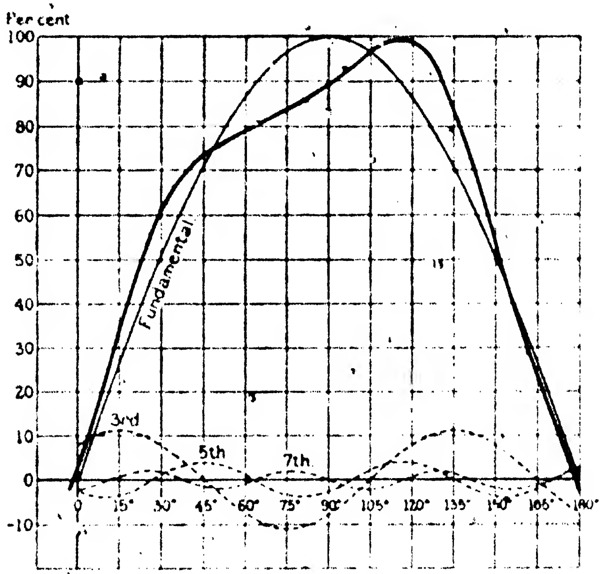


FIG. 80. Flux curve (distorted under load) in terms of fundamental and harmonics.

discussion at a later stage. But proceeding therefrom the virtual E.M.F. of the phase is

$$E_a = \frac{c}{\sqrt{2}}\sqrt{(k_{n1}B_{e1})^2 + (k_{n3}B_{e3})^2 + (k_{n5}B_{e5})^2 + \ldots}$$

$$= \frac{ck_{n1} \cdot B_{e1}}{\sqrt{2}}\sqrt{1 + \left(\frac{k_{n3} \cdot B_{e3}}{k_{n1} \cdot B_{e1}}\right)^2 + \left(\frac{k_{n5} \cdot B_{e5}}{k_{n1} \cdot B_{e1}}\right)^2 + \ldots}$$

Taking the highest values for B_{e3} and B_{e5} that are likely to occur in practice, viz. ± 24 per cent. and ± 10 per cent. respectively of B_{e1}, it can then be shown[1] that with the values for k_{n3} and k_{n5}, etc., given by normal three-phase windings, the value for the radicle is only slightly increased above unity and that there is but little

[1] See "Note on the Use of the Fundamental of the Flux Curve in E.M.F. Calculations," by S. Neville, in *Papers on the Design of Alternating Current Machinery*, by Hawkins, Smith, and Neville (Pitman & Sons), p. 138.

error[1] in identifying E_a with the first term only, $\frac{c\,k_{s1}\,.\,B_{g1}}{\sqrt{2}}$, which is the virtual E.M.F. from the fundamental only of the flux-density wave. Hence within the limits of accuracy required in practice, the higher harmonics have so little effect on the virtual E.M.F. of the phase that we may say

$$E_a = \frac{1}{\sqrt{2}} k_{s1} B_{g1} \frac{Z}{N_{ph}\,q} \frac{N}{60} \pi D L \times 10^{-8} \text{ volts} \quad . \quad . \quad (37)$$

which should be compared with equation (5a) for the average torque. The area of the fundamental of the flux curve, or the fundamental flux taken out of the complete Φ_a curve is $\Phi_1 = \frac{2}{\pi} B_{g1} Y L$, and $Y = \frac{\pi D}{2p}$, so that $\Phi_1 = B_{g1} D L / p$, and in terms of Φ_1

$$E_a = \frac{\pi}{2\sqrt{2}} k_{s1} \,.\, 2p\Phi_1 \,.\, \frac{Z}{N_{ph}\,q} \,.\, \frac{N}{60} \times 10^{-8} \text{ volts} \quad . \quad . \quad (37a)$$

where $\frac{\pi}{2\sqrt{2}}$ is the form factor of the E.M.F. wave $= \frac{\text{virtual value}}{\text{average value}}$.
The last equation should be compared with equation (5d.)

It must here be specially noted that if the total flux Φ_a is inserted and the value k_{s1} appropriate to a sinusoidal field is retained, a greater error is introduced than occurs with the above close approximation.[2]

§ 15. The E.M.F. equation of the heteropolar alternator in terms of the total flux. The fundamental equation of the E.M.F. induced in the armature of a heteropolar alternator can, however, be given in another equally important form, which covers in one term the effect of different shapes of flux curve or E.M.F. wave.

Using the same symbols as in Chapter IV, § 6, the average density of the flux cut by a conductor is $k\,B_{g\,max}$ and its average E.M.F. is $k\,B_{g\,max} L V \times 10^{-8}$. The *gross average E.M.F.* induced by the two belts of a complete coil of t turns would therefore be $2 t k B_{g\,max} L V \times 10^{-8}$. This average E.M.F. would be practically realized in a toothed or tunnelled armature with concentrated winding, the wires of each belt being then wound all in the same slot or hole, and the mean pitch of the belts being equal to the pole-pitch. But with grouped or distributed winding, a distinction must be drawn between the gross average value of the E.M.F. of all the conductors on the supposition that there is no differential action at all between them,

[1] Even when the 3rd harmonic is not eliminated from the line-pressure by "star" interlinking of the phases of a three-phase alternator, the above statement is true; still smaller is the error, when it is eliminated.

[2] S. Neville, *loc. cit.*

and the *net average E.M.F.* which is the arithmetical mean of all the instantaneous values of a half-wave of the actual E.M.F. The latter is less than the gross average obtained on the supposition that each active conductor always assists every other, by some amount depending on the extent to which the E.M.F. of some conductors is at times neutralized or lessened by that of others. As already shown in §§ 4 and 11, there must be some differential action as each belt passes across the neutral line between two poles, and especially will this be the case if the width of the belt exceeds the width of the gap between two neighbouring poles of opposite sign. Again, whether the width of the belt be large or small, there will be differential action between the two belts unless their mean pitch be exactly equal to the pole-pitch. The effect on the average E.M.F. must, therefore, be discounted by multiplying the gross average by a certain differential factor, or as it is also called "winding factor," k_d, which is analogous to k_{d_s} of § 11, but is not confined to the case of a sinusoidal field.

The net average E.M.F. of the single undivided coil as a whole is then $k_d \cdot k \cdot 2B_{g\ max} tLV \times 10^{-8}$, and the net average E.M.F. of the pair of divided coils which is the exact equivalent of the single large coil is of course the same. The value of k_d, or the ratio of the net average to the gross E.M.F., like that of k_{d_s}, will depend upon the ratio of the pole-width to the pole-pitch, upon the spacing of the slots in which the conductors of a belt are wound, or the ratio of width of coil-side to pole-pitch, and also upon the mean pitch of the coil-sides, whether equal to or a fraction of the pole-pitch; while it may approach unity, it must always be less than 1 if the coil-side has any width.

The shape of the instantaneous E.M.F. curve of the coil still has to be taken into account in order to find its virtual E.M.F. as given by the square root of the mean square; for with the same average value the E.M.F. curve may vary greatly in form according to the density of the lines in the air-gap where they are cut by the wires. Let k_f = the ratio which the square root of the mean square bears to the mean ordinate of the E.M.F. curve, its value being also dependent upon the relative widths of coil and pole; then the virtual E.M.F. of the single individual coil or of the pair of divided coils is

$$k_f \times \text{the net average E.M.F.} = k_f \cdot k_d \cdot k\, 2B_{g\ max} tLV \times 10^{-8}$$

and $k_f = \dfrac{\text{virtual E.M.F}}{\text{average E.M.F.}}$ is the *form factor*, since it varies with the shape or form of the E.M.F. curve. The more peaked the curve, the higher the form factor.

As in § 11, there are in a phase p such coils which may be divided between q parallel branches, and $t = \dfrac{Z}{N_{ph} \cdot 2p}$. The virtual

E.M.F. induced in one phase of the armature winding is therefore

$$E_a = \frac{p}{q} \cdot k_f \cdot k_d \cdot 2\,kB_{g\,max} \times \frac{Z}{N_{ph} \cdot 2p} \times LV \times 10^{-8}.$$

$$= k_f \cdot k_d \cdot kB_{g\,max} \times \frac{Z}{N_{ph} \cdot q} \times \frac{N}{60} \times \pi DL \times 10^{-8} \text{ volts} \qquad (38)$$

The reason for the retention of $B_{g\,max}$ in the above equation instead of writing $B_{g\,av}$ in place of $kB_{g\,max}$ is that in practice the average density and total flux of a pole-pitch $kB_{g\,max}\dfrac{\pi DL}{2p}$ cannot be greatly modified except by alteration of $B_{g\,max}$, so that the latter is really the limiting factor and will prove to give the greatest amount of information in the processes of design. The interpretation to be put upon $B_{g\,max}$ in the case of a toothed armature will be explained in Chapter XVI.

Since $\Phi_a = kB_{g\,max}\dfrac{\pi DL}{2p}$ (Chapter IV, § 6), for $kB_{g\,max}L\pi D$ may be substituted $2p\Phi_a$, and in terms of the total flux of a pole-pitch, the virtual E.M.F. may also be expressed as

$$E_a = k_f \cdot k_d \cdot 2p\Phi_a \times \frac{Z}{N_{ph} \cdot q} \times \frac{N}{60} \times 10^{-8}$$

$$= k_f \cdot k_d \cdot 2\Phi_a \times \frac{Z}{N_{ph} \cdot q} \times \frac{pN}{60} \times 10^{-8} \quad . \quad . \quad . \quad (38a)$$

which should be compared with equation (5c), or uniting the two factors k_f and k_d into one joint[1] coefficient K

$$= K \cdot 2\Phi_a \times \frac{Z}{N_{ph} \cdot q} \times \frac{pN}{60} \times 10^{-8} \quad . \quad . \quad . \quad . \quad (38b)$$

Here $\dfrac{pN}{60}$ is the frequency of the machine, and $\dfrac{Z}{N_{ph}\,q}$ is the number of wires in series in one phase. As a general rule, there is only one path in each phase, and $q = 1$. The numerical value of the constant K will vary with the shape of the poles as affecting the distribution of the lines and also with the type of machine, yet in the main it depends upon the *relative* widths of the coils and fields as compared with the pitch; certain general cases may therefore be taken which will serve as guides to practice, and in Chapter XXV, the values for such cases will be tabulated. If the distribution of the field were, in fact, sinusoidal, $k_f = \dfrac{\pi}{2\sqrt{2}}$, k_d becomes identical with k_{dw}, $kB_{g\,max} = \dfrac{2}{\pi}B_{g1}$, $\Phi_a = \Phi_1$, and equations (38) and (38a) reduce to equations (37) and (37a).

[1] In many books the numerical constant 2 is likewise thrown into the combined coefficient, which then becomes for a sine curve 2·22.

If the E.M.F. equation of the alternator in its fully developed forms be now compared with the fundamental equation (7), it will be seen that each of the three variables still finds its appropriate equivalent ; instead of the density B, there now appears the total flux of one field Φ_a, the simple length of the one active conductor is replaced by the total number of wires Z, and the velocity of movement reappears in the number of revolutions.

Again, if the expressions (5e) and (5f) for the average torque in watts per rev. per min. be multiplied by N, and divided by $f \cdot N_{pk} \cdot q$, the induced E.M.F. per phase necessarily results, so that the same expressions have already been implicitly arrived at in Chapter IV by another route which proceeded from the fundamental fact of the torque.

CHAPTER X

HETEROPOLAR CONTINUOUS-CURRENT DYNAMOS

§ 1. Continuous-current machines of Class II requiring a commutator.—Continuing with the heteropolar machines which form our main subject, we pass to the last great group, viz., those which give a continuous or direct current, in the external circuit. Since the E.M.F. induced in each armature coil formed of loops alternates in direction, but the terminal voltage is now required to be unidirected, some form of commutator developed out of the simple split-ring of Fig. 15 must be present. Thus by the addition of a commutator, the heteropolar dynamo is as it were artificially made to yield a uni-directed voltage at the terminals, and by proper design as is now to be explained, this voltage can be rendered practically constant, so that it becomes a continuous-current machine.

The purposes to which such machines can be applied are so numerous and so important, that they are used more extensively than any other class of dynamos, although not built in such large sizes as alternators; whether it be for incandescent lighting, charging accumulators, motor driving and transmission of energy over moderate distances, or electro-deposition of metals, they can be used for all, and are well suited for most purposes, while the voltages for which they are made vary from 5 to 3,000 volts. Although delivering a direct current to the external circuit beyond the commutator, yet internally they remain essentially alternators. Their characteristic features will be found to be intimately connected with the nature of the winding of their armatures. In all cases this forms an endless helix which is closed upon itself, or "re-entrant," as it is termed, independently of any connection to the external circuit. So important is this fact and its consequences that they may be specially regarded as machines having a *closed-circuit armature winding*, although such a type of winding, as will be seen later, may be and often is employed in heteropolar alternators.

§ 2. Necessity for short-circuiting a coil.—Since the voltage of machines belonging to the present group is to be steady and free from fluctuations, it is reasonable to try the effect of placing two or more coils on the armature in positions which are not symmetrical relatively to the poles, so that when one coil is giving its maximum E.M.F. and another is approaching zero, the crest of the one wave may be added to the hollow of the other. We have, therefore, first to determine how the E.M.F.'s of such coils can be combined together, their connection with the external circuit being still reversed

at the right moment when the direction of the E.M.F. induced in them changes. Next, and most important, this must be done without opening or breaking the entire circuit. To effect this, whenever a coil is in the position of reversal of E.M.F. between two adjacent poles, it is *short-circuited* on itself. If each of the two brushes of Fig. 15 is made to touch both sectors of the split ring simultaneously at the moment of reversal, the coil is closed upon itself for a short time through the brush, and this allows the current previously flowing round the coil in the one direction to die away, and a current in the reverse direction round it to be started by the

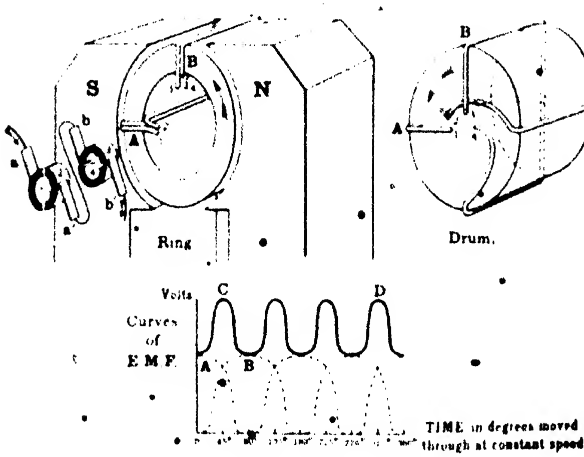


FIG. 81.—Two coils 90° apart, connected in series.

reversed E.M.F. as it approaches the next pole. The brushes must be so set that the short-circuiting takes place approximately at the position of zero E.M.F., and the curve of E.M.F. of the loop, as given in Fig. 15, is thus practically unaltered.

§ 3. Two coils at right angles. Starting, then, in the two-pole case with a pair of coils at right angles to each other, and connected to a pair of split-rings displaced relatively to each other by a corresponding angle of 90°, we must keep the gaps small, give the brushes a sufficiently wide contact surface, and set them in such a position and at such an angle to the split-rings that they bridge over the insulating material between the sectors whenever a coil is in the position of reversal. Next, let one brush of one split-ring be connected with the brush on the opposite side of the other ring; thus Fig. 15 will take the form shown in Fig. 81, which represents a two-coil ring, and by its side the corresponding drum-wound armature; for the sake of clearness, the armature shaft on which both

split-rings would be mounted, and the connecting wires which join the ends of the coils to the sectors are omitted, the connexions being merely indicated by corresponding numerals. Brush *a'* is joined to brush *b*, and the remaining pair of brushes form the terminals to which the external circuit is applied; as shown in the figure, coil *B* is in the position of reversal, and, therefore, is short-circuited through its brushes; the current flows into *a*, through coil *A*, which is alone supplying E.M.F., and out by *a'* into *b* and *b'*. But a moment later, when the armature has moved farther round, both coils will be in action, and neither will be short-circuited by the brushes; the current due to the E.M.F. induced in both will now flow through coil *A*, and thence through coil *B*, leaving the armature by brush *b'*. As the armature continues to rotate, coil *A* will in turn be short-circuited by its brushes when it reaches the second position of reversal, and coil *B* will alone supply E.M.F. to the external circuit; finally, both coils will come again into action, until *B* is again short-circuited at the end of half a revolution.

The armature circuit is thus never broken, and no coil is ever open-circuited; each gives a curve of E.M.F. as shown in Fig. 15, but the important result has been obtained that the two coils are now in series, and the curve of the total E.M.F. acting at the brushes *a* and *b'* will be given by adding together simultaneous ordinates of the two separate curves and plotting their sums as a third curve. This has been done in the lower part of Fig. 81, the curve *C D* showing the effect of adding together in series the component (dotted-line) E.M.F.'s generated at each instant in the two coils. The two coils are now either in series or one of them is short-circuited on itself; as the E.M.F. induced in one coil is diminishing in value, that of the other coil is rising, so that at 45°, 135°, 225°, and 315° the two are equal, and the total E.M.F. then reaches a maximum, while it never falls below the value of the maximum E.M.F. given by one coil when in the position of best action. At once it will be seen that the fluctuations in the third full-line curve of joint E.M.F., though still marked, are very much reduced in value from what they are when each coil acts separately. Expressed as a percentage, the fluctuation, which is roughly 100 per cent. on either side of a mean value in the case of a single coil, is reduced to about 30 per cent. in the case of two coils in series.

A further step towards greater constancy of E.M.F. would be made by arranging still more coils at successive small angles in front of each other, so as to come into and out of action successively; with four coils arranged with angles of 45° between neighbouring pairs, the eight undulations in a revolution would be much smaller, and the fluctuation would be reduced to only 10 per cent. on either side of the average value throughout a revolution.

§ 4. Division of the winding into two halves in parallel.—Next, let the arrangement of Fig. 81 be repeated, so as to obtain four coils symmetrically placed round the armature, and each with its ends attached to its own split-ring commutator (Fig. 82). As in Fig. 81, a current can be collected from coils *A* and *B* by applying the external circuit to brushes *a* and *b*; similarly, a current can be collected from coils *A′* and *B′* by applying the external circuit to *a′* and *b′*. If therefore we join all the brushes consecutively on opposite sides of the commutators, and finally join brush *b′* to brush *a*, both currents can be collected by applying the external circuit to brushes *a* and *b* as before. We now have a dynamo giving an

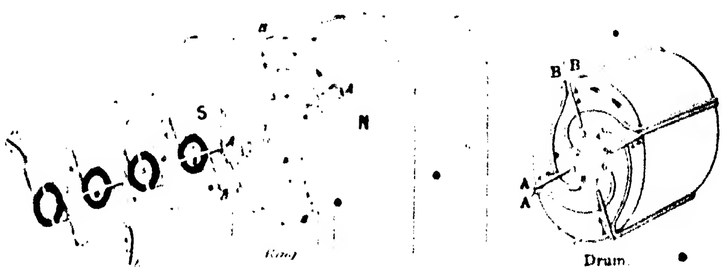


FIG. 82. Four coils 90° apart, connected in two parallels.

E.M.F. curve identical with that of Fig. 81, but carrying twice the current; the two halves of the armature winding are in parallel, and the current entering at *a* divides within the armature into two equal portions, one flowing through coils *A* and *B*, and the other through *B′* and *A′*, the two reuniting to leave at *b*. If the separate brushes and separate commutators are now replaced by one cylindrical structure divided into four sectors insulated from each other, upon which one pair of brushes rests, the effect is to convert the whole winding of the armature into a closed spirally wound coil, joined at intervals to commutator sectors. The end of one coil is consecutively joined to the beginning of the next, as the numerals show, until the end of the last is joined to the beginning of the first. Fig. 83 shows the joint commutator in perspective, and below are shown the connections of the coils; at the right of the diagram the brushes are shown in the act of short-circuiting coils *A* and *A′*. The whole is now symmetrical on both sides of the brushes, and rotation can be indefinitely continued without, at any time, breaking the circuit or leaving any coil open-circuited. As soon as any coil passes away from one pole-piece, it is short-circuited under a brush, and passes over into the other half of the winding. The complete closed-circuit armature

is now seen to justify its name, not only because a coil is never opened, but also because it is itself a closed helix, formed of a number of continuously wound loops; starting from any point, the winding can be traced right round the armature, without any break, until the starting-point is again reached. The brushes must be set at the opposite ends of a diameter nearly corresponding to the neutral line of zero field.

§ 5. Eight-coil bipolar armatures. Having once arrived at an armature winding divisible into two halves in parallel, it is easy to pass from two coils in each half to three or four, or any larger number, in each half; all that is necessary is to introduce more

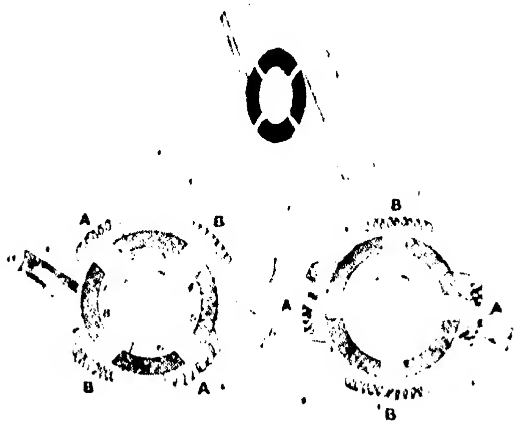


FIG. 83. Four-part commutator.

sectors into the commutator, each forming a connection between the end of one coil and the beginning of another.

Figs. 84 and 85 show two drum bipolar armatures, each with eight coils, but the one lap-wound and the other wave-wound. The difference in the winding will be dealt with in the next Chapter, but if the development of the winding at the foot of each diagram be traced out, it will be seen that each is closed on itself and is divisible into two halves. The letters, *a*, *b*, *c*, etc., mark corresponding portions of the end-connectors, where they would join if the armature were again bent up into a cylinder. The loops short-circuited by the brushes, being situated near to the neutral line of zero field, are left blank, while the direction of the E.M.F. in the rest is marked by crosses and dots or arrows in the developments.

§ 6. The nature of the closed-circuit armature.—The fundamental condition being that in all the active conductors under, say, the

N. pole the E.M.F.'s induced along their lengths are in the opposite direction to the E.M.F.'s in the active conductors under the S. pole, an inspection of Figs. 84 and 85 shows that, by both methods of winding, the small E.M.F.'s induced simultaneously in the several coil-sides are added together in each half of the winding, so that it is their sum which causes half the total armature current to flow through each path from one brush to the other. Every bipolar closed-circuit armature over-wound with coils may be divided diametrically into two halves by a line corresponding to the diameter of commutation, or, in other words, by a line at right angles to the general direction of the lines of flux as they cross from the one pole-piece to the other; along the outside of the half which is on the one side of the dividing line, say under the N. pole, there is a sheet of current flowing in all the conductors in the one direction as viewed by an observer at either end; while along the outside of the other half, the sheet of current flows in the opposite direction. The brushes remaining stationary on the revolving commutator form the contacts, by which the current is collected from both halves of the winding, and passed into or returned from the external circuit; while further, by short-circuiting the coils which are in the position of zero E.M.F., they serve also to "commute" the current, or enable a coil to pass from the one half of the armature over to the other, where the current round it is in the reverse direction. The commutator sectors, which are at any moment situated between the brushes, serve merely as junctions between the end of one loop or coil of many loops and the beginning of the next, but their true function is called out as rotation continues, and they pass successively under the brushes.

To decrease still further the fluctuations of the E.M.F., more coils or "sections" can similarly be wound on to the armature, with a correspondingly increased number of commutator sectors. The amount of fluctuation with a given number of sectors depends upon the shape of the curve of E.M.F. yielded by each section of the winding, and so upon the ratio of the polar arc to the pitch. The more pointed or triangular the shape, the less the fluctuation, but if the pole-width is not more than about 0·70 of the pitch, as in practice is usually the case, any number of sectors more than 15 per pole may be regarded as giving a sensibly constant E.M.F. at the brushes. With a large number of commutator parts, the simple tubing divided as shown in Fig. 83 has to be replaced by a built-up structure, consisting of a number of wedge-shaped bars or *sectors* of hard-drawn copper placed side by side, but completely insulated from each other by intervening strips of mica, so as to form a smooth cylinder upon which the brushes rest, the whole being insulated from the sleeve and collars by which the sectors are held tightly together.

§ 7. Multipolar continuous-current armatures.—From the bipolar armatures of Figs. 84 and 85 are immediately derived the multi-polar

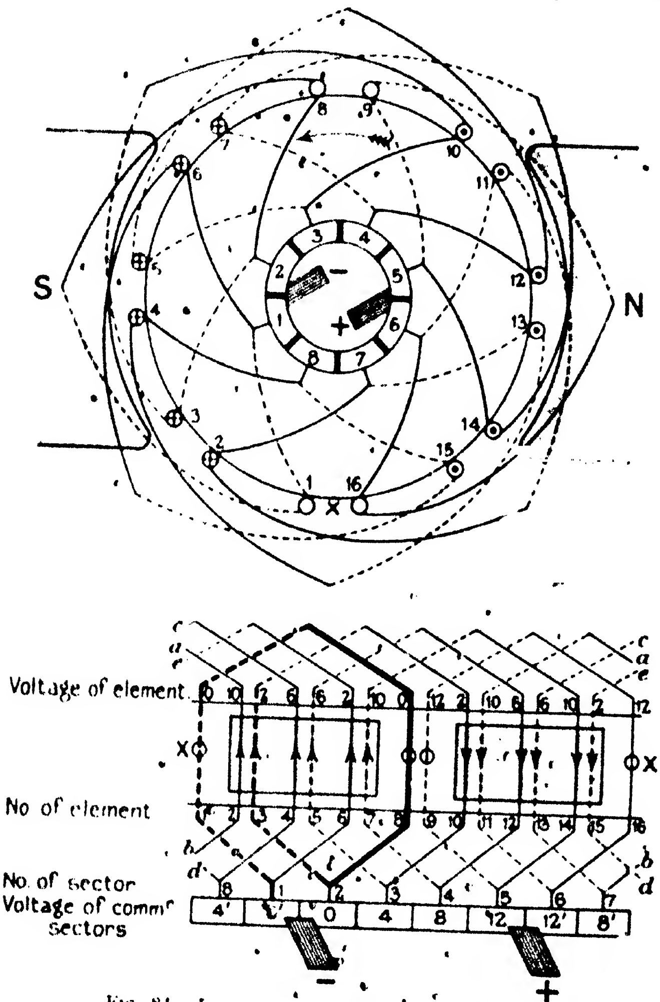


FIG. 84.—Lap-wound two-pole drum armature, $C = 8$, $y_b = 7$, $y_f = -5$, $y_c = m = 1$.

armatures of Figs. 86 and 87. The dotted lines show the former armatures repeated, the back and front pitches being retained unaltered. The winding will be discussed in the next Chapter.

but the diagrams are here introduced to illustrate two types of multipolar armatures in which the winding is divisible, not into two halves in parallel, but into as many paths in parallel as there

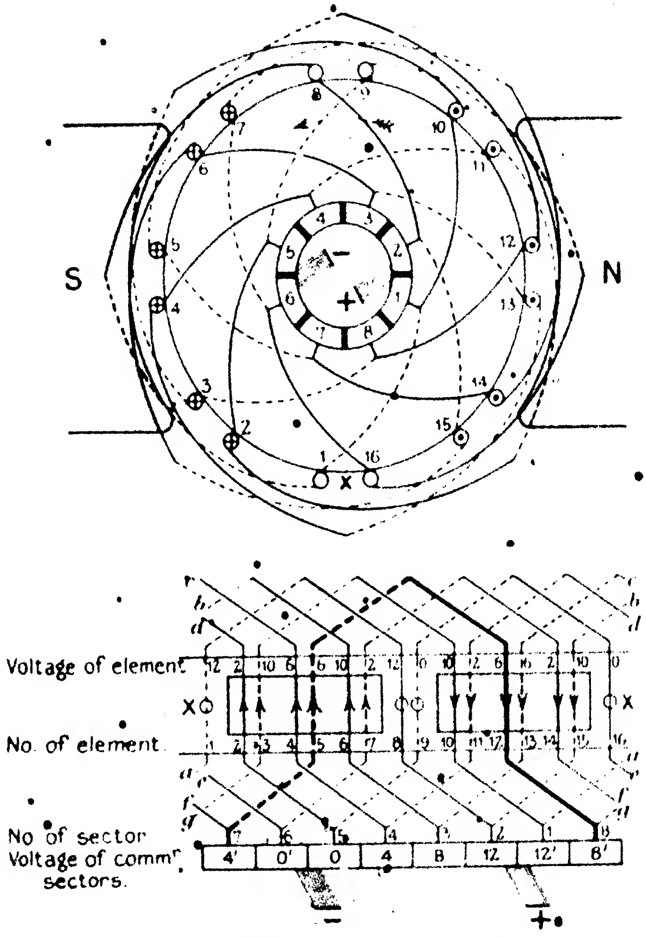


FIG. 85.—Wave-wound two-pole drum armature, $C = 8$, $y_b = 7$, $y_f = 7$, $y_c = 7$.

are poles. This division is effected by the brushes, of which there are as many sets as there are poles, the alternate brushes of + sign and the alternate brushes of − sign being respectively joined together to form a pair of terminals to which the external circuit is

applied. As in the 2-pole dynamo there are two neutral lines where the coils have their E.M.F. reversed, and two corresponding

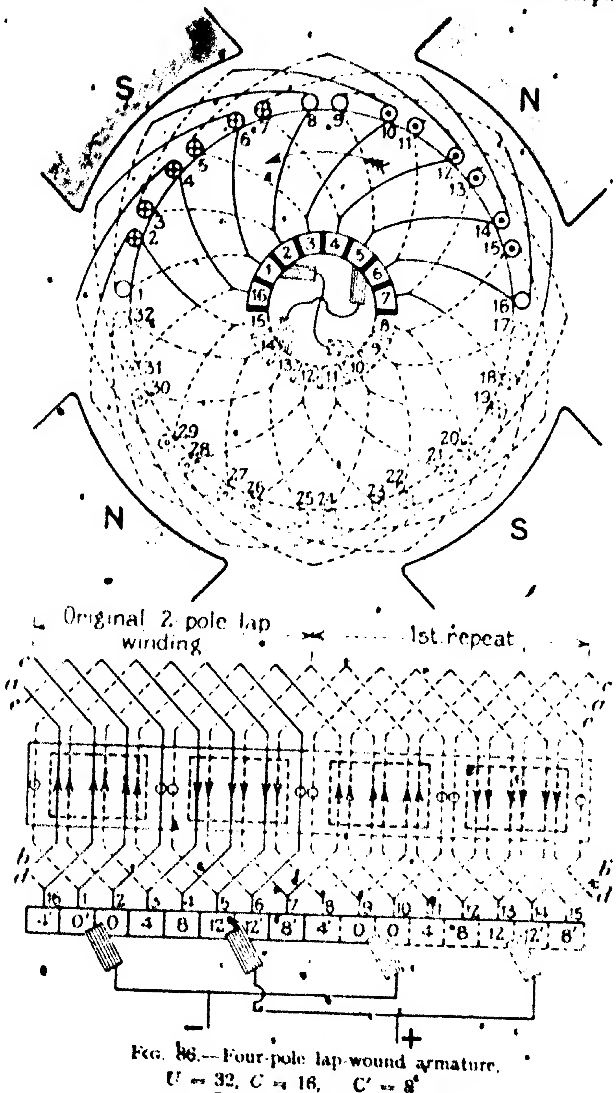


FIG. 86.—Four-pole lap-wound armature.
$U = 32,\ C = 16,\ C' = 8$
$y_b = 7,\ y_f = -5,\ y_c = m = 1.$

points on the commutator where the current is collected, so now there are $2p$ such points; and each of the 4-pole machines of Figs.

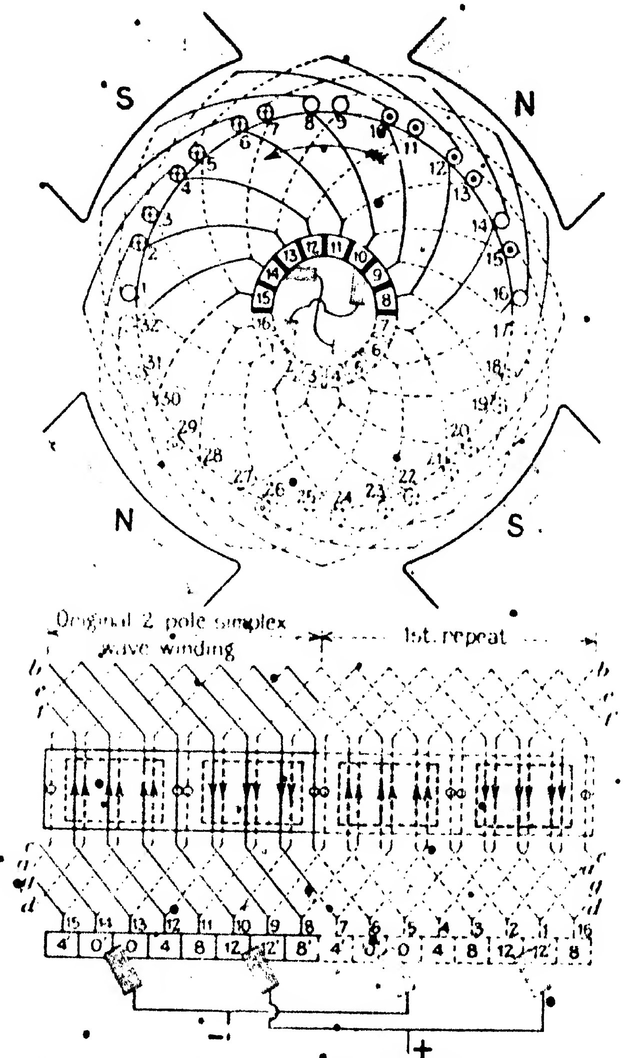


FIG. 87.—Four-pole duplex wave-wound armature ($a = p = 2$).
$U = 32$, $C = 16$, $C' = 8$,
$y_b = 7$, $y_f = 7$, $y_c = 7$.
Single helix (y_c an odd number prime to C')

86 and 87 is essentially two bipolar machines in parallel, together capable of carrying four times the current that each quarter of the winding could carry separately. If q be the number of paths in parallel from − brushes to + brushes in the continuous-current machine, q is here $= 2p$.

Thus, as contrasted with the alternator, in which the winding may be open-ended, the number of parallel paths q in the armature of the continuous-current machine with its closed-circuit winding can never be *less* than 2, and, if more than 2, must be a multiple of 2. Hence in general, $q = 2a$, where a is the number of pairs of parallel paths from − brushes to + brushes.

Of the windings above shown with $q = 2p$, owing to certain disadvantages in the wave equivalent (Chapter XI, § 20), the lap form is by far the most common—so much so that it may be regarded as the winding which gives as many pairs of parallel paths through the armature as there are pole-pairs. It is, in fact, the essential characteristic of lap-winding that it must under all circumstances give *at least* as many pairs of parallel paths as there are pole-pairs, although, as will be seen later, it may give more.[1]

§ 8. Rise and fall of potential round armature and commutator.—The winding of closed-circuit armatures being continuous, at any moment there are points on it which are abreast of each neutral plane where the flux changes direction relatively to the armature. These points are therefore points on the armature winding *fixed relatively to the poles*; whether a commutator is or is not attached to the winding, at one such point on a neutral line the potential of the armature is at its lowest and on another such point at its highest, whence it again falls. There is, therefore, a *difference of potential* between two such successive points in the winding coinciding with neutral lines of the field, and the continuous rise or fall of potential from one to the other has now to be examined.

With q paths in parallel from + brushes to − brushes, the number of conductors in series in each is $= Z/q$, and of coils in each is $Z/2qt = C/q$, where t is the number of turns in a coil. Consider a finite number of coils $2X = 2C/q$ disposed at equal angular distances of μ electrical degrees round the entire double pole-pitch of a smooth armature in a two-pole field. Thus $\mu = \pi/X$, and it will be shown in the next Chapter how μ will equally well represent the relative displacement of the coils in the field of a double pole-pitch used as a standard of reference in the case of a multipolar machine, and for wave-winding,[2] as well as for lap-winding. Although each

[1] The simplex form of lap-winding is here alone considered; its further development into multiplex forms is treated in Chap. XI, § 15.

[2] When the coils successively passed through in each tour of the armature are inserted in their proper sequence in the standard reference field of a double pole-pitch.

coil may have t turns, the width of a coil-side is assumed to be so narrow that it may be regarded as concentrated on a single line. Let the flux-density curve be resolved into its fundamental and harmonics as—

$$B_\theta = A_1 \sin a + A_3 \sin 3a + \ldots \ldots$$
$$+ B_1 \cos a + B_3 \cos 3a + \ldots \ldots$$

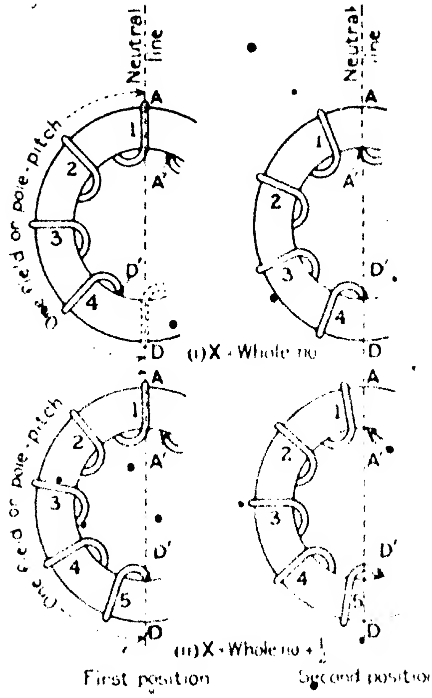


FIG. 88.—Positions for minimum and maximum E.M.F.'s.

Let $\mathbf{a}_1$, $\mathbf{a}_3$ . . . , $\mathbf{b}_1$, $\mathbf{b}_3$ . . . be the maximum value or amplitude of the sinusoidal E.M.F.'s due to each component of the field, *i.e.* $= (2t\,LV \times 10^{-8})$ multiplied by $k_{d1}''\,A_1$ or $k_{d3}''\,A_3$ or $k_{d1}''\,B_1$, etc., as the case requires, where k_{d1}'', k_{d3}'' . . . are the appropriate differential factors for the particular pitch spanned by a coil or the mean distance between its sides (*see* Chapter IX, § 12, equation (33)).

As the starting-point, let the moment be taken when the axis of a drum coil is at right angles to a neutral plane of the supposed

two-pole field. This is represented by the first positions of Fig. 88, although here for simplicity each coil is regarded as of full-pitch and is diagrammatically represented by a ring loop actually coinciding with the neutral line when in the starting position of zero E.M.F.

When X = a whole number, whether even or uneven (Fig. 88, i), a second critical position is found when the coils have moved through half their angular pitch, *i.e.* through $\mu/2$, and all coils are in action. After this, the values of the instantaneous E.M.F.'s begin to repeat in reverse order, and a cycle is completed when the angle μ has been moved through. A new coil has then replaced the first coil, and the original end coil has passed into the other half of the complete winding. The two chosen positions will, therefore, give the limits between which the E.M.F. of coils or sectors varies. For the present, commutator and brushes are supposed absent, and it is merely desired to find the instantaneous potential along the coils, or the total difference of potential between the two points on the winding helix which at any moment are situated on the neutral lines of the field.

In the first position, the instantaneous total E.M.F. between A and D is

$$
\begin{aligned}
= \; & \mathbf{a}_1 \{\sin 0^\circ + \sin \mu + \sin 2\mu + \sin 3\mu + \ldots + \sin (X-1)\mu\} \\
+ \; & \mathbf{a}_3 \{\sin 0^\circ + \sin 3\mu + \sin 6\mu + \sin 9\mu + \ldots + \sin (X-1)3\mu\} \\
+ \; & \mathbf{a}_5 \{\sin 0^\circ + \sin 5\mu + \sin 10\mu + \sin 15\mu + \ldots + \sin (X-1)5\mu\} \\
+ \; & \ldots \\
+ \; & \mathbf{b}_1 \{\cos 0^\circ + \cos \mu + \cos 2\mu + \cos 3\mu + \ldots + \cos (X-1)\mu\} \\
+ \; & \mathbf{b}_3 \{\cos 0^\circ + \cos 3\mu + \cos 6\mu + \cos 9\mu + \ldots + \cos (X-1)3\mu\} \\
+ \; & \mathbf{b}_5 \{\cos 0^\circ + \cos 5\mu + \cos 10\mu + \cos 15\mu + \ldots + \cos (X-1)5\mu\} \\
+ \; & \ldots
\end{aligned}
$$

The addition of the first vertical column (including the constant terms) will give the E.M.F. of coil 1, of the second vertical column the E.M.F. added by coil 2, and so on. The addition of the first two columns, therefore, will give the potential of the junction between coils 2 and 3, reckoned from zero on the first neutral line.

For x coils (including No. 1) out of the number X per path, the general expression for the potential at the end of the xth coil or junction between the xth and $(x+1)$th coil is given by substituting 0° for α, $n\mu$ (where n is the order of the harmonic) for ψ and x for h_1 in equations (29) and (30). Thence it is

$$
= \frac{\sin x \frac{n\mu}{2}}{\sin \frac{n\mu}{2}} \left\{ \mathbf{a}_n \sin (x-1) \frac{n\mu}{2} + \mathbf{b}_n \cos (x-1) \frac{n\mu}{2} \right\} \quad . \quad (39)
$$

Or in full the E.M.F. is—

$$\mathbf{a}_1 \frac{\sin (x-1)\frac{\mu}{2} \cdot \sin x \frac{\mu}{2}}{\sin \frac{\mu}{2}} + \mathbf{a}_3 \frac{\sin (x-1)\frac{3\mu}{2} \cdot \sin x \frac{3\mu}{2}}{\sin \frac{3\mu}{2}} + \mathbf{a}_5 \frac{\sin (x-1)\frac{5\mu}{2} \cdot \sin x \frac{5\mu}{2}}{\sin \frac{5\mu}{2}} +$$

$$+ \mathbf{b}_1 \frac{\cos (x-1)\frac{\mu}{2} \cdot \sin x \frac{\mu}{2}}{\sin \frac{\mu}{2}} + \mathbf{b}_3 \frac{\cos (x-1)\frac{3\mu}{2} \cdot \sin x \frac{3\mu}{2}}{\sin \frac{3\mu}{2}} + \mathbf{b}_5 \frac{\cos (x-1)\frac{5\mu}{2} \cdot \sin x \frac{5\mu}{2}}{\sin \frac{5\mu}{2}} + \ldots$$

In the second position, the total instantaneous E.M.F. is

$$\begin{aligned}
&= \mathbf{a}_1 \left\{ \sin \frac{\mu}{2} + \sin \frac{3\mu}{2} + \ldots\ldots + \sin (2X-1)\frac{\mu}{2} \right\} \\
&+ \mathbf{a}_3 \left\{ \sin \frac{3\mu}{2} + \sin \frac{9\mu}{2} + \ldots\ldots + \sin (2X-1)\frac{3\mu}{2} \right\} \\
&+ \mathbf{a}_5 \left\{ \sin \frac{5\mu}{2} + \sin \frac{15\mu}{2} + \ldots\ldots + \sin (2X-1)\frac{5\mu}{2} \right\} \\
&+ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&+ \mathbf{b}_1 \left\{ \cos \frac{\mu}{2} + \cos \frac{3\mu}{2} + \ldots\ldots + \cos (2X-1)\frac{\mu}{2} \right\} \\
&+ \mathbf{b}_3 \left\{ \cos \frac{3\mu}{2} + \cos \frac{9\mu}{2} + \ldots\ldots + \cos (2X-1)\frac{3\mu}{2} \right\} \\
&+ \mathbf{b}_5 \left\{ \cos \frac{5\mu}{2} + \cos \frac{15\mu}{2} + \ldots\ldots + \cos (2X-1)\frac{5\mu}{2} \right\} \\
&+ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}$$

Hence for this position the general expression for the potential at the end of the xth coil is obtained from equations (29) and (30) by substituting $n\mu/2$ for α and $n\mu$ for γ. This reduces to—

$$\frac{\sin x \frac{n\mu}{2}}{\sin \frac{n\mu}{2}} \left\{ \mathbf{a}_n \sin x \frac{n\mu}{2} + \mathbf{b}_n \cos x \frac{n\mu}{2} \right\} \quad . \quad . \quad (40)$$

Using the expression for the no-load flux-curve of equation (35), the instantaneous results for the eight-coil armatures[1] of Figs. 84 and 85 are shown in Fig. 89 to the same scale for the two positions. Here the thinner lines mark the rise and fall of the potential along the winding, the dots marking the junction of adjacent coils. Since the commutator sectors are connected to these junctions, the heavier stepped curves drawn through the dots show the potential of commutator sectors. As the number of coils per pole is increased, the two curves draw together, and both approximate more nearly to a sine curve.

If a polygon of vectors, each representing the maximum E.M.F.

[1] When arranged as a two layer winding, so that the pitch of the coils can be full, and $k_s' = \cos n \frac{\epsilon}{2} = 1$

of a coil, be drawn for the fundamental or any order of harmonic in the flux curve, as in Fig. 90, and it be imagined to rotate about its centre at the corresponding speed, then if any two points be taken, such as *e.g.* B, D' or A', D' at the ends of vectors (*i.e.* in the winding at the junctions of coils and therefore accessible points to which a circuit can be applied), the projection of the closing vector BD' or $A'D'$ upon a vertical time-axis AD will give the instantaneous value of the voltage between B and D' or A' and D', these being two points fixed in the winding and moving with it. But when the instantaneous voltage between two neutral lines,

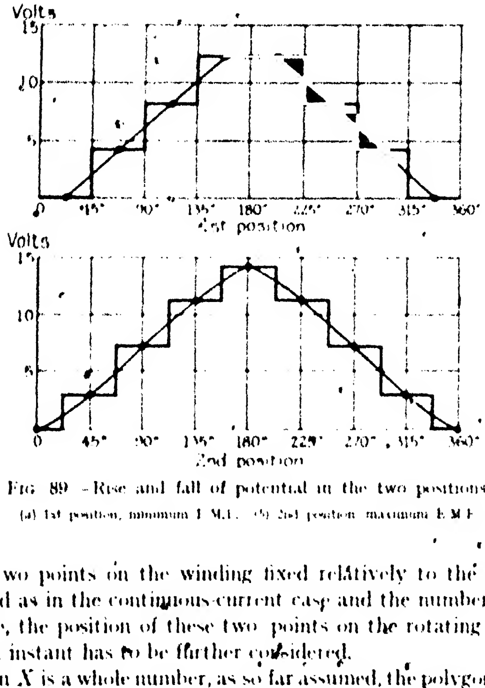


FIG. 89.—Rise and fall of potential in the two positions
(a) 1st position, minimum E.M.F.; (b) 2nd position, maximum E.M.F.

being two points on the winding fixed relatively to the poles, is required as in the continuous-current case and the number of coils is finite, the position of these two points on the rotating polygon at each instant has to be further considered.

When X is a whole number, as so far assumed, the polygon for the fundamental component E.M.F.'s and also for any harmonic closes and is symmetrical. In this case the E.M.F.'s round the closed circuit exactly balance for each harmonic, and although the E.M.F. between two neutral lines of the field fluctuates, there is no circulating current round the armature at no load, and no unequal division of the total armature current under load. In the first position of Fig. 88, i, with coil 1 on the neutral line, it adds no E.M.F., and the potential of points A and D on the winding abreast of the neutral lines are respectively the same as the potentials of the points A'

and D', being junction points of coils in the winding, diametrically opposite to each other. The voltage between A' and D' then grows just as the voltage between A and D, until the second position of Fig. 88, 1, is reached, when A' coincides with A and D' with D, after which it declines. Thus for each polygon the instantaneous voltage

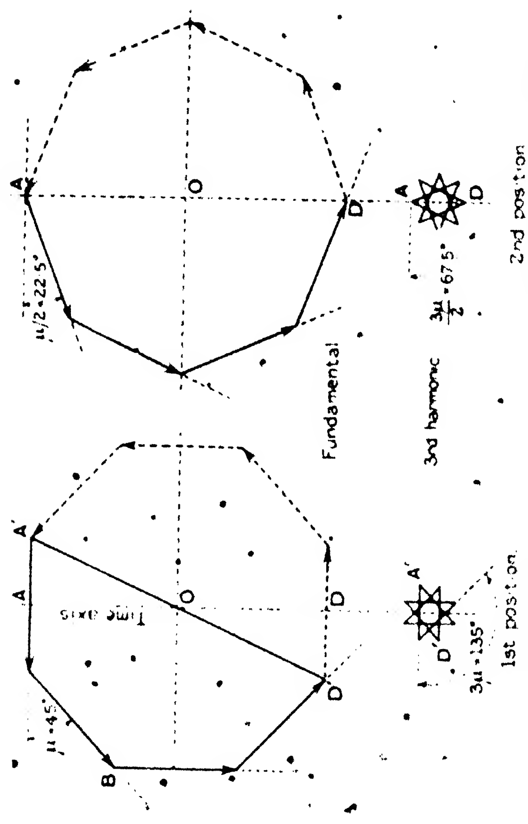


FIG. 90.—Rotating vector polygons for fundamental and 3rd harmonic E.M.F.'s for 8 coils ($X = 4$).

between two neutral lines or between brushes set thereon is given by the projection of $A'D'$ upon the vertical time axis, while $A'D'$ is passing across the vertical from its first position in Fig. 90 to a similar position on the opposite side of the vertical, when another similar cycle commences. It is in fact a small piece of the curve of voltage between two coil-junctions fixed diametrically opposite to each other in the winding, continually repeated. The complete instantaneous voltage between two neutral lines is the sum of the

projections of $A'D'$ for each polygon (fundamental and harmonics) upon the vertical time-axis AD. Thus the continuous series of values of the continuous-current E.M.F. can be traced as each coil passes from the first position to a third position similar to the first, when a new coil coincides with neutral line A, and a cycle is complete. Fig. 90 shows the fundamental polygon for the preceding case of $X = 4$, when thus treated, and to *double* the scale the third-harmonic polygon. For the latter the speed of rotation about the centre is, of course, three times that of the fundamental.

§ 9. The percentage fluctuation of E.M.F.—It is, however, the maximum and minimum total E.M.F.'s and the percentage fluctuation on either side of the mean that possess the chief interest.

For the total set of coils in one pole-pitch of the standard reference field, *i.e.* inserting $x = X$ in (39) and (40) and remembering that $\mu = \pi/X$, the expressions proportional to the minimum and maximum E.M.F.'s become respectively—

$$\mathbf{a}_1 \cdot \frac{\cos\frac{\pi}{2X}}{\sin\frac{\pi}{2X}} + \mathbf{a}_3 \frac{\cos\frac{3\pi}{2X}}{\sin\frac{3\pi}{2X}} + \mathbf{a}_5 \frac{\cos\frac{5\pi}{2X}}{\sin\frac{5\pi}{2X}} + \ldots\ldots \quad (41)$$

and

$$\frac{\mathbf{a}_1}{\sin\frac{\pi}{2X}} + \frac{\mathbf{a}_3}{\sin\frac{3\pi}{2X}} + \frac{\mathbf{a}_5}{\sin\frac{5\pi}{2X}} + \ldots. \quad (42)$$

In explanation of the disappearance of the $\mathbf{b}$ terms, it should be noticed in the first case that the first coil with $\cos 0° = 1$ gives their maximum values, *i.e.* $\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 + \ldots.$ which must add up to nought, since the origin of the curve is the neutral line of zero field; all the other coils except the first either then give zero E.M.F. or can be paired with opposite signs, so that they cancel out. In the second case, all the $\mathbf{b}$ terms cancel out in pairs or are zero. Therefore, in either case no $\mathbf{b}$ terms remain in the total E.M.F., whether or no there are such terms in the components.

Under load when the flux-curve is distorted, the percentage fluctuation on either side of the mean, or $100\,\frac{\text{max.} - \text{min.}}{\text{max.} + \text{min.}}$ calculated for increasing values of X cannot be satisfactorily compared with the results of experiment, since in fact the brushes must have appreciable width of contact to carry the load current, and each coil when short-circuited by them is virtually withdrawn from the circuit for a more or less prolonged time, with complex secondary effects obscuring the simple case. The above will, however, serve to show that when X is 15 or more, the fluctuation is very small.

On no load, the fluctuation on either side of the mean, *i.e.* $\frac{\text{max.} - \text{min.}}{\text{max.} + \text{min.}}$ if only the fundamental were present would be

$$\frac{1 - \cos\frac{\pi}{2X}}{1 + \cos\frac{\pi}{2X}} = \tan^2\frac{\pi}{4X}.$$

But actually this is increased by the presence of harmonics and if the field be regarded as completely represented by the fundamental, third and fifth harmonics, it is—

$$\frac{a_1\left(1 - \cos\frac{\pi}{2X}\right) + a_3\left(1 - \cos\frac{3\pi}{2X}\right)\dfrac{\sin\frac{\pi}{2X}}{\sin\frac{3\pi}{2X}} + a_5\left(1 - \cos\frac{5\pi}{2X}\right)\dfrac{\sin\frac{\pi}{2X}}{\sin\frac{5\pi}{2X}}}{a_1\left(1 + \cos\frac{\pi}{2X}\right) + a_3\left(1 + \cos\frac{3\pi}{2X}\right)\dfrac{\sin\frac{\pi}{2X}}{\sin\frac{3\pi}{2X}} + a_5\left(1 + \cos\frac{5\pi}{2X}\right)\dfrac{\sin\frac{\pi}{2X}}{\sin\frac{5\pi}{2X}}}$$

The presence of harmonics may then appreciably increase the fluctuation. Thus for the no-load flux-curve previously assumed, the addition of the third and fifth harmonics with $X = 4$ raises the fluctuation from 3·96 per cent. for the fundamental alone to as much as $7\frac{1}{4}$ per cent. in all. Yet when the number of coils or sectors per pole is made $X = 15$, the fluctuation above and below the average for the same field curve becomes $\pm$ 0·45 per cent., or less than $\frac{1}{2}$ per cent.

§ 10. The case of X = a whole number + a fraction. When $X = \frac{c}{q}$ is not a whole number, it is evident that each path will at times include one more coil than at other times, and that if at any moment one path includes more than the average, its complementary path includes less than the average. The simple case of X = a whole number + $\frac{1}{2}$ is here considered by means of Fig. 88 (ii) and Fig. 91 for $X = 4\frac{1}{2}$, *i.e.* for 9 coils per pole pair. A cycle is now completed by an angular movement of $\mu/2$, so that the second position of maximum E.M.F. is given by a movement through $\mu/4$. Correspondingly the fluctuation is reduced for the fundamental alone to

$$\frac{1 - \cos\frac{\pi}{4X}}{1 + \cos\frac{\pi}{4X}} = \tan^2\frac{\pi}{8X}$$

$=$ in our case 0·765 per cent., or much less than for the higher number of $X = 5$.

In contrast to the case of § 8, there are no two coil-junctions in the winding between which there is always the same voltage as between the neutral lines.

But now as soon as the armature moves past the 1st position of Fig. 88 (ii), the one path between A and D includes 5 coils or 5 vectors, while the complementary path has only 4. Consequently as the polygon of Fig. 91 rotates about its centre at O, although projection on the usual vertical time-axis passing through that centre will give the voltage between coil-junctions and the ends of complete vectors, the vertical line upon which by projection is

measured the instantaneous E.M.F. between the neutral lines must be made to slide backwards and forwards across the centre in correspondence with the movement of the vectors. The range of the required movement of translation on either side of the centre corresponds to $\mu/4$, and one limit corresponding to $\mu/4$ and maximum E.M.F. is shown by AD in the right hand portion of the

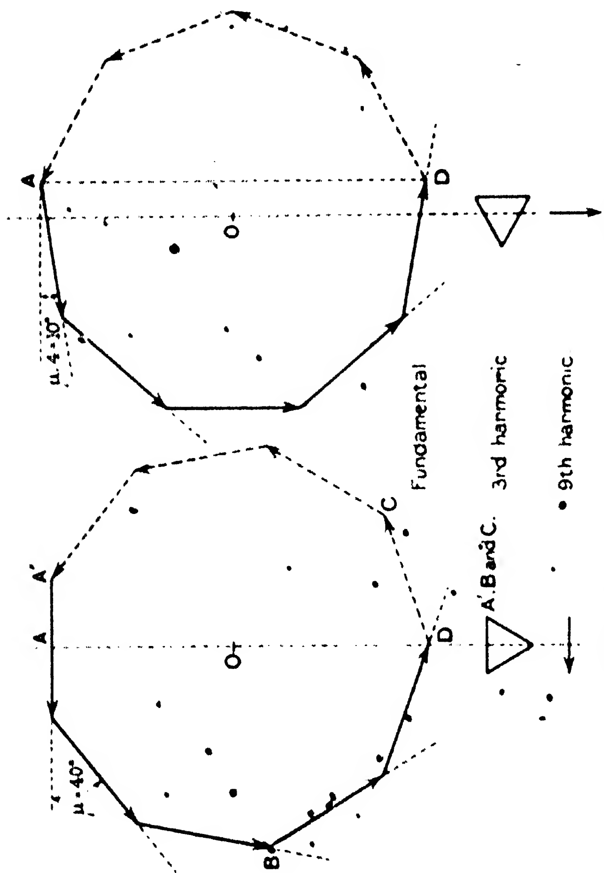


Fig. 91.—Rotating vector polygons for fundamental, 3rd and 9th harmonic E.M.F.'s for 9 coils ($X = 4\frac{1}{2}$).

diagram. Again owing to the polygon closing there is no circulating current on no load of fundamental frequency.

As regards the 3rd and higher harmonics of the flux-curve, a distinction must be drawn between the cases of a smooth and a toothed armature. In the former case with the winding in one layer, the span of the coils may be exactly a pole-pitch, so that $\varepsilon = 0$, and $k_{sn}' = \cos n\frac{\varepsilon}{2} = 1$. In these circumstances there is again no circulating current from the 3rd, 5th, and 7th harmonics if such are present in the flux-curve, the polygon for the first-mentioned being a triangle twice traced. But if a 9th harmonic is present

in the flux-curve, the same principle of construction shows that the polygon reduces to a straight line which revolves through a circle when the armature moves through μ degrees. There is thus on no load a circulating current having a frequency which is 9 times or $2X$ times that of the fundamental E.M.F.; its direction round the closed circuit alternates, as the E.M.F. of one or the other half of the winding preponderates under the action of the 9th harmonic in the flux-curve.

But if the armature be toothed, the span of the coils must be a whole number of slots, say 4 (cp. Fig. 19, ii), and the minimum deviation from an exact pole-pitch will be half a slot-pitch, or $\epsilon = 20°$. So far as the fundamental E.M.F. is concerned, this will imply only a small reduction in its value. But for the 3rd, 5th, and 7th harmonics the reduction increases rapidly, until finally, for a 9th harmonic in the flux-curve, $k_{s9}' = \cos n\frac{\epsilon}{2} = \cos 90° = 0$, and no E.M.F. from it is possible. Thus by the action of the coil-span factor, harmonics of the orders 9, 27, 45 . . . or of S', $3S'$, $5S'$, where S' is the number of slots, are entirely suppressed in the E.M.F., and no circulating current is present therefrom.

§ 11. The E.M.F. equation of the continuous-current heteropolar dynamo. When once it is established as in § 9, that the amount of fluctuation of E.M.F. between two points of the winding on the neutral lines of the standard double pole-pitch is negligible on full-load and no-load, the simple equation which is usually given for the E.M.F. equation of the continuous-current dynamo can be logically deduced.

The assumption that the E.M.F. between the two points on the armature winding that at any moment are crossing the neutral lines of the standard two-pole field is constant, so that the instantaneous and average values coincide, is identical with the supposition that for a finite value of Z, X is made infinitely large and $t = \frac{Z}{2C} = \frac{Z}{2qX}$ correspondingly small, or more strictly that the armature winding is distributed uniformly over the whole double pole-pitch, and further that the subdivisions to be considered are carried past the stage of coils or sectors down to elemental portions of a turn. In these circumstances in expressions (41) and (42), for $\sin n\frac{\pi}{2X}$ may be written $n\frac{\pi}{2X}$, and both $\frac{\cos n\frac{\pi}{2X}}{\sin n\frac{\pi}{2X}}$ and $\frac{1}{\sin n\frac{\pi}{2X}}$ become equal to $X \times \frac{2}{\pi} \times \frac{1}{n}$.

Expressions (41) and (42) therefore become identical, and equal to

$$\frac{Z}{q} LV \times 10^{-8} \times \frac{2}{\pi} \left(k_{n1}'' A_1 + \tfrac{1}{3} k_{n3}'' A_3 + \tfrac{1}{5} k_{n5}'' A_5 + \ldots\right)$$

Turning to the consideration of the total flux of a pole-pitch, the areas corresponding to the B components always cancel out

and add nothing to the net flux. For each A component the areas of $(n-1)$ half-waves in the single pole-pitch also cancel out, leaving only $\frac{1}{n} \times \frac{2}{\pi} YLA_n$ as the net addition to the actual flux of a pole-pitch.

Hence $\qquad \Phi_a = \frac{2}{\pi} YL (A_1 + \frac{1}{3} A_3 + \frac{1}{5} A_5 + \ldots)$

$$\text{or } kB_{g\,max} = \frac{2}{\pi}(A_1 + \tfrac{1}{3} A_3 + \tfrac{1}{5} A_5 + \ldots)$$

Let $k_{s1}'' A_1 + \frac{1}{3} k_{s3}'' A_3 + \frac{1}{5} k_{s5}'' A_5 + \ldots = k'(A_1 + \frac{1}{3} A_3 + \frac{1}{5} A_5 + \ldots)$ where k' only becomes unity in the case of a full-pitch loop for which $k_{sn}'' = \cos n\frac{\varepsilon}{2} = \cos 0° = 1$. Then the maximum E.M.F. between the two points on the winding situated at any moment on the two neutral lines of the double pole-pitch used as the standard of reference, is—

$$k'\, k\, B_{g\,max} \times \frac{Z}{q} \times L \times \frac{\pi DN}{60} \times 10^{-8}$$

But the position of the points on the winding, with which the positive and negative brushes respectively make contact and between which the E.M.F. of the continuous-current machine is to be found, may not be, and in practice seldom are, exactly coincident with the neutral lines where the field changes direction; for during short-circuit under the brush the current in a section of a closed-circuit armature has its direction forcibly changed, and this may require the section to be moving in a *reversing* field, after it has crossed the neutral line. In whichever direction the brushes are moved away from the neutral lines, for the same total flux the E.M.F. declines as is shown by a curve of the rise and fall of the potential such as Fig. 89*b*. If this curve were a sine-curve to which in fact it approximates, the effect for a movement of the brushes through an angle of λ_e electrical degrees would be given by multiplying the preceding E.M.F. by a reduction factor $\cos \lambda_e$. But in general it may be grouped with k' to form a joint differential factor k_{dc} less than k'.

The E.M.F. of the continuous-current armature is then

$$E_a = k_{dc} \,.\, kB_{g\,max} \times \frac{Z}{q} \times \frac{N}{60} \times \pi DL \times 10^{-8} \text{ volts} \quad . \quad . \quad (43)$$

Substituting $2p\Phi_a$ for $kB_{g\,max}\pi DL$, in terms of the total flux

$$E_a = k_{dc} \,.\, 2\Phi_a \times \frac{Z}{q} \times \frac{pN}{60} \times 10^{-8} \quad . \quad . \quad . \quad . \quad (43a)$$

The algebraic sum of the lines cut by an active conductor as the loop of which it forms one side passes from one position of zero E.M.F. to another is always equal to the amount of the flux which

is included within the loop when its axis is at right angles to the neutral plane, and this is $k'\Phi_a$. When the setting of the brushes does not correspond with short-circuit of the loop in this position, the average E.M.F. of a conductor is only proportional to the *net* flux included within a loop at the moment of short-circuit. Further differential action then reduces the average E.M.F. to proportionality with $k_{dc} . \Phi_a$. In practice the span of the drum coil is made so nearly equal to the pole-pitch that k' may be reckoned as unity, and further, the position of the brushes closely corresponds to short-circuit of the sections as they pass the neutral lines of the field, so that finally k_{dc} becomes very nearly equal to unity. Inserting $2a$ for q, equations (43) and (43*a*) take the simplified forms (compare equations (4*a*) and (4*d*))

$$E_a = k B_{g\,max} \times \frac{Z}{2a} \times \frac{N}{60} \times \pi D L \times 10^{-8} \text{ volts} \quad . \quad . \quad (44)$$

and in terms of the total flux of a pole-pitch

$$= \frac{p}{a} \times \Phi_a \times Z \times \frac{N}{60} \times 10^{-8} \quad . \quad . \quad . \quad . \quad . \quad (44a)$$

which may be adopted as the standard forms for use in continuous-current machinery design, although the possibility of a reduction of Φ_a to $k_{dc} . \Phi_a$ must be borne in mind.[1]

That perfectly uniform distribution of the winding over the whole pole-pitch is in reality assumed is evident from the absence of any term defining the moment of time in a sector-cycle that is considered. When the number of sections is large and their distribution round the periphery of the armature sufficiently close, the withdrawal or addition of one section per path as it enters or leaves the condition of short-circuit under a brush, produces a negligibly small effect on the total E.M.F. It is only in these circumstances that the instantaneous E.M.F. of a single section as it moves is always reproduced at any one moment by the E.M.F. of a section occupying its successive positions, and the E.M.F. of the armature as a whole becomes constant and equal to the average. Strictly speaking, the identity of the average and instantaneous E.M.F.'s can only be true if the number of sectors were infinitely large, so that the moment chosen for consideration becomes immaterial. But with the comparatively large numbers that occur in practice, the pulsation of the E.M.F. and current when the speed of rotation is constant is reduced to such a small amount that it is only discernible by the oscillograph or telephone. It may in fact give rise to trouble by interfering with telephonic communication.

§ 12. Comparison of the E.M.F. equations of the alternator and continuous-current machine.—The correspondence in the form of

[1] Cp. Dr. S. P. Smith, *Notes on Theory and Design of Continuous Current Machines*, pp. 6-8.

equations (44) and (44*a*) with the similar ones for the heteropolar alternator (equations 38 and 38*a* Chapter IX, § 12) is evident, and the latter may, in fact, be employed as the general expressions to cover both alternators and continuous-current machines. The form-factor k_f for the heteropolar alternator disappears, or more strictly is not expressed since it is unity, owing to the identity of the virtual and average values of an E.M.F. which is constant. k_d has the special value k_{dc}, and $\frac{pN}{60}$ is the frequency of the current in the armature conductors before it is commuted, which is of importance in the continuous-current dynamo from its bearing on the loss by hysteresis in the core and by eddy-currents in the armature as a whole. The term N_{ph}, which was present in the alternating E.M.F. equation, now disappears, since in the continuous-current machine there is no division of the winding into different phases, while q, which in the alternator may be $= 1$, must in the closed-circuit armature be 2 or a multiple of 2.

It will further be evident that while in the alternator the E.M.F. is the *voltage induced between fixed points in the winding, i.e.* in a certain portion or the whole of the armature winding whatever the position which it occupies at any moment with respect to the poles, the E.M.F. of the continuous-current machine is the *voltage induced between points which are fixed with respect to the poles, i.e.* from the portion of the armature winding which at any moment occupies one and the same position in the field system. The former alternates, but the latter is continuous.

When both are present, as in the rotary converter, the ratio of their magnitude is easily followed, when a sinusoidal distribution of field is assumed, and the number of coils is large. When reduced to a two-pole form on a basis of 180° = one pole-pitch, the continuous-current voltage between brushes of opposite polarity is represented by the vertical diameter of a circle (cp. Fig. 90). For any group of consecutive coils covering an angle of χ radians, the E.M.F. is a maximum when the centremost coil is opposite the centre of a pole and its vector parallel to the vertical diameter, so that the projection thereon of their closing vector is a maximum. The coils being closely distributed, if e be the amplitude of the E.M.F. from one elemental coil, by equation (32*a*) the maximum alternating E.M.F. of the group of coils is $eC \times \frac{\chi}{2\pi} \times \frac{\sin \chi/2}{\chi/2} = eC\frac{\sin \chi/2}{\pi}$ and the continuous-current voltage in the same sinusoidal field is $eC \times \frac{1}{2} \times \frac{\sin \pi/2}{\pi/2} = \frac{eC}{\pi}$. The ratio of the two is therefore $\frac{E_{ph}}{E_{cc}} = \sin\frac{\chi}{2}$.

For diametric tappings with any even number of slip-rings $\chi = \pi$, and the maximum alternating voltage between the tappings at the opposite ends of a diameter is of course the same as the continuous-current voltage since it occurs when the tappings cross the brush diameter. The virtual value of the alternating voltage is then $E_{pA} = \frac{\mathbf{E}_{pA}}{\sqrt{2}} = 0{\cdot}707\ E_{cc}$.

If there are N slip-rings, $\chi = \frac{2\pi}{N}$, and between any pair of adjacent tapping points, the ratio of the virtual value of the alternating voltage to the continuous-current voltage is $\frac{\sin \pi/N}{\sqrt{2}}$.

§ 13. The voltage between adjacent commutator sectors and adjacent coil-sides.—If V_b be the voltage between adjacent sets of brushes of opposite polarity, then since there are $C/2p$ commutator sectors between the brushes, it follows that the average difference of potential between any two adjacent sectors is $\frac{V_b}{C/2p} = \frac{2p \cdot V_b}{C}$. But it has been shown that a higher proportion of the volts is generated in the sections of the winding under the poles, and even here it may vary greatly when the flux-curve is distorted under load. If the flux-curve were symmetrical and the rise and fall of potential were strictly sinusoidal, the maximum voltage between commutator sectors would be $\frac{\pi}{2}$ times the average, and therefore $= \frac{\pi \cdot p \cdot V_b}{C}$. Roughly speaking, it may be reckoned that the maximum difference of potential will be about $1{\cdot}5 \times \frac{2p \cdot V_b}{C}$, while the average difference $2p \cdot V_b/C$ in ordinary dynamos ranges from about 2 to 10 volts, and should not exceed, say, 15 volts, or 20 volts at the most.

Figs. 84 and 85, typical of the two forms of drum winding, serve also to show that in either form the difference of potential between adjacent coil-sides is least under the centre of a pole, and thence rises to a maximum between each pair of adjacent coil-sides situated on or near to the diameter of commutation. The difference here amounts to as much as the full E.M.F. of the machine, for it will be seen that the two short-circuited loops, 1–8 and 9–16, which are at opposite potentials, one being under the positive and the other under the negative brush, are contiguous to one another; and similarly, between the short-circuited loops and the neighbouring loops on either side the pressure of nearly the full voltage of the machine exists.

§ 14. Necessity for equal resistance and symmetry in each coil of the closed-circuit armature.—In a closed-circuit winding without

external load if the sum of all the E.M.F.'s induced in the coils right round the circle is zero at every instant, of course no current circulates round it, and it is immaterial whether the component E.M.F.'s summed up for each pole-pitch separately are equal, or if different, how they are distributed, or whether the resistances of each coil are alike. But as soon as an external circuit is applied either at fixed points of the winding by means of slip-rings or at points fixed relatively to the poles by means of brushes and commutator, these questions become of vital importance. Dealing more particularly with the latter or the continuous-current case, the stationary external circuit is brought into contact with the armature winding at a points of + potential and a points of − potential. No adjustment of these points can be made to meet any temporary inequality of the resistance of the parallel paths, such as might be possible in the static case of a number of batteries joined in parallel so as to supply current to an external circuit; for the actual coils of the armature as it rotates are progressively passing from one parallel path to another.

The first requisite, therefore, is complete equality in the resistance of each and every coil in order to secure permanent equality in the resistances of the parallel paths. If at any moment their resistances r_1, r_2 . . . are unequal, and the internal E.M.F.'s induced in each branch are equal, then as soon as the external circuit is closed, since the brush voltage V_b must be the same for all branches

$$V_b = E - i_1 r_1 = E - i_2 r_2, \text{ etc.,}$$

whence $\frac{i_1}{i_2} = \frac{r_2}{r_1}$. The total armature current I_a will therefore be divided unequally among the branches, in inverse proportion to their resistances. Although perhaps not sufficient to cause serious overheating, some waste of power must result, since the sum of $i_1^2 r_1 + i_2^2 r_2$, etc., must necessarily exceed the possible minimum loss $I_a^2 R_a$ for equal division of the current. Each coil must therefore consist of the same number of turns or loops, with the same resistance and wound similarly; they must all be symmetrical relatively to the axis radially and circumferentially.

There is still another property besides their resistance in which the separate sections should be precisely alike, and this is their inductance. During the rapid change of the direction of the current in the short-circuited coils, the inductance which has already been described in connection with alternating currents also comes into play, and, as will be explained in Chapter XX, upon this depends to a great extent the exact position at which the brushes should be set so as to short-circuit the coils at the right moment. Since the brushes can only be adjusted to suit the average coil, it is very

important that the divergence of any section from the average in both resistance and inductance should be as small as possible.

The first requirement is easily obtained in the modern coil- or bar-wound drum armature with winding in two layers, and only slightly less perfectly when there are four or more layers. The second requirement is less strictly fulfilled when, as is often the case, the number of coils, c, grouped in a slot is two or more. But the required symmetry still holds for the groups as a whole, and each group may be treated as a unit, having the same resistance and yielding the same vector of E.M.F. as every other unit under like conditions.

§ 15. Equality of E.M.F.'s in the paths in parallel. Any inequality in the total flux of one pole-pitch, or difference in its distribution, as compared with that of other pole-pitches, can only be due to causes which must for our present purpose be regarded as abnormal. Some of the disadvantageous consequences arising therefrom will be traced in Chapter XII, § 1, and the Note added to that Chapter.

The sole condition that must be fulfilled to ensure that no parasitic current circulates round the closed winding of the armature on no load, and that any load current divides equally between the paths placed in parallel by the brushes is that the sum of the E.M.F.'s induced between two adjacent brush contact points is at any instant the same, whichever way round the closed circuit the E.M.F.'s are summed. The maximum E.M.F. of each little unit group of c coils in a slot can be found vectorially and their joint vector treated as equivalent to the vector of a coil, as considered in §§ 8 and 10. With the substitution of S, the number of slots, for C, the number of coils, the results of those sections hold. Granting then equality of the resistances of the several units and that the same E.M.F. is induced in each for similar positions relatively to poles of the same sign, it now appears that, so far at least as the fundamental E.M.F. is concerned, the above-mentioned condition is *automatically* secured, when, as in practice is the case, the slots are pitched at uniform distances apart round the entire armature periphery, and all are equally filled.

Lastly, when S is substituted for C, it follows that in order to limit the fluctuation of voltage, the number per pole, $S/2p$, must itself not be reduced below some minimum value analogous to that already named for $C/2p$. In practice $S/2p$ should not be allowed to fall below 10 in small machines, or 15 in machines of moderate and large size.

CHAPTER XI

ARMATURE WINDING

THE leading principles governing the armature winding of drum heteropolar machines, whether for alternating or continuous current work, are so closely allied that the subject will first be considered in a general way, applicable to both types.

GENERAL

§ 1. Drum armature winding and its two forms.—The function of an armature winding is to add up the E.M.F.'s of conductors or of loops which are displaced relatively to one another in the magnetic system. Starting with a number of single bars or conductors disposed round an armature core, it has been already shown that in the formation of a single loop, the span of the loop at the far or "back" end of the drum must be as nearly as possible equal to the pole-pitch, in order that the loop may at some instant embrace nearly the whole of the flux of a field, and as a result yield a maximum of E.M.F. Provided, however, that the span of the loop exceeds the polar arc, and it is not, therefore, subjected to direct differential action, it is not necessary in practice that the span should be precisely equal to the pole-pitch.

In Fig. 92 conductor *a* is joined at the back by a connection (shown dotted) to another conductor *a'* situated nearly 180 electrical degrees apart. Two distinct methods now present themselves, by which the loop may be completed and a second loop started.

§ 2. (I) Lap-winding. By the first or *lap-winding* (Fig. 92, i) the end of loop *a* – *a'* is brought back and joined by a connection which also leads backwards to another conductor that lies within the first loop; the direction of traversing the back connection when tracing out the first loop is here regarded as forwards. Thus, when the process is further repeated, the winding keeps on returning on itself, forming a number of overlapping loops (as shown by such a developed plan as that of Fig. 84), whence its name originates. Yet the winding as a whole gradually "creeps" round through the magnetic field, in order to add up the E.M.F.'s of the conductors. In the 2-pole case, for economy in copper and to avoid crossing, the front end-connector will evidently be made to pass round the shaft on the same side as the back end-connector when the span of the loop is less than diametric.

§ 3. (II). Wave-Winding.—By the second method or *wave-winding* of the drum (Fig. 92, ii), the end of the first loop *a* – *a'* is connected in the 2-pole case to another conductor *b* lying outside

the first loop, whence a second loop can be started. The direction of traversing the end-connections from the original starting-point is now continuously forwards, both at the back and front. It must be particularly noticed that in the two-pole wave-wound drum in order to avoid crossings, the back and front end connectors of the wave loop must be on opposite sides of the shaft, and in order to retain the same direction of "creep" of the winding as a whole round the armature core in the two-pole wave as in the lap-wound drum, it is the back end connector that must be made to differ.

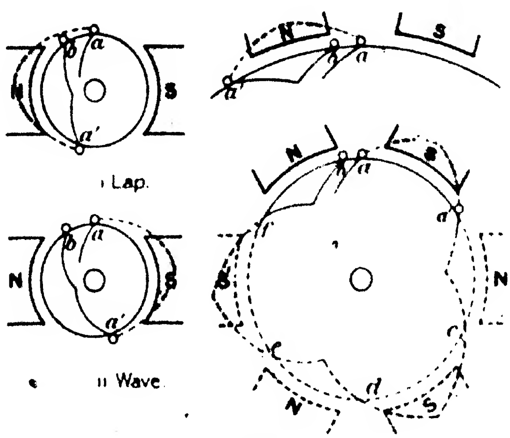


FIG. 92. Bipolar and multipolar lap- and wave-wound loops.

Hence, the direction of tracing through the back end connector being regarded as forwards in both lap- and wave-winding, the direction of the "creep" through the field in the two-pole wave-wound drum is backwards. In other respects, two-pole wave-winding in its result is electrically equivalent to two-pole lap-winding, the same "creep" being obtained in both cases.

But when wave-winding is extended to multipolar fields an entirely new feature is introduced. If the original two-pole machine is cut through to the centre, opened out, and multiplied, it will be seen from the multipolar extension shown dotted in Fig. 92 (ii) that wave-winding enables other conductors, such as *c*, *d*, *e*, *f*, lying in the intervening fields to be picked up in an orderly progression before the winding returns to *b*. When any such complete winding is followed out in the developed plan, such as Fig. 85 or 87, it is seen to work continuously forwards round the armature by large strides in a zigzag "wave," whence its name. At the same time, whenever it has nearly made or has just exceeded an exact tour of the armature, the starting-point for the next

tour will be found to have retreated backwards or advanced forwards through the original field.

§ 4. The "creep" of the armature winding.—Thus by both methods the winding "creeps" through the magnetic field, and the amount of the "creep" or *displacement backwards or forwards through the field per coil traversed* is the fundamental fact upon which all the characteristics of a winding depend, since it is due to it that the E.M.F.'s of coils relatively displaced can be added together. It may be measured as a distance or as an angular advance or retrogression, or it can be expressed in terms of coils or, in continuous-current machines, of commutator sectors. When so expressed, the symbol m is employed for it.[1]

All coils or loops being assumed to be alike one with another, its value may be obtained by considering the displacement of the side which forms the start of each loop or coil, or to avoid considering the sides, it is better defined as the relative displacement of the axes of the coils per coil traversed.

While easy to follow in the case of lap-wound coils, the idea is not at first sight so simple in multipolar wave-winding. It is, however, rendered equally simple if the "creep" per coil traversed is referred to an imaginary double pole-pitch as a common standard. Actually the displacement takes place, not in the same pair of fields, but in successive repeats of the first pair. Thus, in Fig. 92 if the two dotted pairs of poles are imagined to be rotated into coincidence with the original pair of poles (shown in full lines), the "creep" between a and c and again between c and e *becomes measurable in the original double pole-pitch.*

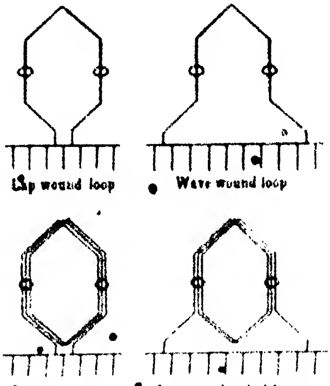
This further serves to bring to light the most important characteristic of the wave-winding. The loop-sides c and e may, and in most cases do, fall *between* a and b, so that if the distance between a and b is a sector-pitch, the "creep" can be a fraction of a sector-pitch, whereas in a lap winding the "creep" may be a multiple, but never a fraction of a sector-pitch.

§ 5. The drum-winding element.—In the finished armature the winding may be regarded as made up of a certain number U of "elements," which are spaced at definite distances apart and are coupled together after a systematic method. Each such winding element may be either a single bar, in which case $U = Z$, or the t conductors of the side of a coil of t turns, whether wound side by side or on the top of one another, so as to form a separate little group of wires, when $U = \frac{Z}{t}$. To define the order in which the elements must finally appear as connected together, winding rules can then be given, expressed in terms of the numbers marking the

[1] Hence, in Chapter X §§ 8 10, the Greek letter μ was chosen as the corresponding symbol for the displacement as an angle.

consecutive spaces into which the elements in the finished armature fall.

Fig. 93 shows a lap and a wave-wound loop connected to commutator sectors as they would be in a continuous-current machine, and also 3-turn coils of the same two types, with the winding elements marked with a ring. It will thence be recognized that the lap and wave connections are to be distinguished from the internal formation of the coil in both alternator and continuous-current machine. The coil may be wound concentrically on itself, or with a regular progression of the turns after lap fashion, and in either case when imagined to be wound with the turns on the top of one



Each separate element marked with a ring

FIG. 93.—Drum winding elements.

another or reduced down to a single loop, it becomes essentially either a lap or a wave coil according to its connection to other similar coils. Thus in Fig. 71 the concentric coils are wave-connected.

By some writers the loop or coil as a whole has been treated as the basis "winding-element" of the drum. Yet for many reasons it is preferable to treat each coil-side as one inducing "element," so that the simple loop of one turn is to be regarded as composed of two elements. And this use of the term is also the most convenient, since many drum armatures are wound with stout bars of copper, each of which is in effect a half-loop or one element; the complete loop which is finally the characteristic feature, is not in evidence at the outset when the process of interconnecting the component parts is begun, and it is just at the commencement of this process that the armature winder requires proper rules for his

guidance. An element of a drum armature will therefore here be taken as either a single bar, if there is one single loop per coil, or the group of conductors forming one coil-side; such a group is treated as if it were a single bar, which amounts to replacing the coil of many turns by the loop of one turn. How many turns there are in the coil or how they are wound need not be considered when only the orderly connection of their ends after a regular law is in question.

§ 6. The component pitches and resultant pitch. The *pitch* of the end-connection of a coil is the number of elements or winding spaces that must be counted off round the armature circumference to find the element which is next in series with the element from which the start is made. Since each of the U elements has a back and a front end, the pitch of the two end-connections must be distinguished respectively as y_b and y_f. Both must necessarily be whole numbers. Starting from some element marked as No. 1, the direction of the back end connection from it is to be taken as giving the positive direction in which the numbering of the elements is to proceed consecutively up to the total of U. The back end of the (x)th element is then joined to the back end of the $(x+y_b)$th element, and the front end of the $(x+y_b)$th element is joined to the front end of the $(x+y_b+y_f)$th element.

The *resultant or total pitch* is the *algebraic* sum of the two component pitches, *i.e.*

$$y = y_b + y_f \qquad (45)$$

It indicates how far in elements or winding spaces the start of the second coil is from the start of the first coil No. 1, and when the elements are distributed in equi-distant groups over the whole of the armature periphery, it measures in terms of elements the actual advance (not in the wave case the "creep" as above defined) through the magnetic system that would be made as each coil is formed. In the lap winding y_f is negative and smaller than y_b; in wave winding, both are positive and may be equal or unequal. Fig. 94 shows the two types of loops as connected to a commutator in the continuous-current dynamo, with the pitches marked on them.[1]

Frequently and especially in continuous-current dynamos with toothed armatures, there are two or more layers of elements in each slot. In such cases for the purpose of calculating the pitch of the winding in elements they must be numbered after a definite order in which they would occur if the armature were made smooth and the coil-sides were spread out evenly over the core, and a distinction must be drawn between the pitch of a coil as reckoned in elements and as reckoned in slots (*cp.* § 12), and also between the

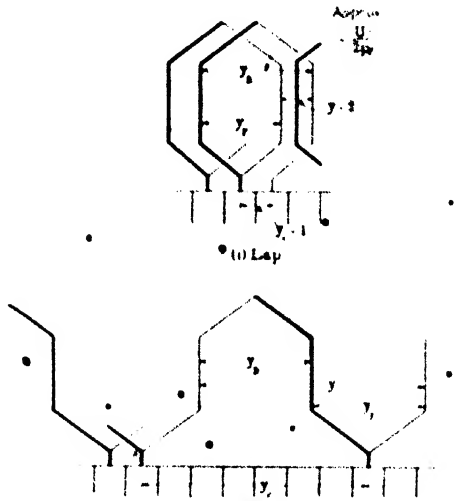
[1] In the lap case, an advance of one commutator sector per coil or $y_c = 1$ is assumed.

"creep" when uniform distribution of the elements is assumed as above, and the true "creep" by long and short steps in a toothed armature.

Lastly, for the reason given in § 1, the back pitch can at once be stated to be in general—

$$y_b \simeq \frac{U}{2p} \text{ or } \frac{U \pm b}{2p} \quad . \quad . \quad . \quad . \quad (46)$$

where b may be zero, or is some small quantity necessary either to make y_b a whole number when $U/2p$ in a multipolar dynamo is not



(i) Lap

(ii) Wave

Fig. 94.—Developments of lap and wave loops.

a whole number or for other purposes to be mentioned in § 14. It is therefore in the front pitch that the difference between the lap and wave-connected coils is made.[1]

Open-Ended Windings

§ 7. Single-phase alternator armature windings in one layer.—Confined in their use to alternators, open-ended windings do not from their nature carry so many conditions or require such exact symmetry of all coils as closed-circuit windings. Fuller details, constructional and otherwise, both for single-phase and polyphase machines, are reserved until Chapter XXIV. It will only here be noted that the alternator being usually designed for high voltages,

[1] It may here once and for all be added that nothing is so instructive as, or can take the place of, the actual drawing of diagrams (including such as those of Figs. 90, 91, and 96) for the purpose of obtaining a clear insight into armature winding problems.

and the wave connection in virtue of its adding up the inductive effect of multipolar fields being best adapted to give such voltages, it forms the basis of many alternator windings in common use. Reduced to its most elementary form of a single bar per pole in the single-phase case, the simple undulating zig-zag of Fig. 95

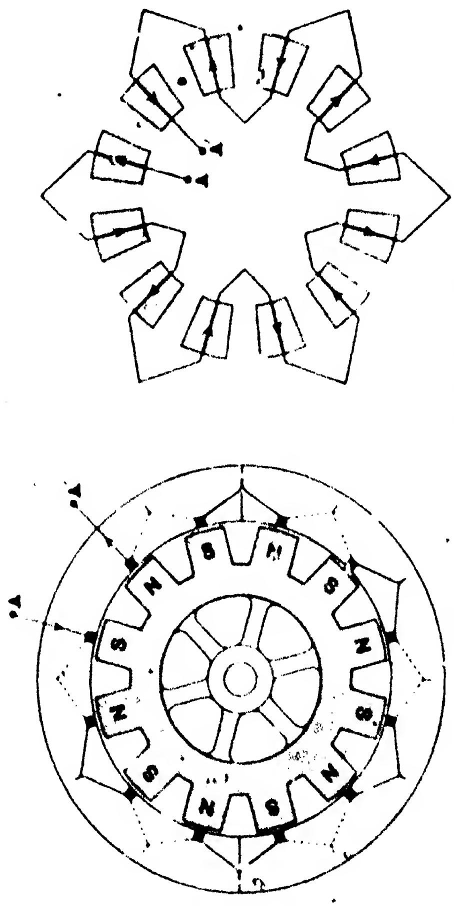


FIG. 95.—Single-phase bar wave-winding.

alone remains. In the left-hand front view the dotted end-connections here and in similar diagrams are at the back of the stator. The front end-connections of the coils which appear in the left-hand portion of each diagram are shown towards the inner periphery of the right-hand portion. The active coil-sides or conductors appear as radial elements, and the back end-connections are on

the outer periphery, so that in the right-hand diagram the drum is, as it were, imagined to be pushed inwards and flattened out. The number of loops being equal to the number of pole-pairs, the winding would close, if continued with the same front-pitch from A' to A, and the "creep" is here zero. But the circuit being open at the terminals A and A', and all conductors having been traversed, there is no need for any "creep" to secure another tour of the armature.

But even when step-up transformers are employed, a single bar per pole per phase would seldom give sufficient voltage, so that as a next step, other tours of the armature might be made as in Fig. 96. Except at B, $y_b = y_f = 4$, but now it will be noticed

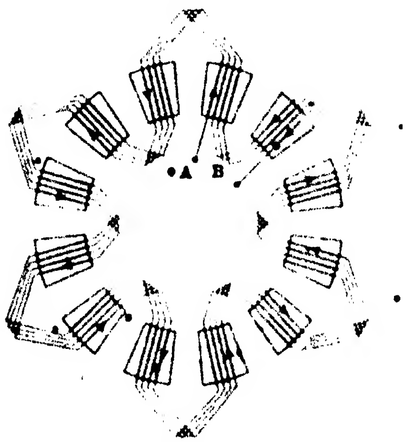


Fig. 96. Wave-wound single-phase armature.

that after *each* tour the pitch of the front end-connector at B is shortened to $y_f = 3$, so that another tour can be started; a "creep" results, and the positions of the bars of each tour referred to the field of a standard double pole-pitch gradually retreat backwards. The practical disadvantage of the arrangement is that everywhere elements of widely different potential occur side by side, and their insulation is proportionately difficult. This disadvantage is, however, removed without effect on the E.M.F. of the phase, if the same conductors are converted into 4-turn lap-wound coils, wave-connected (Fig. 97). The component pitches of the lap-coils are then $y_b = 4$, $y_f = -3$; the "creep" is therefore by one winding space per coil traversed in a lap group, but there is no further "creep" from the addition of other lap groups.

By a further re-arrangement, the component coils without alteration to the E.M.F. could be wound concentrically, while still

retaining the wave-connection, as shown in Fig. 71; the only difference thereby introduced is that the terminals of the winding fall on an inside and an outside end instead of on the outsides of two adjacent coils. The possibility of the concentric arrangement is limited to the

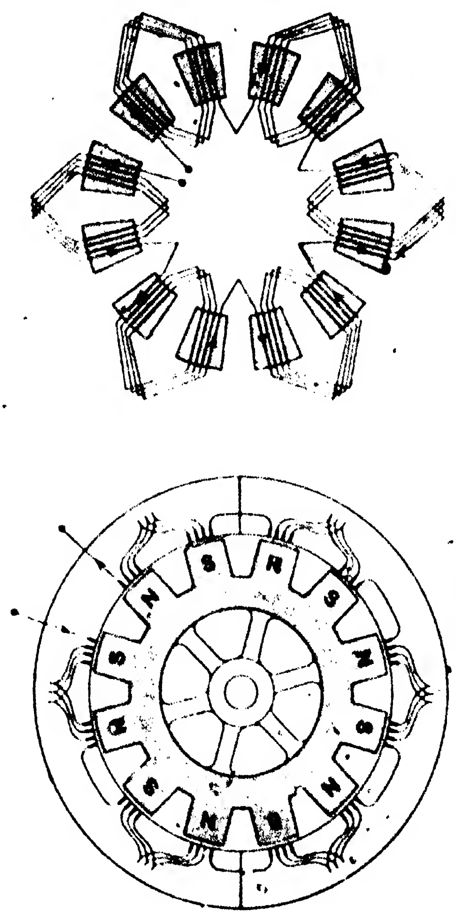


Fig. 97.—4-turn 12 coils, wave-connected.

case of a winding in a single layer, such as has alone been considered above. But the type of end-connector shown in Figs. 96 and 97 whether for lap or wave, is such that the coils could pass one another in order to be arranged in two layers, and the subject of two-layer windings will again be resumed in Chapter XXIV.

Closed-circuit Windings

§ 8. The mechanical conditions for closure.—When we pass to closed-circuit windings, a number of new conditions, which closely define what is possible mechanically and electrically, are introduced. The coils are assumed, as in practice is the case, to be exactly alike, of the same span, and to be symmetrically distributed at equal distances round the entire armature periphery, or to be grouped in similar small groups, *e.g.* within one slot, with equal distances between the several groups. The open-ended windings of, say, Fig. 95 or Fig. 96 might end anywhere with perhaps a half loop. But now since a winding when traced completely through is to close, all the coils, C in number, must be complete, with two coil-sides to each, so that $C' = 2C$ is a multiple of 2, and therefore an even number.

When the coil sides, if in two or more layers, are imagined to be spread out into a single layer as described in § 6, and numbered consecutively round the armature in the direction of the back end-connection of the coil-side which forms the starting-point, every uneven number corresponds to the starting side of a coil, and every even number to an opposite or finishing side. Since a back end-connection joins an element of uneven number to an element of even number, and *vice versa* with a front end-connection, each of the two pitches, y_b and y_f, must be an *uneven number*. The resultant or total pitch, $y = y_b + y_f$, is therefore 2 or a multiple of 2, but of greater importance, the *average pitch*

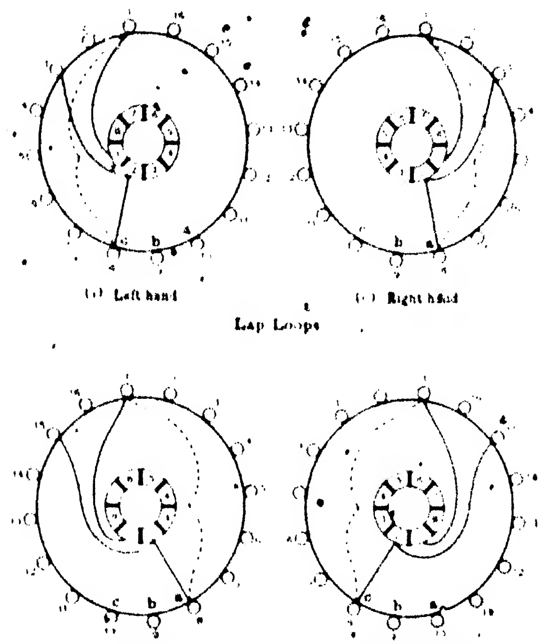
$$y_c = \frac{y}{2} = \frac{y_b + y_f}{2} \quad . \quad . \quad . \quad . \quad (47)$$

must necessarily be a whole number.

The number of joints uniting the finish of one coil and the start of another coil is C, and at each joint a connection is made in the continuous-current machine to a commutator sector. There are therefore as many commutator sectors as there are coils, so that C stands indifferently for the total number of either. The average pitch is then also the *commutator pitch*, or the distance measured in commutator sectors between the start of one coil and the start of the next coil in succession. In closed-circuit windings, therefore, although these are not confined to continuous-current machines, y_c measures, in terms of coils or commutator sectors, the advance made round the commutator per coil traversed or the actual advance through the magnetic field if the coils were spread out evenly into a single layer.

Fig. 98 reproduces Fig. 92 for the more definite case of a closed-circuit winding with commutator attached. Starting a coil from any sector or element marked 1, its second side must be formed by an element of even number separated nearly by a pole-pitch,

e.g. *a* or *c*, reached by an end-connection at the back of pitch $y_b = 7$ or $\frac{16-2}{2}$ by formula (46). Element *b* must not be taken, since being an uneven number it will itself be subsequently wanted to form the starting-point of a coil; whether *a* or *c* be chosen will be found merely to alter the hand of the winding, but in no other way affects its characteristic features. In the present two-pole



(i) Left hand (ii) Right hand

Lap Loops

(iii) Left hand (iv) Right hand

Wave Loops

FIG. 98.—Lap and wave loops on two-pole armature.

case with an even number of commutator sectors and the winding disposed in one layer, the loop spans a chord slightly less than the diameter, *e.g.* 1 – 8, or is in effect wound on one side of the core. Thence from the front end of 8 a second end-connection leads on to a commutator sector, and so to the beginning of the second loop or coil, by the lap or wave method. With an uneven number of coils or sectors per pole-pair, the component loop (cp. Fig. 104) can be diametric, with a back pitch $y_b = U/2p$ as giving a whole uneven number.

§ 9. The number of independent windings.—When all the elements of a winding have been traversed once, that winding

re-enters on itself or closes. It is not, however, necessarily the case that all the elements on the armature as a whole must have been traversed before any one winding that is being traced closes. Let $C = Ah$ and $y_c = Bh$, where h is the highest common factor between the number of coils and the average or commutator pitch, and may be 1 or more. Since in traversing one coil from its start to the start of the next, y_c coils or sectors are passed by, in joining up A coils $A\,y_c$ coils or sectors have been passed by, round the periphery of the armature or commutator. But $Ay_c = \frac{C}{h}y_c = C \times B$, and B is by assumption a whole number. Hence when A coils have been joined up, the commutator has been passed round a whole number of times B, and also for the first time the circuit has closed, the starting-point having again been reached.

If, therefore, there is no common factor between y_c and C greater than unity, *i.e.* if $h = 1$, the commutator pitch $y_c = B$, the number of complete tours round the armature made before the winding closes. That in tracing out a winding we may have to pass round the armature or commutator several times is at once evident in the case of the wave winding (Fig. 85 or 87). In so doing, the number of coils joined up is $A = C$, since $h = 1$, *i.e.* it is the whole of the coils, and there is only a single helix closed on itself.

But if $h = 2$ or more, the number of tours round the armature or commutator before a winding closes is $B = \frac{y_c}{h}$; only $\frac{C}{h}$ coils have been joined up, and the remainder form one or more entirely independent helices each closed on itself.

The H.C.F. of y_c and C thus in all cases gives *the number of independent closed helices* on the armature as a whole. In the simple cases illustrated in Figs. 84–87, y_c and C had no common factor greater than 1, so that only a single helix resulted, and this is the most usual case. But the possibility of a second independent helix or even of two or more such additional helices, interleaved between the first, must be recognized when we pass to general winding formulae.

§ 10. Electrical conditions. The preceding §§ 8–9 have dealt with purely mechanical conditions that arise in the orderly coupling up of a number of similar coils into a closed-circuit winding or windings on an armature, and nothing has been said on the electrical side.

In Chapter X, § 15, it has been shown that to avoid any circulating current of fundamental frequency, the only necessary condition is automatically obtained when the coils are disposed round the entire armature with a uniform displacement between each or between each small group of c coils in a slot. Even if the differential factor for the coil-span permits an alternating current of higher frequency

Every time a nearly complete or over-completed tour of the armature is made and we return to a sector near to our starting-point, p coils have been traversed and $p-1$ sectors have been touched at in addition to No. 1. In the multipolar machine, the $p-1$ sectors are then to be thought of as transferred with their coils for insertion where they rightly belong, so far as their displacement in a single standard field is concerned. Attention need only be directed to one side of each of the similar coils, and the diagrams take the forms shown in Fig. 99. The tours of the armature by long strides are thus dispensed with, and the case becomes analogous to that of the two-pole lap-wound machine.[1] If the creep m is 1 or a whole number of sectors (*i.e.* C/p is a whole number), the transferred sectors exactly coincide with those of the original double pole-pitch; thus Fig. 99 (i) gives the re-arrangement of Fig. 87 with p pairs of parallel paths, and the same diagram would equally well represent the lap-wound case of Fig. 86, which shows the equivalence of the two. If m is a proper fraction and $mp = 1$, as in Figs. 105 and 106, the transferred sectors and coils are simply interleaved between those of the original double pole-pitch, as shown in Fig. 99 (ii) and (iii). If m = an improper fraction, say, 1½ in a 4-pole machine, the transferred sectors are again interleaved with the original sectors, but the first interleaved sector from No. 1 does not fall between Nos. 1 and 2, as in Fig. 99 (iii), but between Nos. 2 and 3, as in Fig. 99 (iv); the data for the latter figure are $C = 17$, $y_c = 6$, $m = 1\frac{1}{2}$, so that the winding is triplex, mp being 3.

§ 12. Slot-pitch and element-pitch. Practically all modern continuous-current machines have toothed armatures with the winding disposed in two layers (or in small machines in four layers). On this account even when the elements have been shown in a single layer on a smooth-surface armature, as in Figs. 81–87, they have been grouped in pairs, as they might be in a toothed armature with two coil sides per slot. In such cases it has been stated that the elements must be numbered regularly as they would fall if spread out uniformly into one layer. Thus Fig. 100 shows two arrangements by which the even numbers would fall between the uneven numbers in regular order.

The distinction must then be noted between the pitch in elements and the pitch in slots, as in Fig. 100, where the slot-pitch of the coil is in both cases 5, while the element-pitch in the two cases is 31 and 21. When the winding table in elements for any class of winding

[1] A chief advantage of the method is the assistance it renders in the drawing of rotating polygons such as Figs. 90 and 91 by indicating the true displacement of the vectors of the coils in their proper sequence. A smooth armature is here tacitly assumed, and if it be toothed the vectors of each slot group must be taken together, the true displacement or "creep" through the field not being reproduced in the uniform spacing of the sectors.

has been drawn up, the slots numbered consecutively in the direction of the back pitch from No. 1 which contains No. 1 element at one side, and the order of numbering the elements in a slot settled, it is easy to pass from the winding-table in elements to that in slots. Each element number in the former table must be divided by the number of elements in a slot; if there is no remainder, the quotient gives the number of the slot, and the element is the last one which completes the slot in question; if there is a remainder, the element falls into the next slot to the whole number, and in the place indicated by this remainder.

Before proceeding to consider lap and wave winding in detail, it may here at once be stated that in both types with a winding in two layers, if complete similarity of all coils is to be retained,

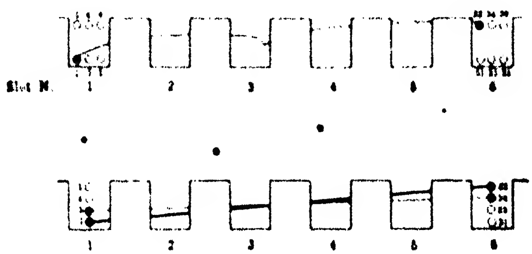


Fig. 100.—Slot pitch and element pitch.

it is a necessity that the back-pitch reckoned in elements must be equal to (a whole number × number of coil-sides in a slot) *plus* one; i.e. $y_b = y^1_s u + 1$, where u is the number of coil-sides in a slot and y^1_s is the back pitch in slots, and this condition of similarity of all coils so universally obtains that it may be regarded as the normal case. The back slot-pitch as so discovered, namely,

$$y^1_s = \frac{y_b - 1}{u} \quad . \quad . \quad . \quad . \quad (49)$$

must then be checked by comparison with the polar arc and the pole-pitch, to see that it avoids differential action; it must exceed the width of the pole-face, and in fact should not approach it closely, and on the other hand it should not greatly exceed the pole pitch $= S/2p$, where S = the total number of slots.

Further, with a winding in one layer it appeared from Fig. 98 that the back pitch of the coil reckoned in elements cannot be quite diametric with an even number of coils or sectors per pole pair, but can be so with an uneven number of coils. But now when reckoned in slots, the reverse is the case in the toothed armature with a two-layer winding: when the number of slots per pole-pair

is even, the back pitch in slots can be diametric or exactly equal to a pole-pitch, while with an uneven number of slots per pole-pair, the back pitch must fall short of the pole-pitch by at least one-half of a slot-pitch—a result which has the important consequence noted in Chapter X, § 10.

(1) Lap-winding

§ 13. The connexion between y_c and m.—In the lap-wound armature, the commutator pitch, y_c, is immediately the "creep" of the winding through the magnetic field per coil traversed, *i.e.*

$$y_c = \frac{y_b - y_f}{2} = m = \frac{a}{p} \quad . \quad . \quad . \quad (50)$$

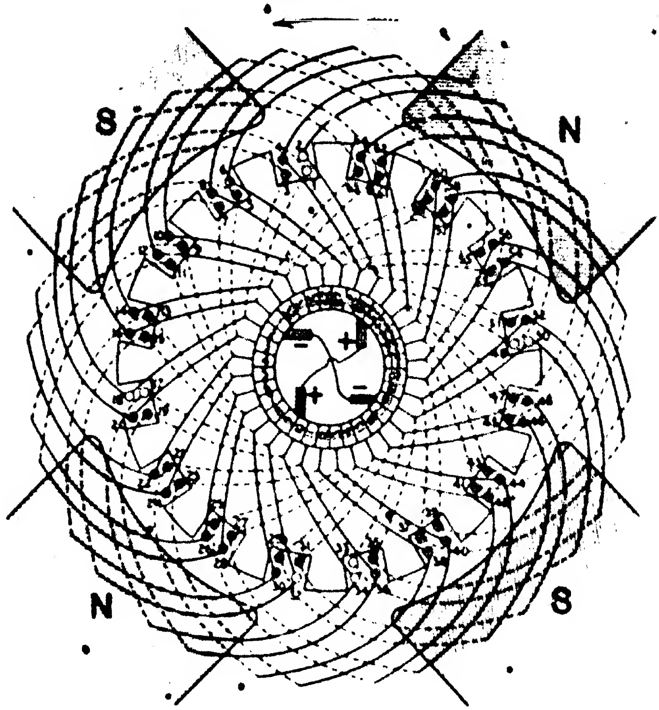
Since y_c must be a whole number, it follows that m must be a whole number, *i.e.* must be 1, 2, 3 . . . . sectors. Therefore, $a = mp$ can only be p, $2p$, $3p$ . . . . *i.e.* the same as the number of pole-pairs, or some multiple of that number. Usually, $a = p$, and the number of parallel paths in the armature, $q = 2a = 2p$, the number of poles.

If y_f is made greater than y_b, m is negative, and the "creep" through the field is retrogressive or against the direction of numbering the elements. But since this involves an unnecessary crossing of the front end-connexions, y_f is always made less than y_b, and therefore in the lap-wound machine, m is positive, and the "creep" is *progressive*, *i.e.* in the same direction as the numbering of the elements.

§ 14. The component pitches for lap-winding. The general formulae for the two component pitches are

$$y_b = \frac{U \mp b}{2p} \text{ and } y_f = \left(\frac{U \mp b}{2p} - 2m\right) \quad . \quad . \quad (51)$$

The function of b is in the first place to make $\frac{U \mp b}{2p}$ a whole uneven number. With an uneven number of sectors per pair of poles, or $C/p = U/2p$ immediately a whole uneven number, b can be zero, and in the 2-pole case the back end-connection then joins diametrically opposite elements. The back end-connector must, however, pass round the shaft on one side or the other, and whichever side it takes, the front end-connector is also disposed on the same side. But in a multipolar dynamo in general $U/2p$ is not necessarily a whole uneven number, and the special function of b is then called out to convert either a fractional value of C/p, or a whole even number C/p, into a whole uneven number. The latter is the more usual case, since, as will be explained in the next chapter, C/p is usually a whole number in order to render equalizing connections possible.



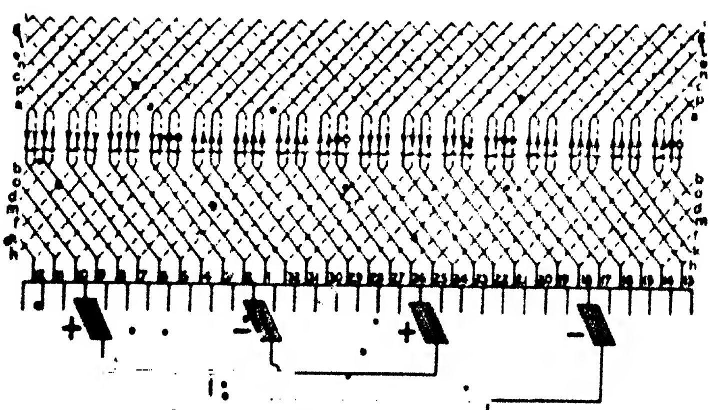


FIG. 101.—Lap-wound toothed drum.
$U = 64$ $y_b = 17$ $y_f = -15$ $y'_a = 4$

In Fig. 84, $y_a = \frac{N - b}{2} = \frac{16 - 2}{2} = 7$, and y_f numerically $= 7 - 2$, so that $y_f = m - 1$. The completed winding table is thus—

Back end.

1 8 3 10 5 12 7 14 9 16 11 2 13 4 15 6 1

Front end.

In Fig. 86 the same pitches are retained, and it is seen how in a 4-pole field, the coils now span approximately a quarter of the circumference. The greater the number of poles, the flatter become the coils, since the arc spanned by the back connections varies inversely to p. It is never advisable with a smooth surface armature to use plus b as unnecessarily increasing the length of copper in both end connectors. But if the multipolar machine have a slotted armature, it may occasionally become necessary to add a positive b, as in Fig. 161, where $y_a = \frac{U + 4}{4} = 17$, so that the real pitch in slots, y_a^1, may be a whole number $= 4$; $y_a = 13$, giving $y_a^1 = 3$, would bring the sides of a coil within the polar arc. Yet as this is not so frequently met with, the plus sign is given in the formulae as the second alternative.

The quantity b has, however, another function, since it enables us at will to shorten still further the pitches below the maxima values given above. The lap wound loop of the smooth 2-pole armature with C even is in effect chord wound, even when the back pitch is as long as is advisable. So long, however, as y_a exceeds the polar arc, it may be made shorter than the pole pitch so as to approximate more nearly to the polar arc, especially if this latter be small and the number of elements large. The front pitch is then also proportionately reduced, so that both end connections are shortened and the winding has less copper and less resistance. But the chief effect is that the two coils which are simultaneously short-circuited at the two brushes no longer lie side by side, but are separated by a small zone of elements carrying the full current in opposite directions. In Fig. 102 with 32 elements the back pitch is 13 and the front 11, and between a pair of short-circuited elements there are 2 elements whose joint magnetic effect is practically nil. If the pitches had been made 11 and 9, the zone would have included 4 elements with currents opposed. It will subsequently be seen that what is called the demagnetizing effect of the armature back ampere turns is thereby reduced, although this advantage is obtained at the expense of the short-circuited coils being rather far from the neutral line of symmetry between the poles. Suffice it here to say that when by an increase of b the pitches are shortened so as to produce

this effect, the method is known as *chord-winding* proper, and this term is extended to cover the analogous arrangements in multipolar lap- or wave-wound armatures. Since in toothed armatures it may

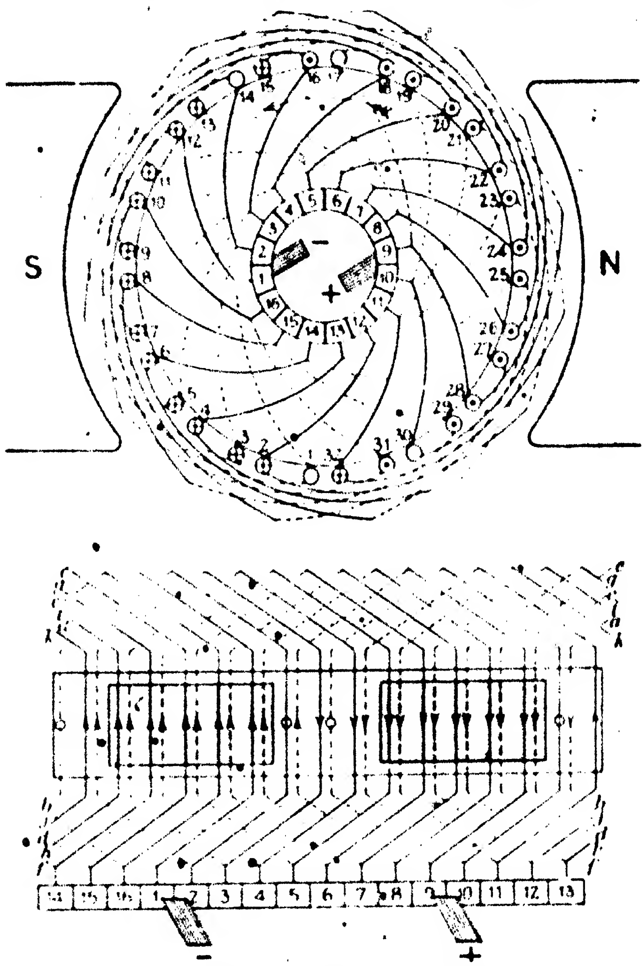


Fig. 102.—Chord winding of lap-wound drum.
$U = 32$ $y_b = 13$ $y_f = -11$.

be advisable to spread the bars which are simultaneously short-circuited over several slots, a moderate amount of chord-winding has more in its favour than in the corresponding smooth-surface armature.

§ 15. Simplex and multiplex lap-wound armatures.—According to the value of m, two main subdivisions are obtained.

(*a*) $m = 1$, so that $a = p$.

The *simple lap-wound armature* which thence results, and which has been described in Chapter X, is by far the most common. It always gives as many parallel paths for the total armature current as there are poles, or $q = 2p$. Each commutator sector is passed through in succession in the positive direction, since $y_c = 1$. U may be any even number, with the provisos that if equalizing connections are to be added it must be divisible by $2p$ without remainder, and that in slotted armatures with two or more layers C must be equal to or a multiple of the number of slots, and, therefore, $U = 2C$ must be an even multiple of the number of slots.

(*b*) $m =$ a whole number greater than 1, so that $a = 2p, 3p$, etc. If at a given speed an output of low voltage but of a large number of amperes is to be obtained, the simplex lap-wound drum may yield too small a number of elements and sectors for satisfactory working. *If it is not desired to increase p, such cases may be met by recourse to the multiplex lap-wound drum*, in which m is some whole number greater than 1. Since the number of parallel branches is $q = 2mp$, the number of armature elements and sectors is m times greater, and the E.M.F. is only $\frac{1}{m}$th of what it would be if the same number of active conductors were connected up as a simplex lap winding.

The difference between the two pitches is now greater than in the simplex lap winding, and the resultant pitch, instead of corresponding to two winding-spaces, is increased m times, *i.e.* $y = y_1 + y_2 = 2m$. Correspondingly $y_c = m$ and instead of the commutator sectors being traversed successively, the winding as it is continuously traced forwards touches, *e.g.* if $m = 2$ at every alternate sector.

Two variations are now found to be possible, both of which comply with the above requirements. If y_c and C have no common factor higher than 1, then as shown in § 9 the winding forms one single closed coil, and when it is traversed throughout it is found to make m passages round the armature, and then to close on itself; yet when the $2p$ sets of brushes are placed on the commutator, it is divided into $2mp$ parallel paths. On the other hand, if the highest common factor of y_c and C is two or more, there result two or more independent windings, each separately re-entrant on itself. In Fig. 103 $y_c = 2$ and $C = \frac{38}{2} = 19$, so that the winding is a single closed coil. In Fig. 104, $y_c = 2$, but $C = \frac{36}{2} = 18$, so that their highest common factor is 2, and two entirely independent windings are obtained. Both the above are instances of duplex

winding, and in either case[1] each set of brushes must have sufficient arc of contact to cover more than one commutator sector with its insulation on either side, so that each coil may be successively short-circuited for a sufficient length of time to enable the current in it to be commuted before it is thrown into connection

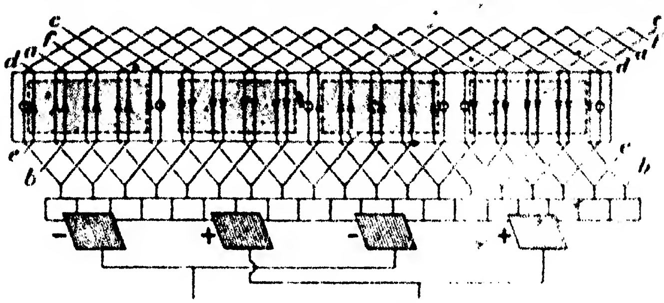


FIG. 103.—Duplex lap-wound 4-pole drum with single helix.
$C = 19$, $y_B = 9$, $y_F = 5$, $y = 2$.

with a new parallel path. With a triplex winding the width of the brushes must exceed the width of two commutator sectors, and so on, so that preferably the width of the brushes is made equal to *m* sectors. As it becomes difficult to maintain perfect uniformity of contact of the brushes over a very wide arc, each set of brushes

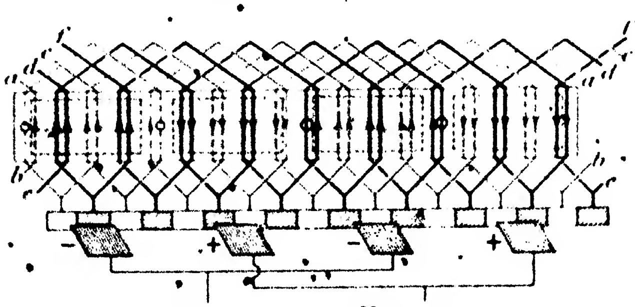


FIG. 104.—Duplex lap-wound 4-pole drum with two independent windings.
$C = 18$, $y_B = 9$, $y_F = 5$, $y = 2$.

may in such cases be staggered in order to obtain the necessary effective width. The sectors are interleaved just as the coils are, but though in either case each coil is changed from one parallel path to another every time that it passes a brush, the

[1] Unless there are two commutators, one at each end of the machine, with brush leads coupled together.

peculiarity of the first form of multiplex winding is that, if any one coil be traced during a revolution, it will be found to pass successively through *each one* of the $2mp$ parallel paths,[1] and this in practice renders it slightly preferable to independent windings. In the second form it is apparent from Fig. 104 that the dotted winding is simply a second winding added to the same armature core, and interpolated between the coils of the first winding; the alternate commutator sectors thus belong solely to the second winding and are accordingly shown shaded.[2]

The induced E.M.F. of the lap-wound armature is thus in general

$$E_a = \frac{1}{m} \times \Phi_a \times Z \times \frac{N}{60} \times 10^{-8} \quad . \quad . \quad (52)$$

with the special form in the ordinary simplex case when $a = p$,

$$E_a = \Phi_a Z \frac{N}{60} 10^{-8} \text{ volts} \quad . \quad . \quad . \quad (52a)$$

(2). Wave-winding

§ 16. The connection between y_c and m. In the wave-connected armature with C coils or sectors and p pole-pairs

$$y_c - \frac{C}{p} = \pm m = \pm \frac{a}{p} \quad . \quad . \quad . \quad (53)$$

y_c is reckoned in the forward direction of numbering, and is always positive. According as y_c exceeds or is less than $\frac{C}{p}$, the "creep" m may be either progressive (positive) or retrogressive (negative), as shown by the double sign attached to m.

If C is exactly divisible by p, y_c must be made to differ from C/p by at least one sector in order that there may be some creep. To make $y_c = C/p$ when this is a whole number means physically that the winding would close after only one tour of the armature and after touching at p sectors only instead of passing through C sectors, the result would then be exactly analogous to making m in the lap-wound armature zero, so that each coil closed on itself.

[1] *Vide* Mather and Platt's Brit. Pat. 768 (1884), and Nebel's Brit. Pat. Nos. 2448 (1890) and 20840 (1891).

[2] By many authors the terms "duplex," "triplex," or "multiplex" are confined to the cases when there are two, three, or more entirely separate windings, and the terms "doubly re-entrant," "trebly re-entrant," etc., are used to express the cases in which the armature is traversed m times by the same winding before closing upon itself. The latter terms appear, however, to be inappropriate and misleading, since no winding can in the strict sense of the term ever re-enter upon itself more than once, and it is just in the case of independent windings that on any given armature there may be more than one point of re-entrancy.

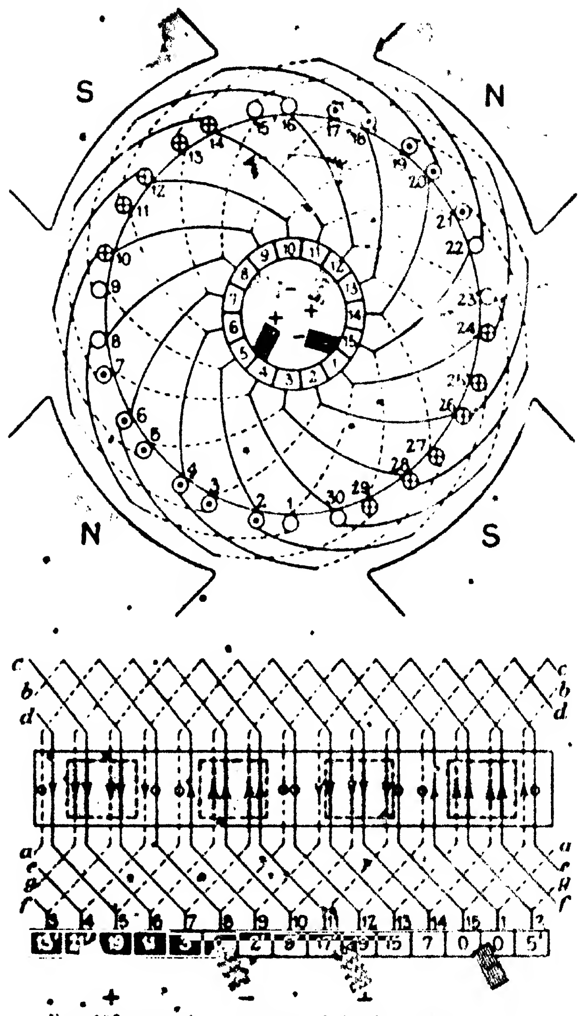


FIG. 105.—Simplex wave wound 4 pole drum (a = 1).
$U = 30$ $y_a = 7$ $y_r = 7$ $y'_c = 7$

When C/p, the number of sectors per pole-pair, is a whole number, m must be a whole number as in the lap-wound armature. But in the wave-wound armature, C/p may be and usually is an improper fraction; in this case, m must itself be fractional or an improper fraction, and in this possibility lies the greater generality of the wave-connected armature. Since $a = mp$, by a suitable choice of coils and commutator pitch to make m fractional, its product with p, i.e. a, can (arithmetically) be made any desired whole number, 1, 2, 3, etc., independently of the value of p.

§ 17. Simplex or two-path wave-winding ($a = 1$). It has been shown (Chapter XI, § 2) that in its application to a 2-pole machine wave-winding is electrically equivalent to lap winding, and in the 4-pole machine derived therefrom (Fig. 87) the resemblance is still maintained, so that $a = p$, as in the 4-pole lap machine. But there still remains in the 4-pole case the second possibility that a may be 1, giving

$U = 2p \cdot y_e \pm 2$, where y_e is any whole number

= for example, $4 \times 7 \pm 2 = 26$ or 30 (Fig. 105) instead of 32 as in Fig. 87.

In this form with $a = 1$ the special characteristics of wave-winding are brought out more clearly. Whatever the number of poles, after having traversed $2p$ elements and so made very nearly one exact tour round the armature, it always returns to the second element in front of or behind the starting-point, just as was also the case in the bipolar wave-wound form. Thus in Fig. 105, starting from, say, element 5 under a N. pole, the back end connection joins it to an element 12 nearly but not quite 90° ahead and under a S. pole; thence the circuit passes in succession to a third and fourth element (19 and 26) nearly 180° and 270° ahead of the starting-point, and next to a fifth element 3, which is the second behind the original starting-point. Hence after traversing as many elements as there are poles, the winding on reaching the next element has always lost or gained two winding-spaces, and by so much the tour of the armature is either incomplete or has been exceeded. Similar tours are made, until the winding finally closes on itself, and the number of such complete tours is always equal to the average of the back and front pitches. The total pitch, being the sum of the two component pitches, or $y = y_b + y_f$, is approximately equal to a double pole-pitch, but the essential relation is that the average pitch $y_a = \dfrac{y_b + y_f}{2}$ must be $= \dfrac{U \pm 2}{2p}$. In Fig. 105 with four poles and a total number of 30 elements, $y_a = \dfrac{30 - 2}{4} = 7$, and the back and front pitches are both $= 7$.

The winding table is therefore as follows—

Back	1—8	15—22	29		
Front	8	15	22	29	
				1st tour completed	
Back	6	13—20	27		
Front	6	13	20	27	
				2nd tour completed	
Back	4	11	18	25	
Front	4	11	18	25	
	3rd tour completed				
Back	2	9	16	23	30
Front	2	9	16	23	30
				4th tour completed	
Back	7—14	21—28			
Front	7	14	21	28	
			5th tour completed		
Back	5	12	19	26	
Front	5	12	19	26	
	6th tour completed				
Back	3	10	17	24	
Front	3	10	17	24	1
				7th tour completed	

The average pitch is not only the number of complete tours of the armature as shown by the winding table, but is also the pitch in commutator sectors, or the number of sectors which must be counted off to find the order in which the winding dips into the sectors.[1]

§ 18. The number of brush sets.—In the multipolar wave-wound armature only two sets of brushes are necessary, and if this minimum number of sets is adopted, there is with four poles a choice of position at which they may be placed, namely, the two pairs of right angles. With six or any greater number of poles, there is a choice as to the angular distance at which they may be set as fixed by the angles between any pair of poles of opposite sign. Thus in the 6-pole machine they may be set at 60° or 180°, and in the 8-pole machine at 45° or 135°. There may, however, be as many sets of brushes as there are poles, and two dotted sets are shown in Fig. 105; that this is permissible will be evident from an examination of the distribution of potential round the commutator of, *e.g.* Fig. 105, which shows that there are two additional points of low and high potential respectively where a − or a + brush could be applied and connected to the existing brushes of the same sign (the divergence of the potentials in the diagram is due to the approximate assumptions as to the E.M.F. of two volts induced in every element without regard to its position in the field, and partly also to the

[1] This is equally true, although not so apparent in the 2-pole form of Fig. 85.

limited number of elements shown). In large machines and wherever the diameter of the commutator is sufficient to give room for several sets of brushes without unduly crowding them or bringing the sets of opposite sign into too close proximity, it is always preferable to employ as many sets of brushes as there are poles. The armature winding is not thereby divided into more parallels, and the current in each conductor still remains half the total armature current. The only change introduced is that the coils are now short-circuited, not only at one set of brushes, but also by the leads connecting brushes of like polarity. If any set of brushes be specially inaccessible, as on traction motors, it may be omitted, and a total number of sets intermediate between 2 and $2p$ be employed.

§ 19. Freedom from effects of magnetic dyssymmetry.—Multipolar wave-wound machines possess one signal advantage, which is not shared by lap-wound machines. Each of the paths through the armature winding consists of elements influenced by all the poles. Hence if for any reason, such as eccentricity of the armature in the bore, or unequal permeance of the different magnetic circuits, the inductive action of one pole and its adjacent neighbour is not equal to the corresponding action of another pole, this will have no effect upon the equality of the volts produced in the different paths of the armature winding. The simplex wave wound multipolar is, therefore, like a bi-polar drum in that there is no fear of inequality of E.M.F.'s in its two armature branches.

§ 20. Selective commutation in wave-wound armatures with 2p sets of brushes.—But if the wave-wound drum is used with as many sets of brushes as there are poles, the advantage which it possesses by reason of the equality of the E.M.F.'s in its parallel branches is to some extent discounted by a drawback which is peculiar to the winding. As soon as the attempt is made to take full advantage of its possibilities by applying $2p$ sets of brushes with a corresponding reduction in the length of the commutator, the difficulty arises that equal division of the current between the sets of brushes which are of the same sign is entirely dependent upon the contact-resistance of the sets being precisely equal. In the wave-wound drum there is no automatic check due to armature reaction to prevent unequal division of the current between the various sets of brushes. Slight differences in their contact-resistances must inevitably occur, and, further, are more or less variable; the division of the current between the sets of one sign will then be unequal and will fluctuate, so that at one moment perhaps one set may be carrying by far the larger part of the current. This effect has been called *selective commutation*, and in consequence there may result trouble from sparking and from overheating of the brush tips. Hence in practice it is not advisable to rely too implicitly upon the current being exactly equally divided, and the commutator

should be given fairly ample proportions of length in order to meet such cases of selective commutation.

§ 21. The component pitches for wave-winding. The division of the total pitch $y = y_a + y_r = \frac{U \pm 2a}{p}$ into its two components

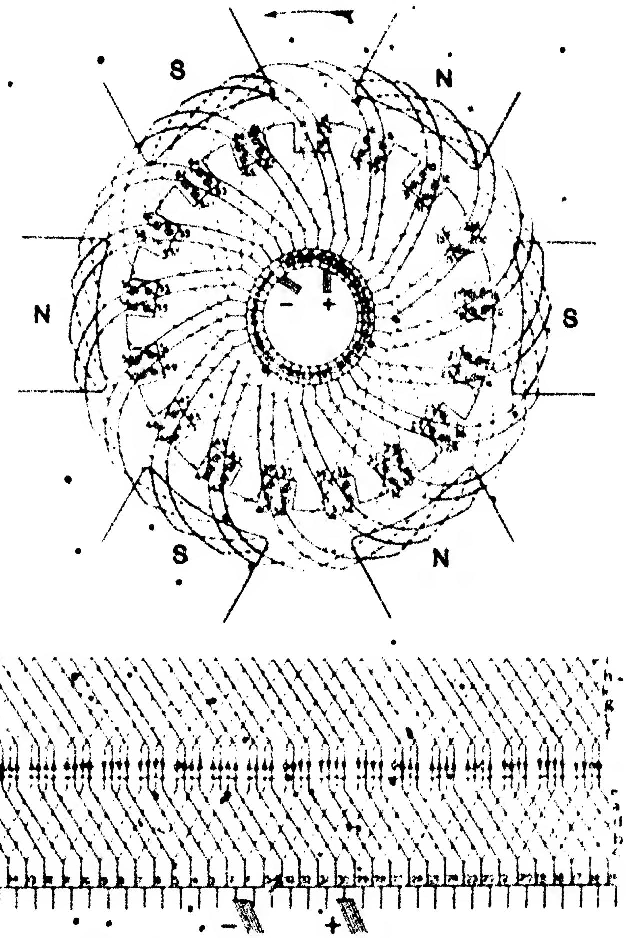


FIG. 108.—Wave-wound toothed drum ($a = 1$); $U = 68$, $y_a = 13$, $y_r = 9$, $y_c = 11$, $y'_a = 3$.

is not laid down by the formula for y_c, and some choice is here permissible. They may in many cases be equal, and each equal to the average pitch of y_c when this is uneven. But when the

desired number of elements U is such that an equal division would lead to y_b and y_f being even numbers and an impossible winding, y_b may be made slightly less or greater (preferably the former) than the pole-pitch $\frac{U}{2p}$, and y_f correspondingly greater or less, so as to secure the necessary average pitch.

If $y_b = y_f$, any two elements which are simultaneously undergoing short-circuit in the same interpolar gap lie side by side; this is also the case if the component pitches only differ by two (as is possible if the number of sectors is not divisible by the number of poles), and the back pitch is the longer when the shorter of the two possible average pitches is taken or is the shorter with the larger of the two possible average pitches. But the same effect of reducing the armature back-ampere-turns as is produced by chord winding in the lap-wound drum can be produced if the two pitches are made to differ by more than the above amounts; a pair of elements which are short-circuited in one interpolar gap are then separated by two or more elements carrying the full current in opposite directions, as is shown in two of the interpolar gaps in Fig. 406.

§ 22. Simplex and multiplex wave-wound armatures.—In the wave-connected multipolar armature, the possible numbers of elements for a given value of a and a given number of poles go up by steps, which introduces certain restrictions. Even the limited choice given by the formula

$$U = 2p \cdot y_c \pm 2a$$

is only obtained by making y_c alternately exceed C/p, so that the winding becomes progressive, as shown by the appearance in the complete formula of the alternative of plus $2a$ or minus $2a$. Further, in the slotted armature in order that all slots may be equally filled, $C = U/2$ must be a multiple of the number of slots and with more than two coilsides per slot or $c > 1$, this introduces a second restriction when the number of slots on a standard machine is fixed. To overcome this latter restriction, use has been made of the device of adding "dead" bars or idle coils; these either have their ends insulated or are connected in parallel with their nearest neighbour, and serve merely to balance the armature and give mechanical symmetry. But since they always introduce some electrical dyssymmetry, their use should be strictly discountenanced.[1]

According to the value of a the winding falls into two classes:

(*a*) $a = 1$.

This gives the *simplex wave-wound drum*, for which

$$y_b + y_f = \frac{U \pm 2}{p}$$

and

$$y_c = \frac{U \pm 2}{2p}, \text{ or } U = 2p \cdot y_c \pm 2 \quad . \quad . \quad . \quad (54)$$

[1] Except possibly in small machines with $a = 1$.

Since y_e and $C = U/2$ have no common factor, a single closed helix always results. The possible number of elements goes up by steps of four in a 4- or 8-pole machine, and by steps alternately of four and two in a 6-pole machine, and of four and six in a 10-pole dynamo.

Since y_e must be a whole number and in the usual case of a winding in two or more layers $= \frac{cS \pm 1}{p}$, where c is the number of coils or sectors per slot, it follows that in the multipolar machine, the number of slots S must not be divisible by p.

(b) a = any whole number greater than 1.

The number of pairs of parallel paths may in the wave-wound drum be made equal to any whole number by suitably choosing the number and pitches of the elements. When $a > 1$, there results the *multiplex wave-wound drum* (also called "series parallel wound drum"), which retains many of the characteristics of the multiplex lap-winding in that it increases the number of paths and sectors for a given voltage, and so lends itself to large outputs at low voltages. But its distinguishing feature is that the number of armature paths which it yields, although a multiple of two, bears no relation to the number of pairs of poles. Hence a may be either $\gtrless p$, or $= p$. After one nearly complete tour of the armature passing through $2p$ elements, the winding now returns, *e.g.* if $a = 2$, to the fourth element or to the second sector behind or ahead of the starting-point, instead of to the second element or the adjacent sector behind or ahead of the starting-point, as in the simplex wave winding with $a = 1$. Commencing from a sector on which a + brush rests, when we have passed through $\frac{U}{2a}$ elements, we reach a − brush, and have passed in continuous sequence through the sectors opposite or corresponding to one magnetic field ahead of or behind our starting-point. The same characteristics, namely, that each circuit from brush to brush is subjected to the inductive effect of every pole and therefore is practically independent of magnetic dissymmetry, and that the use of as many sets of brushes as there are poles may be accompanied by selective commutation, are still retained as in the simplex form. If only two sets of brushes are employed, each set must cover more than ($a - 1$) sectors, so that no circuit may be broken, and preferably covers a or $a + 1$ sectors, but if a number of sets are used, each can be of less width, the short-circuiting of the coils being effected through the brushes and the leads connecting those of the same sign. A reduction of the number of sets of brushes below $2a$, *i.e.* a positive and a negative, is not to be recommended, so that, although when $a = 1$ two sets are quite admissible with any number of poles, this is not the case when, *e.g.* $p = 9$ and $a = 3$. In such a case, with all the brushes in use and

with, *e.g.* all the *positive* brushes numbered from 1 to 9, brushes 1, 4, 7 would have the same potential, 2, 5, 8 would have the same potential, and so also 3, 6, 9. Between 1 and 4 there may be omitted the brushes 2 and 3, and also between 4 and 7 may be omitted the brushes 5 and 6, and so on. If p_w = the number of brushes of the same sign which are omitted in continuous sequence, which will be, as a maximum 2 in the above example, then during short-circuit there are $(1 + p_w)$ coils thrown into series at this spot, and their voltage is $(1 + p_w)$ times the voltage of a single coil.'

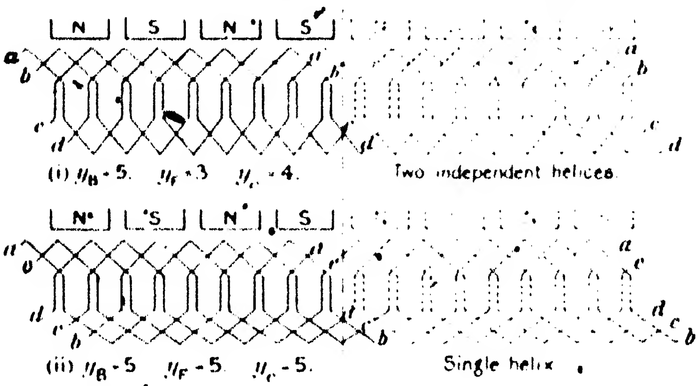


FIG. 107.—8-pole duplex symmetrical wave windings ($a = 2$) derived from 4-pole parent machines with simplex wave winding ($a = 1$).

$S' = 2n + 1 = 9$, $c = 1$, $C' = 9$.

$S = 4n + 2 = 18$, $c = 1$, $C = 18$. H.C.F. of S and $p = t = 2$.

(i) H.C.F. of y_c and $C = 2$. (ii) H.C.F. of y_c and $C = 1$.

As in the analogous case of the multiplex lap-wound drum, two variations are possible. If y_c and C have no common factor other than 1, the winding forms a single closed helix; thus in the lower part of Fig. 107 $y_c = 5$, and $C = 18$. On the other hand, if the highest common factor of y_c and C be two or more as in the upper part of the same Figure where, with the same value of C, y_c is made 4, the H.C.F. of y_c and C gives the number of totally independent windings.

Obviously, if $a = p$, the winding bears a close resemblance to a multipolar lap-wound drum, and when C/p is a whole number it may be obtained from the 2-pole wave wound drum by opening it out and repeating it p times, as in Fig. 87, which is the duplication of Fig. 85. But the great advantage of the winding lies in the fact that it enables us to make $a \gtrless p$, *e.g.* with 8 poles there may be 4 parallel paths in the armature as in Fig. 107, or with 12 poles we may have 4 or 6 parallel paths, which may often enable a standard

pattern of field-magnet to be used for widely divergent voltages and outputs without alteration.

The induced E.M.F. of the wave-wound drum thus retains the general form of equation (44a), viz.

$$E_a = \frac{p}{a} \times \Phi_a \times Z \times \frac{N}{60} \times 10^{-8} \quad . \quad . \quad . \quad (55)$$

with the special form for the simplex case when $a = 1$

$$E_a = p\Phi_a Z \frac{N}{60} 10^{-8} \quad . \quad . \quad . \quad . \quad (55a)$$

Since with the same multipolar field the number of elements required for a given voltage is in the wave-wound armature only $1/p$ of that in the lap-wound armature, it follows that with the same number of commutator sectors if the lap coils are of 2 or more turns, the coils in the wave machine can be reduced either to a single turn, or generally, to $1/p$ of the lap turns if more than a single turn thence results,[1] with a corresponding reduction of the space lost in insulation of the wires, and a saving in the time taken to wind them. But when the comparison lies between single-turn coils in both cases, it is the sectors that must be reduced in the wave machine; a limit in this direction is then often reached by the average difference of potential between neighbouring sectors becoming too high. On this account wave-winding is, generally speaking, better suited to fairly low speeds and small machines.

§ 23. The values of a or of $m = a/p$ in the general case of C/p whole or fractional. Whatever the number of poles and slots, and whatever the value of c, the number of coils or sectors per slot, the coils of a toothed armature can always be coupled up to form a regular wave-winding when all slots are equally filled without recourse to the device of "dead" bars or coil-sides. But in each case there necessarily result particular values of a, the number of pairs of armature paths, and particular types of wave-winding with a single closed helix or two or more independent helices, and these values of a may or may not be desired, and may or may not be desirable for electrical reasons. The value of a must, therefore, be specially considered.[2]

If S = the number of slots in a toothed armature with a winding in two or more layers,[3] the total number of coils or commutators sectors is $C = cS$. Let the number of slots, S, be expressed as $pn_1 \pm x$, where x is *zero* if S/p is a whole number, and if S/p is fractional, n_1 is the *nearest* whole number to S/p, so that x is then a positive or negative integer, the numerical value of which may be

[1] And a fractional result is for the moment taken as feasible.

[2] Cp. Dr. S. P. Smith, "The Theory of Armature Windings," *Journ. I.E.E.*, Vol. 55, p. 81.

[3] Or half the actual number of slots in the rare case of a single-layer winding.

equal to but cannot exceed $p/2$. When $x = p/2$, there are two whole numbers n_1 and n_1', one on either side of S/p, and equally near to it; it is then a matter of indifference whether we work with $pn_1 + x$, where x is positive, or with $pn_1' + x$, where x is negative, and $n_1' = n_1 + 1$.

Analogously to the expression $S = pn_1 + x$ for the slots, the number of sectors per slot may be expressed as $c = pn_2 + y$, where y may be zero or any integer, positive or negative, less than or equal to $p/2$. n_2 may be zero or any positive integer, but with the proviso that if y is negative, n_2 must be such an integer that $pn_2 + y$ yields a positive value for the sectors per slot. When p is high, n_2 is, of course, usually zero, and y is then $= c$. Inserting the above expressions for S and c,

$$\frac{cS}{p} = \frac{(pn_1 + x)\ (pn_2 + y)}{p}$$

$$= \frac{pn_1c + pn_2x + xy}{p}$$

$$= n_1c + n_2x + \frac{xy}{p}$$

$\frac{xy}{p}$ may be positive or negative, and may be a proper or improper fraction. If a proper fraction less than $\frac{1}{2}$ or an improper fraction leaving a remainder less than $\frac{1}{2}$, it may be expressed as $z + \frac{k}{p}$, where z may be zero or an integer. If a proper fraction greater than $\frac{1}{2}$ or an improper fraction leaving a remainder greater than $\frac{1}{2}$, the next higher number being z', $\frac{xy}{p}$ may be expressed as $z' - \frac{k'}{p}$; thus by the use of the next higher number z' the fraction is always to be reduced to its minimum value. When $\frac{xy}{p} = \frac{1}{2}$ or an improper fraction having a remainder equal to $\frac{1}{2}$, the use of either z or z' leads to the same result, as will be shown later.

The minimum value of a for any combination of S and c is of chief practical importance, and this will correspond to the minimum value of m. It follows from equation (53) that the minimum value of m or a is always given by that value of the commutator pitch which is the whole number nearest to $\frac{C}{p} = \frac{cS}{p}$, but not equal to it when C/p is itself a whole number. The nearest whole number is now given by $n_1c + n_2x + z$ (or z', as the case requires); deducting from this $\frac{C}{p} = n_1c + n_2x + z + \frac{k}{p}$ or $n_1c + n_2x + z' - \frac{k'}{p}$,

we have for the minimum value of m in the former case $-\frac{k}{p}$, and $+\frac{k'}{p}$ in the latter case. Hence

$$a_{min} = -k, \text{ when } \frac{xy}{p} = z - \frac{k}{p}$$

$$= +k' \quad " \quad " \quad = z' - \frac{k'}{p}$$

the signs of z and k or z' and k' being determined by the sign of xy, and either z or z' being chosen to make k or k' a minimum.

From this starting-point of a_{min}, other possible whole numbers for the commutator pitch are obtained by shortening or lengthening the pitch by one or more sectors. Let s be the [illegible] of sectors by which the pitch is lengthened (s with positive sign) or shortened (s with negative sign) from an original length $n_1c + n_2x + z$ (or z'). Thence—

$$m = n_1c + n_2x + z + s - n_1c - n_2x - z - \frac{k}{p} = -\frac{k}{p} + s$$

$$\text{or} \quad = n_1c + n_2x + z' + s - n_1c - n_2x - z' + \frac{k'}{p} = \frac{k'}{p} + s$$

The complete series of values for a is, therefore, as shown in Table II on pages 230 and 231.

By inserting z and k or z' and k' with their appropriate signs, positive or negative as the case may be, according to the sign of xy, the above Table answers every question, and the sign which finally results for m or $a = mp$ informs us whether the direction of "creep" through the magnetic field is progressive (positive) or retrogressive (negative). In the former case, py_c sector-pitches exceed, and in the latter case fall short of, an exact tour round the armature.

The first effect of shortening or lengthening the central pitch by one sector is to add on p pairs of armature paths algebraically, which in one direction means that the actual number is the difference between p and k or k'. After this, every successive reduction or increase of y_c adds on p pairs of paths.

When $\frac{k}{p} = \frac{k'}{p} = \frac{1}{2}$, *i.e.* when xy/p is equidistant between the two whole numbers z and z', the two series coincide thus—

$y_c =$	$n_1c + n_2x + z - 1$ or $n_1c + n_2x + z' - 2$	$n_1c + n_2x + z$ or $n_1c + n_2x + z' - 1$	$n_1c + n_2x + z + 1$ or $n_1c + n_2x + z'$	$n_1c + n_2x + z + 2$ or $n_1c + n_2x + z' + 1$
$a =$	$-1\frac{1}{2}p$	$-\frac{p}{2}$	$+\frac{p}{2}$	$+1\frac{1}{2}p$

$$= a_{min}$$

TABLE II.—

When $\frac{xy}{p} = z + \frac{k}{p}$

	Shorter by 1, 2, 3 . . . x vectors ←		
When y_e is	$n_1c + n_2x + z - x$ . . .	$n_1c + n_2x + z - 2$	$n_1c + n_2x + z - 1$
$a =$	$(xp + k)$	$(2p + k)$	$(p + k)$
$a =$	$(xp - k')$	$(2p - k')$	$(p - k')$
When y_e is	$n_1c + n_2x + z' - x$ . . .	$n_1c + n_2x + z' - 2$	$n_1c + n_2x + z' - 1$

When $\frac{xy}{p} = z' - \frac{k'}{p}$

Since the values of a now repeat on either side of the value a_{min}, the pitch will be chosen so as to make the winding retrogressive and thereby avoid unnecessary crossings.

When C/p is a whole number to start with,

either (1) S is exactly divisible by p, and $x = 0$,

or (2) c „ „ „ „ $y = 0$,

or (3) p is composed of two factors, b and d, each higher than unity, of which b is a factor in S, and therefore in x, and d is a factor in c, and therefore in y, so that xy/p is a whole number $= z$. Therefore if z is the above whole number, or if it is zero, the fact that $k = 0$ makes Table II reduce to

$y_e =$	$n_1c + n_2x + z - x$ . . .	$n_1c + n_2x + z - 2$	$n_1c + n_2x + z - 1$
$a =$	xp	$2p$	$p = a_{min}$

The only possible values of a are then p or a multiple of p, and again as in the preceding case since the series repeats, the "creep" will be made retrogressive or a negative.

§ 24. General expression for the values of a.—The general values for a are from Table II

$$a = -k \pm sp, \text{ when } xy/p = z + \frac{k}{p} \qquad (56)$$

$$\text{or } k' \pm sp, \text{ when } xy/p = z' - \frac{k'}{p}$$

These may also be put in the single form

$$a = -xy \pm sp$$

but the minimum value of a is not then immediately indicated.

From either expression the possible values of a for given numbers

VALUES OF a.

and $\frac{k}{p}$ is ... $\frac{1}{2}$.

Longer by 1, 2, 3 ... sectors

$n_1 c + n_2 x + z$	$n_1 c + n_2 x + z + 1$	$n_1 c + n_2 x + z + 2$	$n_1 c + n_2 x + z + s$
$= k$	$(p + k)$	$(2p + k)$	$(sp + k)$
$= a_{min}$			
$+ k'$	$(p + k')$	$(2p + k')$	$(sp + k')$
$n_1 c + n_2 x + z$	$n_1 c + n_2 x + z' + 1$	$n_1 c + n_2 x + z' + 2$	$n_1 c + n_2 x + z' + s$

and $\frac{k'}{p}$ is ... $\frac{1}{2}$.

of poles, slots and sectors per slot can be quickly written down, as in the specimen cases of $p = 6$ and 7 given below in Table III.

It will be seen that in each set of numbers the vertical columns and horizontal rows keep on repeating in different order. When $y = \pm 1$ or is prime to p, the possible values of a in a joint vertical column form a consecutive series from 1 upwards without any intermediate whole numbers being missed. The same holds for any value of y when p is itself a prime number, 2, 3, 5, 7, 11 . . . . When y is a factor in p, the possible values of a diminish, until with $y = p/2$, we are reduced to $a = p/2$ or a multiple of $p/2$.

When p is high, the values of a mount up by large jumps, a

$n_1 c + n_2 x + z$	$n_1 c + n_2 x + z + 1$	$n_1 c + n_2 x + z + 2$	$n_1 c + n_2 x + z + s$
0	p	$2p$	sp
impracticable	$= a_{min}$		

TABLE III. VALUES OF $a = k$ (OR $= k'$) $= sp$, FOR $p = 6$ AND 7.

Number of pole pairs.	Number of slots	Number of sectors per slot, $c = pn_1 \pm y$			
p	$S = pn_1 \pm r$	$y = 0$	$y = \pm 1$	$y = \pm 2$	$y = \pm 3$.
6	$6n_1$	6 or multiple of 6	6 or multiple of 6	6 or multiple of 6	6 or multiple of 6
	$6n_1 \pm 1$	6 6	1, 5, 7 . . .	2, 4, 8 . . .	3, 9 . . . .
	$6n_1 \pm 2$	6 6	2, 4, 8 . . .	2, 4, 8	6 or multiple of 6
	$6n_1 \pm 3$	6 6	3, 9	6 or multiple of 6	3, 9 . . . .
7	$7n_1$	7 or m. of 7	7 or m. of 7	7 or m. of 7	7 or m. of 7
	$7n_1 \pm 1$	7 7	1, 6, 8	2, 5, 9	3, 4, 10 . . .
	$7n_1 \pm 2$	7 7	2, 5, 9	3, 4, 10 . . .	1, 6, 8 . . . .
	$7n_1 \pm 3$	7 7	3, 4, 10 . . .	1, 6, 8	2, 5, 9 . . . .

difference in the commutator pitch of one sector making a large difference in py_s sector-pitches, so that the cumulative total falls far short of or much exceeds an exact tour once round the armature.

The further use of the above expressions is given in § 15 and Table V of the next Chapter

Lap- and Wave-winding

§ 25. The E.M.F. equations for lap- and wave-wound armatures contrasted.—A summary of the foregoing results brings out clearly the inherent differences of the two great divisions, lap and wave-winding, of the drum continuous-current armature; by the side of the E.M.F. equation is added in each case the corresponding value of the current in any one armature path, or $J = \dfrac{I_a}{q}$

Lap.	Wave.
$a = mp$, and m must be some whole number.	$a \neq mp =$ any whole number.
(1) If $m = 1$, *simplex* lap	(1) If $a = 1$, *simplex* wave
$E_a = \Phi_a \dfrac{N}{60} Z \times 10^{-8}$ $\qquad J = \dfrac{I_a}{2p}$	$E_a = p\Phi_a \dfrac{N}{60} Z \times 10^{-8}$ $\qquad J = \dfrac{I_a}{2}$
(2) If $m > 1$, *multiplex* lap	(2) If $a > 1$, *multiplex* wave
$E_a = \dfrac{1}{m} \Phi_a \dfrac{N}{60} Z \times 10^{-8}$ $\qquad J = \dfrac{I_a}{2mp}$	$E_a = \dfrac{p}{a} \Phi_a \dfrac{N}{60} Z \times 10^{-8}$ $\qquad J = \dfrac{I_a}{2a}$

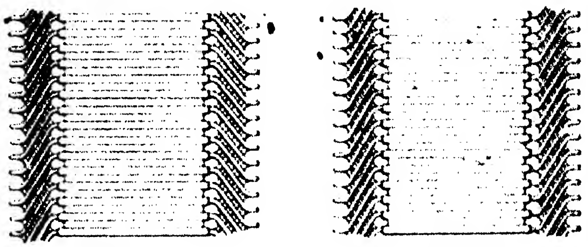
§ 26. Lap and wave-wound armatures contrasted in appearance.—As will be more fully described in Chapter XIII, the most usual type of winding employed on a toothed armature core is known as "barrel-winding," and consists of lozenge-shaped coils, the sides of which fall into two layers as shown in Figs. 101 and 106 (*cp.* Figs. 93 and 94).

In the lap- and wave-wound toothed armature represented in Figs. 101 and 106, the lower elements are marked with uneven and the upper elements with even numbers. The end-connectors of such armatures with two layers of bars are V-shaped, and themselves fall into two layers. The developed plans of Figs. 101 and 106 then show that in the lap-wound armature the slope of the upper layer of end-connectors at each end of the armature is aslant to the armature core in opposite directions, while in the wave-wound armature their slope is parallel. From this external difference it is at once easy to determine by inspection whether a toothed armature is lap- or wave-wound (Fig. 108).

§ 27. Brush-polarity.—By reason of convenience in manufacture the connections to the commutator sectors will be made at the part of a front coil-connector which is nearest to the commutator. With coils as above described, this is in the lap-wound drum the apex of the V-shaped end-connections exactly midway between the two coil-sides (Fig. 101), and in the wave-wound drum midway

between two coils at the end of one and at the beginning of the next (Fig. 106).

Since the brushes are so arranged as to effect short-circuit of the coils when their sides are in the interpolar gaps and the coils face the poles, the positions of the brushes in both cases fall nearly opposite to the centre of the poles. It thence results that in both cases, whether lap or wave, *with a counter-clockwise direction of*



Lap Wave

Fig. 108. External appearance of lap- and wave-wound toothed armatures contrasted.

rotation in a dynamo (when viewed from the commutator end) a positive brush is opposite to a N. pole, and a negative brush is opposite to a S. pole, and vice versâ *with a clockwise direction of rotation*, whatever the "hand" of the winding. It results that the "hand" of the armature system as a whole[1] becomes immaterial, so far as regards the self-excitation of the dynamo for a given direction of rotation, a given direction of coiling in the field-magnet bobbins and connection of them to the positive and negative sets of brushes in accordance therewith.

[1] For a fuller statement, see Chapter XI, § 17, in the 5th edit. of *The Dynamo*.

CHAPTER XII

CLOSED-CIRCUIT ARMATURE WINDING (*continued*)

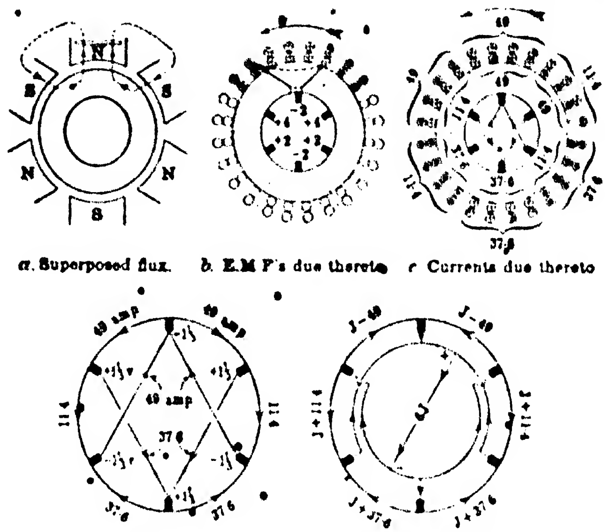
PARENT AND DERIVATIVE MACHINES

§ 1. Inequality of the E.M.F.'s in the lap-wound multipolar.—In the lap-wound multipolar since the alternate sets of brushes are connected together, the commutator sectors or points in the winding with which each joint set of brushes of the same sign at any moment makes contact should be at precisely the same potential, and the same also applies in a lesser degree to the wave-wound multipolar when as many sets of brushes are employed as there are poles.

As a matter of fact in the lap-wound multipolar it is not easy to secure absolute equality of the E.M.F.'s of the different branches which are in parallel, owing to slight differences in the permeability of the several magnetic circuits or to the armature not being exactly in the centre of the bore. The consequences arising therefrom will, therefore, now be traced.

In the lap-wound multipolar drum, each branch of the armature winding between a pair of adjacent brushes is acted on by two adjacent poles. Suppose now that for some reason the flux from one pole and into the halves of adjacent poles is less than the normal owing to the pole not being as permeable as the others, or owing to the length of its air-gap being greater. The effect on the armature currents is most simply studied by assuming an equal flux from or into each pole, and upon this superposing a local flux in such amount and direction as to give the actual distribution of flux in the real case. The equal E.M.F.'s due to the assumed equal fluxes, together with the terminal voltage and the external current therefrom may then be mentally dismissed, and attention directed at first solely to the E.M.F.'s and current due to the superposed flux. It will be found in all such cases that between brushes of the same sign which should be at the *same* potential, an E.M.F. is set up by the local flux. The guiding principle must then be as follows—since the brushes of the same sign are in reality joined by short-circuiting connections of practically zero resistance, such local currents must flow as will exactly absorb the local E.M.F.'s, and again leave no difference of potential between brushes of the same sign. Finally, under load the local currents are to be combined with an equal division of the load current to obtain the actual currents in each branch of the armature winding. Thus, in Fig. 109*a* let the steel casting of the upper N. pole of a 6-pole magnet

have a concealed cavity within it, by reason of which the flux is reduced below the normal. The imaginary local flux is then directed as shown, causing E.M.F.'s and brush potentials as shown in Fig. 109*b*. The shading off of the lines is for simplicity neglected, and each active bar is credited with an E.M.F. of 1 volt. Let the resistance of each branch of the armature winding be 0·0533 ohm, and of each brush set be $\frac{1}{15}$th of this amount. Let the short-circuiting brush connections of zero resistance now be applied; the system



a. Superposed flux. *b*. E.M.F.'s due thereto. *c* Currents due thereto

d System of circulating currents. *e* Combination with normal currents

FIG. 109. Unequal E.M.F.'s and division of current due to defect in one pole of a 6 pole lap wound machine.

of local equalizing currents shown in Fig. 109*c* and *d* will then result from the E.M.F.'s of Fig. 109*b*, and once again all brushes of the same sign are brought to the same potential. The difference between the potentials of the two brush sets (2⅔ volts) is in the reverse sense to the normal, and corresponds to the lower terminal voltage arising from the cavity. The currents due thereto would then be as shown in Fig. 109*d*, and the extra loss over the armature resistance, apart from the extra loss over the brushes, would be 420 watts, compared with a normal loss at 540 amperes of $540^2 \times \frac{0{\cdot}0533}{6} = 2590$ watts. Actually certain secondary reactions from the armature ampere-turns very greatly check any such unequal division of current and unequal pole-strengths. The mechanism by which this reduction is automatically effected in the

lap-wound multipolar is explained in detail in a note to the present Chapter.

§ 2. Equalizing connections.—It will be seen that in such a case as the preceding the equalizing-currents flow through the brushes. But this could be very largely prevented if the difference of pressure was equalized by the addition of a subsidiary set of cross-connections analogous to the stationary connections of the brushes and coupling up points in the winding or on the commutator, which should be at the same potential, but of lower resistance than the brush connections which incidentally serve the same purpose. No very large number of such *equalizing connections* need be applied, and usually in practice only about 6 to 10 points in every pole-pair are connected to as many rings, or, say, one ring for every 6 to 10 sectors. The principle is shown diagrammatically in Fig. 110, where in a 6-pole machine four similar points in the winding in each double pole-pitch are connected to 4 rings; each ring is therefore connected to three points in the winding situated at equal angles of $120° = \frac{360°}{p}$ apart.

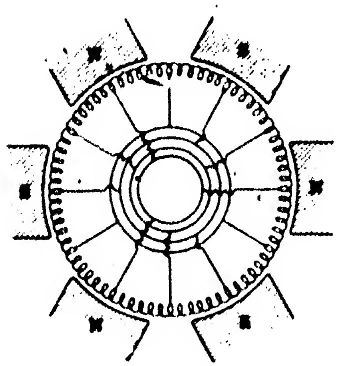


FIG. 110. Equalizing connections.

The connections may be made to commutator sectors at the back of the commutator, but more commonly they are tapped off from the winding proper at the back of the armature (*cp.* Figs. 195 and 196). It must be clearly understood that they in no way prevent the evil of unequal E.M.F.'s in the different branches of the armature, but owing to their low resistance as compared with that of the brushes they practically short-circuit the alternative path offered by the connecting lead between the brushes of the same sign. The excess current of any stronger branch is thus shunted through the auxiliary by-pass, and does not have to pass through the brushes where it increases the difficulty of commutation. The currents in the equalizing connections alternate, and since their flow implies a certain loss of watts, the use of equalizers does not remove the necessity for great care in the centring and "electrical balancing" of the armature in multipolar fields. (Chapter XV, §§ 14 and 15.)

In order that equalizing connections may be legitimately added to an armature, it is evident that under normal conditions when there is no magnetic reason for unequal E.M.F.'s, points of equal

potential must exist in the winding which may be joined together without any current passing through the connection. The sum of the E.M.F.'s between a pair or between two groups of such points must be zero. The possibility of points which strictly fulfil this condition existing in a winding turns upon the question of whether after the rise or fall of the E.M.F. has been traced through some portion of the winding, exactly the same rise or fall is found to "repeat," as will be explained in the following paragraphs.

§ 3. The conditions under which repetitions of the E.M.F. arise.—A coil-side A situated at any moment at a particular spot on an armature relatively to a pole of, say, N. sign generates a certain instantaneous E.M.F. If on the same armature there is at the same instant another coil-side A' occupying a precisely identical position relatively to another pole of the same sign, it must also generate the same instantaneous E.M.F., the assumption being, of course, made that in the multipolar machine every pair of poles yields a flux equal in amount and similar in distribution.

It is further the case that in all windings such as are employed with closed-circuit armatures, which are mechanically symmetrical and close naturally (*i.e.* without any irregular connection causing what may be termed forcible closure), the span of the coils is uniform and their connections are made after a uniform law, *i.e.* the back and front pitches of the coils are the same throughout the whole winding or windings. In these circumstances, when the winding is followed through, the coil-side B next in succession to coil-side A will generate an instantaneous E.M.F. which will be the same as that of coil-side B' the next in succession to A'. Similarly the third coil-sides C and C' must generate the same instantaneous E.M.F., and so on. From the given starting-points then the gradual summation of the E.M.F.'s will proceed equally until in each case, as we trace out the winding, we reach either the original starting-point A or another coil-side A' in the same winding occupying at the same instant identically the same position as A, but relatively to another pole of the same sign, when the process begins again.

Proceeding from the position of zero E.M.F. in a coil and tracing the winding until it has crept past a double pole-pitch, the first result is a rise of the induced E.M.F., say, to a positive maximum e_1 when one pole-pitch has been crept through; thenceforward the sign changes and a negative induced E.M.F. rises to a maximum $-e_1'$. As further explained in § 6, the possibility of some inequality between the two E.M.F.'s must be considered as the general case, and from the difference, if any, *i.e.* from $e_1 - e_1'$ will arise a circulating current, alternating in direction round the closed circuit as rotation proceeds and the second part of the winding assumes the same

position relatively to the second pole-pitch that the first part had initially to the first pole-pitch. It must now be noted that as soon as a point of repetition of E.M.F. is reached in the winding, exactly the same circulating current will be carried on through the next stage and so on, until the ring closes, and whether each stage corresponds to one or to more pole-pairs, the magnitude and direction of the current will be just such that the volts lost in its passage over the ohmic resistance exactly absorb the differences $e_1 - e_1'$, and leave the points marking the termination of stages at precisely the same potential. Thus if between any pairs of points the same E.M.F.'s $e_1 - e_1'$ are induced, these points will remain at the same potential, whether or no there be any circulating current or its equivalent, an unequal division of the armature current under load. There are then points of equal potential in the winding which may be legitimately joined by equipotential connections. When so joined, no current will flow through the connecting leads, unless there be inequalities in the pole-strengths, from the effects of which on the brushes it is the function of the equipotential connections when acting as equalizers to protect the machine.

If, therefore, out of the p points on a multipolar armature which at any instant may be similarly situated relatively to poles of the same sign, the number f are, in fact, occupied by coil-sides, equalizer connections become legitimate, and each such connection may join together f corresponding points, and must join them if it is required to load all the f pairs of armature paths equally from slip rings. The equipotential pitch or the distance on the armature which separates one point from the next successive point of equal potential, when measured in coil-sides or elements, will be U/f, and when measured in coils or sectors will be C/f.

§ 4. The case of the slotted armature.—In the above general statement of the circumstances under which it becomes possible to divide an armature winding into two or more portions which at every instant give an E.M.F. equal in amount and in phase, no specific mention has been made of the slotted armature. When the armature of a multipolar machine is slotted, there will not be f points which at any instant are similarly situated relatively to poles of the same sign, unless there are f slots so situated. In order that there may be slots occupying at one instant similar positions relatively to poles of the same sign, S and p must have a common factor $f > 1$. When S is prime to p, there are no two slots which at any given moment have the required similar positions.

The sole condition then for repetitions of E.M.F. and for the legitimacy of equipotential connections in a slotted armature as normally constructed is a common factor $f > 1$ between S and p, and it is the importance of this factor in the case of the slotted armature which was first brought out by Dr. S. P. Smith in his

paper on "The Theory of Armature Windings."[1] The pole-pairs are then divisible into f groups of $p/f = p'$ pole-pairs; the slots are similarly divisible into f groups of $S/f = S'$ slots, and the two sets of groups exactly correspond to one another. The number of slots which at any instant occupy exactly the same positions relatively to poles of the same sign is f. This granted, it follows that in each of the f slots there is a coil-side occupying the same position within the slot, since all slots in the assumed regular winding are equally filled. There are, therefore, f coil-sides fulfilling the conditions of § 3, and the equipotential pitch measured in slots becomes S/f. If, as is the case in practice, S exceeds p, the H.C.F. f between S and p may be equal to p, but this is its maximum value.

All slotted armatures can thus be divided into two classes, according to whether S has not or has a factor common to p and higher than unity. In the former class shown on the left-hand side of Table V for pole-pairs up to 12 in number there are no repetitions of E.M.F.; but armatures in the latter class, shown on the right-hand side, possess true points of equal potential, give one or more repetitions of the same E.M.F.'s round the closed circuit or circuits, and admit of equalizing connections.

§ 5. The distinction between parent and derivative machines. — The general statements of §§ 3 and 4 have made no definite reference to the nature of the winding of the armature. Indeed they have been framed to apply equally to windings forming a single closed helix or two or more independent helices, and in each case whether lap or wave-wound. They therefore require to be supplemented by a further consideration of the actual circumstances of each class of winding.

In this connection it will be found that many interesting points in the nature of multipolar armature windings are rendered clearer by considering whether they can be reduced to a simpler form by the process of removing one or more sets of pole-pairs and the same number of repeats of the armature winding until there is left a machine which may be regarded virtually as the unit by the multiplication of which the actual machine is obtained.

The first requisite will be a knowledge of the "parent" or original types which cannot be reduced by the above-described process to anything simpler.

In the multipolar machine repetition of E.M.F. has been shown in § 4 to arise from the multiplication of an original number of slots S' with their corresponding coil-positions within them, and an original number of pole-pairs p', so that $S = fS'$ and $p = fp'$, while $c = c'$ remains unchanged; and this suggests what is in fact the case that the machines tabulated on the left-hand side of Table V

[1] *Journ. I.E.E.*, Vol. 55, p. 18.

furnish the parent numbers of slots and pole-pairs from which other machines (tabulated on the right-hand side) can be derived by the converse process of cutting the originals through to the centre, inserting one or more additional sets of p'' pole-pairs (each set being equal in number to the original set) and connecting in a corresponding number of exact repeats of the original armature winding. Every time that the same system of conductors occupying slot positions similarly situated relatively to poles of the same sign is repeated, there will be an exact repetition of the instantaneous E.M.F. of the original winding in the repeat winding. Whatever the component E.M.F.'s or resultant E.M.F. as a whole induced between the severed ends of any continuous length of copper in the original winding, it follows from the nature of the multiplying process that there must be a corresponding length of copper and a corresponding E.M.F. or E.M.F.'s in the first or any number of sets of repeat coils. If, therefore, the multiplication is carried out f times, that is, if $f-1$ sets of p' pole-pairs and $f-1$ repeats of the winding are inserted, then in the derived machine the E.M.F., whatever it may be, which is induced in each portion into which the original winding has been severed, must re-occur $f-1$ times, so that there are in all f occurrences of it; and any f corresponding points of the complete winding may be joined together by an equalizer connection as being points of equal potential.

Bipolar Parent Types

§ 6. The two-pole parent machine.—The essential characteristic of the two-pole machine, whether lap or wave-wound, is that m must be a whole number.

With any number of sectors $cS' = n_1n_2$, where n_1 and n_2 are any whole numbers, let the winding be traced through, starting from one side of a coil, the axis of which stands at right angles to the neutral plane of zero field, and being careful to follow the winding in the direction of the E.M.F., *i.e.* from a negative brush. Thus in Fig. 84 the start may be made either from sector 1 down element 1 and so on, or from sector 2 down element 8, any little E.M.F.'s in these two elements being balanced and in the same direction away from the observer. Or in Fig. 85 the start may be made either from sector 5 through elements 9 and 16 in which the E.M.F.'s are balanced on to element 7, or from section 6 down element 16 and up 9 to element 2. The only reason for so choosing the start is that in this way the result is the gradual attainment of the E.M.F. e_1 in its entirety, followed in proper sequence by the whole of $-e_1'$. It has already been shown in § 10 of Chapter X that, though in the toothed 2-pole drum there can be no circulating current, such a current will arise in the smooth-core armature with

an uneven number of slots S' if there are harmonics in the flux-curve of the order S' or any multiple thereof. Under load conditions the complex processes of commutation may also affect the above statement, so that as already mentioned if some difference $e_1 - e_1'$ be assumed as the general case, there will arise a circulating current, passing through a cycle of values at a frequency considerably greater than the fundamental frequency. But when the voltage absorbed by the ohmic resistance in relation to this current is taken into account as a negative E.M.F., the sum of the E.M.F.'s is rendered exactly zero; in fact the circuit of the armature is closed at the point reached at the end of the tracing-out process, and the circulating current or its equivalent, the unequal division of an armature load current, absorbs at any instant any difference between e_1 and e_1'.

Lap and wave-winding will now in turn be considered separately, although it will be seen later that there is but little electrical distinction between them in their two-pole application which is alone now under consideration.

A. Lap-winding

§ 7. (1 a) $m = 1$.—Let an additional pole-pair and a repeat of the winding now be inserted and joined up as shown in Fig. 86. Starting from sector 1 as before, and in the same direction, we again obtain e_1, and at the further end the original winding will take in portions of the inserted winding up to sector 9, which being similarly situated will again complete the E.M.F. $-e_1'$. The remainder of the inserted winding will in turn be completed by linking up with the portions of the original winding so far left out, and will yield a second and exactly equal $e_1 - e_1'$. Hence brushes on the commutator at 1 and 9 may be joined in parallel, as being strictly at the same potential, or the rotating points 1 and 9 of the winding may be joined by an equalizer connection as strictly equipotential points; and the same holds for other corresponding intermediate points in the winding or for the intermediate pair of commutator brushes of opposite sign. As already stated, the possible presence of an alternating current circulating at some frequency higher than the fundamental round the closed winding will not in any way affect the equality of potential of the above-described points. Its continuance round the enlarged circle of the winding is exactly compensated at each instant by the recurrence of the same difference $e_1 - e_1'$, and the same ohmic loss under the second pair of poles.

The same process may again be repeated by the addition of another pole-pair; since in the first instance the original winding, after crossing the double pole-pitch once, returned to its original starting-point, no change is caused by the insertion of either one

or more repeats. In every case there results a single closed helix, which also follows at once from the fact that $y_c = m = 1$ has been kept unchanged, and this can have no common factor higher than unity with the new number of sectors $C = pcS'$. The derived machine is the simplex lap-wound multipolar with $a = p$, and it is seen that it is the parent two-pole machine multiplied p times. The number of slots in the derived machine being $S = pS'$, the H.C.F. of S and p is $f = p$, and there are p repetitions of the E.M.F. $e_1 - e_1'$, the slot positions at any moment in one double pole-pitch being exactly repeated in every other double pole-pitch.

Thus any simplex lap-wound multipolar ($y_c = 1$) which has a number of slots S exactly divisible by the number of pole-pairs (*i.e.* $S = pS'$) is a derivative from a parent 2-pole lap-wound machine having the same value of c and S' slots. The equipotential pitch measured in slots is $y_{eq} = S' = S/p$, and in coils or commutator sectors is $cS' = C/p$, or in elements is U/p. The number of points at any given potential reaches the maximum value p, and to all of these any equalizer connection will be attached, as in Fig. 110.

§ 8. $m = 1$ has a common factor with cS' greater than unity.—Next let the commutator pitch of the original 2-pole lap-wound machine be increased to $y_c = m > 1$, so that the winding becomes multiplex, $a' = m p'$, m being a whole number, 2 or more.

(1*b*) First let $m > 1$ be a factor in $C' = cS'$. Then in the parent machine there are m independent helices, interleaved between one another, each of which, considered by itself, will follow exactly the same laws as the simplex winding of case (1*a*). For every pole-pair and winding inserted there must be a repetition of the E.M.F. of the first helix in its repeat, and of the E.M.F. of the second helix in its repeat, and so on. There are therefore at any instant p repetitions of E.M.F. in each of the m independent helices of the derived machine. The equipotential pitch is as before. But it may be asked whether there is any exact repetition of the E.M.F. of one of the original helices by that of another, or in the derived multipolar machine between portions or the whole of the independent helices. Since for any one position of a coil-side in a slot and relatively, say, to N poles, all the p corresponding points are occupied by one helix, this can at once be answered in the negative.

Equalizer connections in the derived multipolar therefore can only connect points in each of the windings independently—a condition which is automatically cared for by the use of the formulae for the equipotential pitch given above (§ 7).

For the same reason the points joined on the commutator by any brush are never strictly at the same potential. Yet the difference of phase which causes this cannot be alleged as any objection to the multiplex winding; on the contrary it would be the ground for its adoption for the following reason. If in the course of design we are limited to a two-pole lap-wound machine to give a very large current, the progressive commutation of the contents of a slot due to the multiplex winding and following as a corollary from the difference of phase above mentioned would be the cause of its superiority to a simplex winding.

Thus any multiplex lap-wound multipolar with $y_c = m > 1$, in which the number of slots S is exactly divisible by p, and the number of sectors C is exactly divisible by mp, has m independent windings, each possessing p repetitions of E.M.F. but with none between windings; and it may be traced back to a parent 2-pole machine lap-wound with m independent helices, and having the same values of c and of $y_c = m > 1$, but with S' slots.

§ 9. $m > 1$ having no C.F. with cS' higher than 1.—(1, *c*) Let m, although > 1, have no common factor higher than 1 with $C' = cS'$, so that the original

multiplex winding forms a single closed helix. Thus in Fig. 111 let the winding be triplex, with $m = 3$. Neither c nor S' can have any factor higher than 1 with m, and therefore S' cannot be divisible by $a' = m$, but this cannot be regarded as any objection, for the same reasons that have been given under (1*b*); it only implies that one or more of the pairs of armature paths must contain two coil-sides less than the other or others.

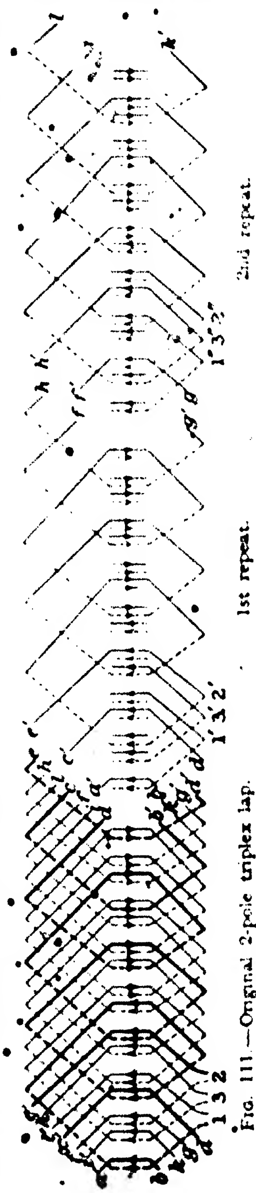


Fig. 111.—Original 2-pole triplex lap.
$C' = 16$ $S' = 8$ $c = 2$
$y_b = 15$ $y_f = 9$ $y_c = m = 3$
m prime to C'

Such a 2-pole winding creeps past the double pole-pitch m times; and, when cut through and opened out, the m sections of the windings can be traced through from the interpolar line of bisection separately. Following the same rules as before, for the first tour of Fig. 111 the starting point will be the coil-side marked 1; the second tour starting from 2 begins with the next but one coil-side in the direction of advance, and the third tour starting from 3 begins with the coil-side adjacent to the original starting point. The first tour embracing 12 elements or 6 coils will yield an E.M.F. $E_1 - e_1'$, the second and third each embracing 10 elements or 5 coils will yield E.M.F.'s $e_2 - e_2'$, and $e_3 - e_3'$, with differences of phase between all three, and all unequal.

Now since the winding is a closed helix and after each tour returns to a different starting point, it follows that when another pole-pair and a repeat winding are added on, the first half-tour of the whole machine will end at 2' and its continuation onwards completing the tour will be from 2' to 3, yielding the same E.M.F. $e_2 - e_2'$ as the original second tour from 2. The second half-tour from 2 will end at 3', and its continuation onwards completing the second tour will be from 3' to 1, yielding the same E.M.F. $e_3 - e_3'$ as the original tour from 3. Lastly the third half tour from 3 will end at 1', and its continuation onwards will be from 1' to 2, yielding the same E.M.F. $E_1 - e_1'$ as the first tour of the original winding.

But now let a second pole-pair and repeat winding be inserted, making 3 in all, so that $p = m$. Then at once $y_c = m$ and $C = cp S'$ will have the common factor $m > 1$, and we return to m independent helices, but now each giving the same total E.M.F. at any instant. In series with 1 - 2' - 3" will be found a continuation from 3" to 1 closing one helix. In series with 2 - 3' - 1" will be found a continuation from 1" to 2 closing the second helix. In series with 3 - 1'-2"

will be found a continuation from 2″ to 3 closing the third helix. We thus have—

			Original 2-pole.		First repeat.		Second repeat.
1st tour.	From coil	1.	$E_1 - e_1'$	+	$e_2 - e_2'$	+	$e_3 - e_3'$
2nd tour.	„ „	2.	$e_2 - e_2'$	+	$e_3 - e_3'$	+	$E_1 - e_1'$
3rd tour.	„ „	3.	$e_3 - e_3'$	+	$E_1 - e_1'$	+	$e_2 - e_2'$

Hence from the parent multiplex two pole machine with $y_e = m - 1$ and *a single closed helix*, although p repetitions of each component E.M.F. must always occur between different portions of the windings, the derived machine assumes three different forms.

If p is prime to m, the winding remains a single closed helix. If the actual winding be traced through from a given starting-point, an equipotential point is found after mS' slot-pitches have been passed by, but since m is prime to p, this always reduces down to S slots passed a whole number of times *plus* or *minus* S' slot-pitches, *i.e.* the span of the equalizer connection is S' slot pitches, so that again the same formulae for the equipotential pitch remain true, *i.e.* in slots $y_{eq} = S'$ and in sectors $= cS'$.

If $p = m$ or a multiple of m, say km, there are m independent helices; in this case when $p = m$, they are without repetitions in themselves, and an equalizer connection joins the several helices together, and when $p = km$, they have k repetitions in themselves and an equalizer connection joins both points in each helix and the several helices.

If m and p have a common factor f' greater than unity but less than m or p, there result f' independent helices, each of which is now itself multiplex and making m/f' tours of the armature, there are then p/f' repetitions in each helix. In both the last two groups, there is always complete identity between the E.M.F.'s of the f' helices, and it is evident that the second group when $p = m$ or km is the extreme case of the third group when $f' = m$, so that only the third or general case is tabulated in Table IV (p. 247).

Thus multipolar machines having a multiplex lap winding with $y_e = m - 1$ and a number of slots S exactly divisible by p, but a number of sectors C not exactly divisible by mp, although they may have independent windings or a single helix, reduce to a parent two-pole multiplex machine having a single closed helix.

B. Wave-winding

§ 10. $m = 1$.—Starting with a two-pole simplex wave-wound machine such as Fig. 85, let an additional pole-pair and a repeat of the winding be inserted and joined up, as in Fig. 87. The ends of each continuous piece of wire into which the original winding, when opened out, is cut are never on the same level. Hence when the 4-pole winding is traced through, it skips alternate coils of both the original and inserted windings.

In contrast with the similar lap machine in which the winding always remains a single closed helix, the same two cases which arose first in § 9 already become possible in the multipolar derivatives from the wave machine with $m = 1$. If as in Fig. 85 y_e is a prime number to start with, the winding when multiplied always remains a single helix (*e.g.* with 4 poles in Fig. 87). But if y_e is not a prime number to start with, the multiplication of S' or of C' or of m by p may cause S or C to have a common factor h greater than unity with y_e, or mp to have the same common factor with $C = pC'$. There then result h independent wave-wound helices.

Since in the 2-pole machine with $a' = 1$, after one coil has been traced through, the distance traversed has exceeded or fallen short of one exact tour of the armature by one sector, the excess or deficiency after two coils have been traced through in the 4-pole derivative will be 2 sectors, and after three coils have been traced through in the 6-pole derivative will be 3 sectors. Or in general, if $(p-1)$ pole-pairs are inserted and p coils are traced through, it will be p sectors. But since $m = a/p$ has been retained unchanged and this is $= 1$, a must be $= p$, and a multiplex wave-winding with $a = p$ is obtained. The commutator pitch which in the 2-pole case was $y_c = cS' \pm 1$ remains unchanged as $\frac{pcS'}{p} \pm 1$, but there may result a C.F. higher than 1 between mp and C or between $cS' \pm 1$ and cS.

Thus any wave-wound multipolar machine which has a number of slots exactly divisible by p, or $S = pS'$ and $a = p$, is a derivative from a parent 2-pole simplex wave-wound machine with S' slots, $a' = 1$ and the same value of c.

If the actual winding of the derived machine be traced out when it remains a single closed helix, the number of slot pitches passed by in following the winding from a given point to the next equipotential point in succession (where both the position of the winding in the slot and the slot position relatively to a pole of the same sign are repeated)[1] is $y_c \cdot S/p$. The commutator pitch y_c being $cS' \pm 1$, this is equal to $S \cdot \frac{cS'}{p} \pm \frac{S}{p}$. Now in order that y_c may be prime to p, cS' must be divisible by p, i.e. cS'/p = a whole number. Therefore the equipotential pitch in slots is equal to S slots passed by cS'/p times, i.e. to cS' exact tours round the armature which cancel out, *plus* or *minus* the resultant equipotential pitch S/p.

When y_c and p have the common factor h higher than 1, the number of slot-pitches passed by in proceeding from a given point to the next equipotential point *in the same helix* is $S \cdot \frac{y_c}{p} = S \cdot \frac{k_1 h}{k_2\ h} = S \cdot \frac{k_1}{k_2}$. When p is a factor in y_c, this is an exact number of tours and therefore there are no repetitions within any of the helices, but repetition between the helices.

§ 11. $m > 1$.—Next, let the commutator pitch of the original 2-pole wave-wound machine be lengthened or shortened from $y_c = cS' \pm 1$ to $y_c = cS' \pm m$, where m is a whole number > 1. In the 2-pole parent since $p = 1$, there are h' helices if m has a common factor h' with C, and in the multipolar derivative, the H.C.F. of mp and C must at least contain h'; if y_c/h' and p have the further common factor f', the number of helices will rise to $h'f'$, and when $f' = y_c/h$, we finally reach as many as y_c helices, each making one tour of the armature. It will be gathered that y_c in the

[1] Such an analysis forms the basis of an article by the writer on "Repetitions of E.M.F. in Armature Windings" (*Electr.*, Vol. 79, p. 886).

multipolar wave machine takes the place of m in the multipolar lap machine, and it is therefore needless to go through the cases in detail. The results will be followed from Table IV, page 247.

§ 12. The equivalence of the two-pole lap and wave parent machines, and of the derivatives therefrom.—It can now be followed how closely equivalent a wave-winding with C/p or cS/p a whole number is to a lap winding with the same values of c and S, so long as all the $2p$ possible sets of brushes are applied to the commutator of the wave machine. In both there always results a number of pairs of parallel paths, $a = p$ or a multiple of p. In the 2-pole machine with the same numbers of slots and sectors per slot, the same value of m and the same back and front pitches, the same coils are traversed in the same sequence in each armature path from brush to brush, whether lap or wave-connected, and the order of traversing the actual coil-sides can even be made the same. The 2-pole windings are therefore practically identical. In their multipolar derivatives the coils are, of course, not traversed in the same sequence, since the wave-winding proceeds at once in strides from the original to the repeat coils, and with $m = 1$, the derived wave multipolar may have more than one closed helix; but for each coil-side traversed in the lap-machine in one armature path, a corresponding coil-side occupying *exactly* the same position relatively to another pole of the same sign is found in the corresponding armature path of the wave-wound machine, so that the E.M.F.'s and circumstances of the two machines are so far identical.

There remains, however, in the wave-wound derivative the difference that one or more sets of brushes might be lifted from the commutator, when the armature paths of the equivalent wave and lap machines, both derived from a 2-pole parent, would no longer remain alike. And this fact implies not only freedom from the effect of unequal pole-strengths, but also the liability to the objectionable feature of "selective commutation." As mentioned later in § 18, wave-wound machines even with $a = p$, or a multiple of p, are not infrequently fitted with equalizer connections, although they can only then be required to minimize the likelihood of "selective commutation." If so fitted, the wave-wound machines can claim no advantage over the lap-wound machines, in which an automatic check to unequal current division is set by armature reaction. The conclusion, therefore, is that whenever it is desired to have $a = p$ or a multiple of p, lap winding should be adopted in preference to wave-winding.

§ 13. Multipolar parent types.—The parent 2-pole machines have now been disposed of; in all the remaining types of parent machines, S' must have no common factor higher than 1 with p', and for each number of poles there will be one or more parent numbers of slots according to the number of values which x' may

assume in the expression $S' = p'n_1 \pm x'$ without its having a common factor with p', as shown in Table V (p. 248).

The first parent machine of higher order will be the 4-pole machine with $2n_1 + 1$ slots, followed by the 6-pole machine with $3n_1 \pm 1$ slots, and so on. The principles of the preceding sections remain the same, with this difference that each addition must be of a set of pole-pairs equal in number to the original set p'. Instead of p occurrences of the same E.M.F., the number of occurrences can now never be more than f, and the number of points of equal potential that may be joined together by an equalizer connection is f. Correspondingly the equipotential pitch in slots is $y_{eq} = S/f$, or in coils or commutator sectors $= y_{eq} \cdot c = C/f$, and in elements $= U/f$.

In each case when m is a whole number, all the same variants as have already been described in connection with a 2-pole parent will be found, with the exception that f now appears instead of p. But there now appears for the first time the additional possibility *in the wave machine* of m being a proper or improper fraction.

Table IV is extended to include the first such case, viz.:— the 4-pole parent machine with $S' = 2n_1 + 1$ and $c = 2n_2 + 1$ whence $m = a/p = \frac{1}{2}$ or $\frac{3}{2}$ or $\frac{5}{2}$ . . . . is fractional (*cp.* Fig. 107).

TABLE IV

Parent Machines.	No. of helices.	No. of tours of each helix.	Derivative Machines.	No. of helices.	No. of tours of each helix.
General, lap or wave, the H.C.F. of y_c and cS' being h	h	y_c/h	H.C.F. of y_c/h and f is f'	hf'	y_c/hf'
I. **Two-pole**, $p' = 1$, $S' = n_1$, m = a whole number. Lap or wave	h	y_c/h	**$2p$-pole**, $S = pS'$		
			y_c/h prime to p	h	y_c/h
			H.C.F. of y_c/h and p is f'	hf'	y_c/hf'
II. **Four-pole**, $p' = 2$, $S' = 2n_1 \pm 1$. m = a whole no., lap or wave; m = a proper or improper fraction, wave only	h	y_c/h	$p = 2f$, $S = f(2n_1 \pm 1)$		
			y_c/h prime to f	h	y_c/h
			H.C.F. of y_c/h and f is f'	hf'	y_c/hf'
III. **Six-pole**, $p' = 3$, $S' = 3n_1 \pm 1$, and so on.			$p = 3f$, $S = f(3n_1 \pm 1)$ and so on.		

In all multipolar parent machines and in all their derivatives, *i.e.* when $S = pn_1 \pm x$, and x is not zero, although the lap machine giving $a = np$ pairs of armature paths may still have any whole number n_2 of sectors per slot (like the two-pole machine and its derivatives), the wave-wound machine to give the same value of a must have $c = pn_2$, in order that $cS = pn_2 (pn_1 + x)$ may be divisible by p. The wave-wound machines with $S = pn_1 + x$ and $c = pn_2$ shown on the left-hand side of Table V, and the

TABLE V

Parent Machines, no repetitions.				Derivative Machines having t repetitions of E.M.F.					
No. of pairs of parallel paths, a $= mp$ in lap machines $= zy \pm sp$ in wave „	No. of sectors per slot, z $= p'n_2 \pm y'$	No. of slots prime to p, $S = S'$ $= p'n_1 \pm s'$.	No. of pole pairs, p.	No. of slots, S_a $= pn_1 \pm s$	No. of sectors per slot z $= pn_2 \pm y$.	No. of pairs of parallel paths, a $= mp$ in lap machine $= zy \pm sp$ in wave „	No. of repetitions, t $=$ H.C.F. of S and p.	No. of pole pairs in parent machine, $p/t = p'$.	No. of slots in parent machine, $S/t = S'$.
1, 2, 3	n_2 lap or wave	n_1	1						
2 or multiple of 2	Lap n_2; Wave $2n_2$	$2n_1 \pm 1$	2	$2n_1$	n_2 lap or wave	2 or multiple of 2	2	1	n_1
1, 3, 5	„ $2n_2 \pm 1$								
3 or multiple of 3	Lap n_2; Wave $3n_2$	$3n_1 \pm 1$	3	$3n_1$	n_2 lap or wave	3 or multiple of 3	3	1	n_1
1, 2, 4	„ $3n_2 \pm 1$								
4 or multiple of 4	Lap n_2; Wave $4n_2$	$4n_1 \pm 1$	4	$4n_1$	n_2 lap or wave	4 or multiple of 4	4	1	n_1
1, 3, 5	„ $4n_2 \pm 1$			$4n_1 \pm 2$	Lap n_2; Wave $2n_2$; „ $2n_2 \pm 1$	„ „ „ ; 2, 6, 10	2	2	$2n_1 + 1$
2, 6	„ $4n_2 \pm 2$								
5 or multiple of 5	Lap n_2; Wave $5n_2$	$5n_1 \pm 1$; $5n_1 \pm 2$	5	$5n_1$	n_2 lap or wave	5 or multiple of 5	5	1	n_1
1, 4, 6	„ $y' = 1$ or 4								
2, 3, 7	„ „ $= 2$								
6 or multiple of 6	Lap n_2; Wave $6n_2$	$6n_1 \pm 1$	6	$6n_1$	n_2 lap or wave	6 or multiple of 6	6	1	n_1
1, 5, 7	„ $6n_2 \pm 1$			$6n_1 \pm 3$	Lap n_2; Wave $2n_2$; „ $2n_2 \pm 1$	„ „ „ ; 3, 9, 15	3	2	$2n_1 + 1$
2, 4, 8	„ $6n_2 \pm 2$			$6n_1 \pm 2$	Lap n_2; Wave $3n_2$; „ $3n_2 \pm 1$	6 or multiple of 6; 2, 4 . . .	2	3	$3n_1 \pm 1$
3, 9	„ $6n_2 \pm 3$								
7 or multiple of 7	Lap n_2; Wave $7n_2$	$7n_1 \pm 1$; $7n_1 \pm 2$; $7n_1 \pm 3$	7	$7n_1$	n_2 lap or wave	7 or multiple of 7	7	1	n_1
1, 6, 8	„ $y' = 1$ or 6								
2, 5, 9	„ „ $= 2$ or 9								
3, 4, 10	„ „ $= 3$ or 4								

8 or multiple of 8	Lap n_2; Wave $8n_2$		$8n_1 \pm 1$; $8n_1 \pm 3$	**8**
1, 7, 9	.. y_f = 1 or 9			
2, 6, 10	.. .. = 2 or 6			
3, 5, 11	.. .. = 3			
4, 12	.. .. = 4 or 12			
9 or multiple of 9	Lap n_2; Wave $9n_2$		$9n_1 \pm 1$; $9n_1 \pm 2$; $9n_1 \pm 4$	**9**
1, 8, 10	.. y_f = 1 or 8			
2, 7, 11	.. .. = 2 or 10			
3, 6, 12	.. .. = 3 or [illegible]			
4, 5, 13	.. .. = 4			
10 or multiple of 10	Lap n_2; Wave $10n_2$		$10n_1 \pm 1$; $10n_1 \pm 3$	**10**
1, 9, 11	.. y_f = 1 or 9			
2, 8, 12	.. .. = 2 or 12			
3, 7, 13	.. .. = 3			
4, 6, 14	.. .. = 4 or 6			
5, 15	.. .. = 5 or 15			
11 or multiple of 11	Lap n_2; Wave $11n_2$		$11n_1 \pm 1$; $11n_1 \pm 2$; $11n_1 \pm 3$; $11n_1 \pm 4$; $11n_1 \pm 5$	**11**
1, 10, 12	.. y_f = 1 or 10 or 12			
2, 9, 13	.. .. = 2 or 9 or 20			
3, 8, 14	.. .. = 3 or 8 or 25			
4, 7, 15	.. .. = 4 or 15			
5, 6, 16	.. .. = 5 or 6 or 16			
12 or multiple of 12	Lap n_2; Wave $12n_2$		$12n_1 \pm 1$; $12n_1 \pm 5$	**12**
1, 11, 13	.. y_f = 1 or 25			
2, 10, 14	.. .. = 2 or 10			
3, 9, 15	.. .. = 3 or 15			
4, 8, 16	.. .. = 4 or 20			
5, 7, 17	.. .. = 5			
6, 18	.. .. = 6 or 30			

$8n_1$	n_2 lap or wave	**8** or multiple of 8	8	1	n_1
$8n_1 \pm 4$	Lap n_2; Wave $2n_2$; .. $2n_2 \pm 1$	**4**, 12, 20	4	2	$2n_1 + 1$
$8n_1 \pm 2$	Lap n_2; Wave $4n_2$; .. $4n_2 \pm 1$; .. $4n_2 \pm 2$	8 or multiple of 8; **2**, 6, 10 . . . .; 4, 12	2	4	$4n_1 \pm 1$
$9n_1$	n_2 lap or wave	**9** or multiple of 9	9	1	n_1
$9n_1 \pm 3$	Lap n_2; Wave $3n_2$; .. $3n_2 \pm 1$	9 or multiple of 9; **3**, 6, 12	3	3	$3n_1 \pm 1$
$10n_1$	n_2 lap or wave	**10** or multiple of 10	10	1	n_1
$10n_1 \pm 5$	Lap n_2; Wave $2n_2$; .. $2n_2 \pm 1$	10 or multiple of 10; **5**, 15	5	2	$2n_1 + 1$
$10n_1 \pm 4$; $10n_1 \pm 2$	Lap n_2; Wave $5n_2$; .. y_f = 2 or 8 or 12; .. .. = 4 or 6 or 16	10 or multiple of 10; **2**, 8, 12 . . . .; 4, 6, 14	2	5	$5n_1 \pm 1$; $5n_1 \pm 2$
$11n_1$	n_2 lap or wave	**11** or multiple of 11	11	1	n_1
$12n_1$	n_2 lap or wave	**12** or multiple of 12	12	1	n_1
$12n_1 \pm 6$	Lap n_2; Wave $2n_2$; .. $2n_2 \pm 1$	12 or multiple of 12; **6**, 18	6	2	$2n_1 \pm 1$
$12n_1 \pm 4$	Lap n_2; Wave $3n_2$; .. $3n_2 \pm 1$	12 or multiple of 12; **4**, 8, 16	4	3	$3n_1 \pm 1$
$12n_1 \pm 3$	Lap n_2; Wave $4n_2$; .. $4n_2 \pm 1$; .. $4n_2 \pm 2$	12 or multiple of 12; **3**, 9, 15 . . . .; 6, 18	3	4	$4n_1 \pm 1$
$12n_1 \pm 2$	Lap n_2; Wave $6n_2$; .. $6n_2 \pm 1$; .. $6n_2 \pm 2$; .. $6n_2 \pm 3$	12 or multiple of 12; **2**, 10, 14 . . . .; 4, 8, 16 . . . .; 6, 18	2	6	$6n_1 \pm 1$

n_1 = any whole number; n_2 = any whole number or zero.

derivatives from them which occur on the right-hand side, are then closely paralleled by lap-wound machines with the same values of S and with n_1 made equal to p or a multiple of p, in that both give $a = mp$. The resemblance of the two is, however, not complete in the sense that the wave winding picks out the same or similarly situated coil-sides to match those in the lap machine; this is forbidden by the fact that the number of slots in the present cases is never divisible by the number of pole-pairs. Consequently the positions of one set of slots corresponding to a pole-pair are never exactly reproduced by the positions of another set. Now the lap winding passes through each set in succession, while the wave winding proceeds in strides from one set to another, so that in each pair of armature paths the coil-sides of the wave-winding will never be situated exactly similarly to those of the lap winding.

It will be seen from Table V that when $S = pn_1 + x$, and a is to be $= p$ or a multiple of p, the lap-wound machine is less restricted than the wave-wound machine in the possible values of c. Since these cases stand on the same footing in that they equally admit or do not admit of equalizer connections, it may for the reason given in § 12 be now stated that whenever a is to be p or a multiple of p, lap winding is to be preferred to wave-winding. But as soon as a is to be independent of p, wave winding with its possibility of m being a proper or improper fraction alone holds the field.

§ 14. Parent and derived machines.—A genealogical tree could thus be drawn up, showing each parent machine with m a whole number which may be lap or wave-wound, and with m a proper or improper fraction which can only be wave-wound. According as y_e has or has not a common factor with cS' greater than 1, the parent winding consists of independent helices or of a single helix, or in general if h' be the H.C.F. of y_e and cS', there are h' helices. The class of winding resulting from multiplication of the original can then be brought under a single general statement.

Assuming y_e and c to be retained unchanged, if f is the H.C.F. between S and p, then in the derived machine if the H.C.F. of y_e/h' and f is f', there are $h'f'$ helices, each making $y_e/h'f'$ tours of the armature before closing, and $h'f' = h$, the H.C.F. of y_e and C for the derived machine. In shortened form Table IV illustrates the process of derivation, and by inserting the proper value of y_e any case coming under it can be analysed. y_e/h' is a whole number not present as a factor in C', and, therefore, fixing the number of tours of the winding; but when C' is multiplied by f, y_e/h' may then have a common factor f' with f, raising the number of helices to $h'f'$ and correspondingly reducing the number of tours in each.

When either m or mp is contained as a factor in C, h may rise to equality with their values. It is only in the derivatives from an

original 2-pole that in the above statement p can be substituted for f.

§ 15. The possible values of a in both lap and wave machines.—A complete list of parent and derivative machines for all numbers of pole-pairs up to $p = 12$ is given in Table V. Although appearing on opposite sides of the central column 4, it will be understood that the machines on the right-hand side are not derived from those in the same row on the left (the arrangement being purely by pole-pairs), but from the machines given in the same row in columns 9 and 10.

The values of a for all numbers of slots and sectors per slot for both lap and wave machines are added, the sequence of numbers in each row of columns 1 and 7 corresponding to increasing values of m (1, 2, 3 . . .) for lap machines and of varying values of s for wave machines reckoning from a_{min}. In the latter, values of a above 2 or 3 seldom occur in practice, but are included to indicate the result that follows from alteration of the commutator pitch. Further, Table III has already shown that for a given value of p, the same values of a recur with different values for x and y, giving the same product xy, so that in Table V to avoid unnecessary repetition in the case of parent machines with the higher number of poles c is not directly given, but in its place the value which $x'y'$ must have. It will also be understood that any of the alternative bracketed values of S can be employed with a value of c or of $x'y'$ to give a the values which appear in the same horizontal row.

The great number of possible values of a that wave winding allows, due to the possibility in it of a fractional creep, is apparent.

§ 16. Simplex wave-windings ($a = 1$). It will have been noticed from Table V that all cases of simplex wave-windings (i.e. $a = 1$) appear on the left-hand side of the table as original types, and it is of interest to collect these cases and discover what values of c are possible with each value of p as in Table VI (p. 252).

It will be seen that $x'y'$ must be 1 or differ from p' or a multiple of p' by 1. y' must not have any common factor higher than 1 with p', and consequently the same is true for c. There is a good choice in the possible values for c, especially when p' is a prime number, provided that the number of slots is chosen to suit.

But when we pass to duplex, triplex . . . wave windings with $a = 2, 3 \ldots$, it will be found that each value of a appears on both sides of Table V. The difference thereby indicated being that in the one case there are repetitions of E.M.F. and in the other case there are none, an important consequence flowing from it is traced in §§ 18, 20.

§ 17. The choice of a continuous-current armature winding.—The choice between the various alternative methods of drum-winding for a continuous-current armature turns in the first place mainly upon (1) the value of the total current I_a, since upon this depends the question of the number of parallel paths q which it is advisable for the armature to present for the passage of the current, and indirectly the best number of poles. In order to limit the

inductance which retards the commutation of the current in each short-circuited coil as it passes under a brush, it is inadvisable for the current in any one path, or $J = I_a/q$, to exceed 150–200 amperes, and to avoid an unduly long and unwieldy commutator, the current per brush arm should not exceed 400 amperes. With commutating poles J may be raised to 350–400 or even more amperes per path, when special occasion requires it, and the long commutator thereby involved cannot otherwise be avoided.

TABLE VI

SIMPLEX WAVE WINDINGS, $a = 1$

No. of pole pairs p'.	No. of slots S'.		Possible Values of c. Relatively to S'.	Possible Values of c. Joint possibilities relatively to p'.
1	n_1	or n_2	= 1, 2, 3, 4 . . .	= any whole number.
2	$2n_1 \pm 1$	$2n_2 \pm 1$	= 1, 3, 5, 7 . . .	= any uneven no.
3	$3n_1 \pm 1$	$3n_2 \pm 1$	= 1, 2, 4, 5 . . .	= any no. except 3 or multiples of 3
4	$4n_1 \pm 1$	$4n_2 \pm 1$	= 1, 3, 5, 7 . . .	= any uneven no.
5	$5n_1 \pm 1$ $5n_1 \pm 2$	$5n_2 \pm 1$ $5n_2 \pm 2$	= 1, 4, 6, 9 . . . = 2, 3, 7, 8 . . .	= any no. except 5 or multiples of 5
6	$6n_1 \pm 1$	$6n_2 \pm 1$	= 1, 5, 7 . . .	= any no. prime to 6.
7	$7n_1 \pm 1$ $7n_1 \pm 2$ $7n_1 \pm 3$	$7n_2 \pm 1$ $7n_2 \pm 3$ $7n_2 \pm 2$	= 1, 6, 8 . . . = 3, 4, 10 . . . = 2, 5, 9 . . .	= any no. except 7 or multiples of 7
8	$8n_1 \pm 1$ $8n_1 \pm 3$	$8n_2 \pm 1$ $8n_2 \pm 3$	= 1, 7, 9 . . . = 3, 5, 11 . . .	= any uneven no.
9	$9n_1 \pm 1$ $9n_1 \pm 2$ $9n_1 \pm 4$	$9n_2 \pm 1$ $9n_2 \pm 4$ $9n_2 \pm 2$	= 1, 8, 10 = 4, 5, 13 . . . = 2, 7, 11 . . .	= any no. prime to 9
10	$10n_1 \pm 1$ $10n_1 \pm 3$	$10n_2 \pm 1$ $10n_2 \pm 3$	= 1, 9, 11 . . . = 3, 7, 13 . . .	= any no. prime to 10
11	$11n_1 \pm 1$ $11n_1 \pm 2$ $11n_1 \pm 3$ $11n_1 \pm 4$ $11n_1 \pm 5$	$11n_2 \pm 1$ $11n_2 \pm 5$ $11n_2 \pm 4$ $11n_2 \pm 3$ $11n_2 \pm 2$	= 1, 10, 12 . . . = 5, 6, 16 . . . = 4, 7, 15 . . . = 3, 8, 14 . . . = 2, 9, 13 . . .	= any no. except 11 or multiples of 11.
12	$12n_1 \pm 1$ $12n_1 \pm 5$	$12n_2 \pm 1$ $12n_2 \pm 5$	= 1, 11, 13 . . . = 5, 7, 17 . . .	= any no. prime to 12.

Trial of a simplex lap-winding, with each coil consisting of a single turn only, is then to be recommended, and this as the best and simplest of windings should be discarded only when its use is forbidden by the considerations to be named below. The practised designer knows at the outset its limitations and when it must be departed from, but for the beginner and less experienced designer, it is advocated as the starting-point in every case. The further considerations that may necessitate its abandonment are as follows—

(2) The magnet must not prove unduly heavy or the armature core too long; see Chapter XV, § 17.

(3) It may lead to a width of commutator sector less than the minimum that is prescribed by mechanical and manufacturing

reasons. In anticipation of Chapter XIII, § 30, it will here be stated that the peripheral width of a commutator sector at its top should not be less than 0·190", so that with mica 0·020" thick the minimum width of the pitch of the commutator sectors is 0·210", or, say, 5 mm.; further, for preliminary calculation the commutator diameter may be taken as 0·75 D for small or 0·6 D for large machines, where D is the over-all diameter of the armature. To lessen the number of sectors and so increase their width for a given diameter of commutator, the use of coils each with 2 or more turns suggests itself, but the increased difficulty of commutation owing to the inductance of each short-circuited coil increasing roughly as the square of the number of its turns introduces the restrictions next to be mentioned.

(4) In all cases of coils with different numbers of turns on similar machines, the current J must vary nearly inversely to the square of the number of turns to maintain the same freedom from sparking. Generally speaking, in practice

with one turn per section, J = 200 amperes
,, two turns ,, J = 50 ,,
,, three ,, ,, J = 22·5 ,,

(5) The average voltage between two adjacent sectors, $2p \cdot V_t/C$ should not exceed about 15 volts (Chapter X, § 13), and lastly

(6) the number of sectors per pole, $C/2p$, should not fall below a minimum of 15 of the number of slots below 10 even in small machines (Chapter X, § 15).

Whenever the total amperes do not exceed 300–400, it follows from (1) and (4) that q could be 2. In such cases with a simplex lap winding for which $q = 2p$, we are led to a single pair of poles. Yet experience has shown that, except for very small outputs at high speeds, a 4-pole machine is in general cheaper to manufacture, since among other reasons the multipolar field renders possible the use of former-wound coils which are in themselves cheaply produced. Further, the yoke and magnetic circuit of the bipolar machine is heavier, and even though its field copper may be less than that of the 4-pole machine, the total cost of the magnet is greater.[1] A 2-pole magnet is therefore put out of court by (2), and a 4-pole field magnet is much more likely to exist as the standard pattern to which the designer must conform. From the data of the required output and speed the designer can estimate approximately what size of armature core is suitable, and which of the standards at his disposal fulfils the requirements most nearly. It will now be found that up to 300 amperes in most cases a simplex

[1] See Miles Walker, *The Specification and Design of Dynamo-Electric Machinery*, pp. 11, 12.

lap-winding leads to commutator sectors which are too thin, and too numerous for economical manufacture, and if the lap-winding is to be retained, recourse must be had to coils with 2 or more turns. But a single-turn wave-winding in a 4-pole field has the same number of sectors as lap-winding with 2 turns per coil; the former is then to be preferred as cheaper (*cp.* Chapter XI, § 22) and better for manufacturing reasons owing to the size of the conductor being more substantial. Throughout the present range of amperes, therefore, simplex wave-winding ($a = 1$) in general takes precedence over lap-winding. The only limit to this is in the case of such high speeds and low voltages that the number of commutator sectors per pole falls below the minimum of 15, when a simplex lap-winding will be adopted in preference.[1]

When the total amperes exceed 300, and a 4-pole field is still suitable, the transition is made to single-turn simplex lap-winding, the dividing line being drawn according to the circumstances of the particular case.

When I_a is $>$ 400 amperes, the number of pairs of armature paths or of pole-pairs need not exceed $a = I_a/400$, but as an empirical expression in better agreement with average cases may be taken the next larger whole number to $a = I_a/300$, and the next step is to try provisionally the corresponding number of poles, viz. $p = a$. Assuming then a magnet frame with the required number of poles to be available and that it does not contravene condition (2) above given, single-turn simplex lap-winding will henceforth be in all cases adopted, but with the limitation that those combinations only are to be used which appear on the right-hand side of Table V with a definite preference for those which permit of the full use of equalizing connections. The lap-winding ensures more positively than wave winding equal division of the total armature current between the several sets of brushes of the same sign, owing to its freedom from "selective commutation," and the reason for its adoption in preference to wave-winding with $a = p$ in both cases has already been stated in § 12, apart from the fact that the winding of the wave machine in such cases is slightly more expensive. The assumptions made before adopting the single-turn simplex lap winding have, however, indicated the cases when wave-winding again becomes the best, but now in its multiplex form with $a > 1$ and $< p$. These, with other exceptions to the general adoption of simplex lap winding, have now shortly to be considered.

(*a*) The size of machine necessarily increases with the output per rev. per min., and it is not practically advisable to increase the length beyond certain well-defined limits (*cp.* condition (2) above and Chapter XV, § 16), so that it is the diameter rather than the

[1] Occasionally a 3-turn lap-winding may work in more conveniently than either a single-turn or 2-turn wave between which it falls.

length of the armature core in which the increase is actually made to meet increasing outputs. Next, for every diameter of core there is a *minimum* number of poles below which it is not advisable to go, owing to the great weight of iron in the yoke (*cp.* Chapter XV, § 16). Thus whenever an output of high voltage at a low speed demands a machine of large diameter but is combined with such a comparatively small current that $p = I_a/300$ gives a number of poles below the minimum for the diameter, a multiplex wave-winding will be adopted.[1]

(*b*) The same alternative of multiplex wave-winding ($a < p$) also enables the designer to meet the case when a pattern already exists in which the number of poles is larger than the current per path calls for, and the coils or bars and sectors of the lap-winding become too numerous and small for cheap and easy production. Even when the discrepancy between the number of pole-pairs that gives a light and economical magnet-system and that given by $I_a/300 = a$ is not very great, between each pair of consecutive pole-numbers there often occur cases of outputs for which single-turn simplex lap-winding with the smaller number of poles makes the machine heavy and long and with the next larger number gives too many commutator sectors. The larger number of pole-pairs with multiplex wave-winding is then to be adopted. Thus a four-path winding ($a = 2$) can be used in an 8-pole machine,[2] and a 4- or 6-path winding in a 12-pole machine. The combinations of a 6-path wave-winding ($a = 3$) in the 8-pole machine, and of a 4-path wave-winding ($a = 2$) in a 6-pole machine are here not included for a reason to be mentioned in § 18.

It will be noticed that the cases under (*a*) and (*b*) are in fact analogous to our first use of simplex wave-winding with four poles in preference to a bipolar lap-wound machine.

(*c*) Next, if the armature current be large, the voltage low and the speed comparatively high, as in machines for electro-deposition or electrolytic purposes, and the number of poles required by simple lap-winding is beyond that which the existing pattern of field-magnet possesses or exceeds the limit usually called for in commercial practice, two alternative courses are open to the designer. A compromise may, in the first place, be effected by adopting the smaller number of poles and a multiplex lap-winding giving $2mp$ parallel paths. *E.g.* if the simple lap-winding leads to eight poles, but four poles are otherwise more advantageous, an 8-path

[1] A good example is found in the design of a 500 H.P. 500-volt rolling-mill motor running at the slow speed of 32 revs. per min., given by Miles Walker, *The Specification and Design of Dynamo-electric Machinery*, p. 511.

[2] *Cp.* S. P. Smith, *Notes on Theory and Design of Continuous-current Machines*, p. 38, where the *pros* and *cons* for the winding of a 350 kW. 500-volt machine at 200 revs. per min. are clearly set forth; also Miles Walker, *loc. cit.*, p. 513.

we take $N = 3$, 4, or 6 as being the usual numbers of phases required in practice, the condition is satisfied by making $S' = 3n_1$ or $4n_1$ or $6n_1$ in the three cases.

But each multipolar simplex wave-winding is also an original type which cannot be obtained by multiplication of the pole-pairs from the two-pole wave-wound machine. In these cases again S' must be exactly divisible by N, and since S' now $= p'n_1 \pm x'$, $\frac{p'n_1 \pm x'}{N}$ must be a whole number. If p' and N are identical, $\frac{p'n_1 \pm x'}{N}$ cannot be a whole number since $\frac{x'}{N}$ would then $= \frac{x'}{p'}$, and this is fractional; and if p' and N have any common factor greater than 1, $\frac{p'n_1 \pm x'}{N}$ will only yield a whole number, when x' and N have the same common factor as p' and N. Otherwise, taking out the common factor of p' and N would leave $\frac{p'n_1}{N}$ with a smaller denominator than $\frac{x'}{N}$, and their sum cannot be a whole number.

Taking $N = 3$, 4, or 6, and applying these criteria to the simplex cases of Table VI, it is found that—

A machine with $p' = 2, 4, 8$, or 10 cannot be tapped for 4 or 6 phases;
" " $p' = 3$ or 9 " " " 3 or 6 "
" " $p' = 6$ or 12 " " " 3, 4, or 6 "

and within the range considered only machines with $p' = 1, 5, 7$, or 11 can be tapped for either 3, 4, or 6 phases.[1] Since S' and p' have no common factor higher than 1, and S' is to be exactly divisible by N, it follows that p' must not be exactly divisible by N.

In the possible cases, to determine the number of slots divisible by N and conforming to a given type $p'n_1 \pm x'$, the simplest method is to determine in the first place the lowest value of $p'q \pm x'$ which is divisible by N, *i.e.* q is to be the lowest integer which will make $\frac{p'q \pm x'}{N}$ = a whole number, say $\pm k$; then to the number of slots $\pm kN$ can always be added any number n_1 of groups of $p'N$ slots. The complete expression for the total number of slots is then for N phases—

$$S' = p'Nn_1 \pm kN.$$

Thus, for $N = 3$, let $p' = 4$; then the lowest value of $4q \pm 1$, which is divisible by 3, is $4 \times 1 - 1 = 3$, and the general expression for S' is $p'Nn_1 \pm kN = 4 \times 3n_1 \pm 3 = 12n_1 \pm 3$. But if $p' = 5$, and $x' = 1$, the lowest value of $5q \pm 1$ divisible by 3 is $5 \times 1 + 1 = 6$, and $p'Nn_1 \pm kN = 5 \times 3n_1 \pm 6 = 15n_1 \pm 6$, so that the possible numbers of slots go up by alternate steps of 12 and 3.

We thus find for simplex wave windings divisible into N phases the values of S' given in Table VII,[2] which also covers the 2 pole lap armature. The alternatives correspond to the alternative values of $p'n_1 \pm x'$ in Table V.

The slots having been determined, the phase-pitch is y_{ph} in slots $= S'/N$, or in coils or sectors $= cS'/N$, so that—

Phase I is connected to sector or coil 1
" II " " " " 1 + y_{ph}
" III " " " " 1 + $2y_{ph}$

and

Phase N " " " " 1 + $(N - 1)\, y_{ph}$

[1] S. P. Smith, "The Theory of Armature Windings," *Journ. I.E.E.*, Vol. 55, pp. 22 and 32.
[2] S. P. Smith, *loc. cit.*

TABLE VII

No. of pole-pairs. p'.	Number of Slots, S', when: $N = 3$.	$N = 4$.	$N = 6$.	Type of Winding.
1	$3n_1$	$4n_1$	$6n_1$	Lap or wave 2-pole.
2	$6n_1 + 3$	—	—	Simplex multipolar wave-windings.
3	—	$12n_1 \pm 4$	—	
4	$12n_1 \pm 3$	—	—	
5	$15n_1 \pm 6$	$20n_1 \pm 4$	$30n_1 \pm 6$	
	$15n_1 \pm 3$	$20n_1 \pm 8$	$30n_1 \pm 12$	
7	$21n_1 \pm 6$	$28n_1 \pm 8$	$42n_1 \pm 6$	
	$21n_1 \pm 9$	$28n_1 \pm 12$	$42n_1 \pm 12$	
	$21n_1 \pm 3$	$28n_1 \pm 4$	$42n_1 \pm 18$	
8	$24n_1 \pm 9$	—	—	
	$24n_1 \pm 3$	—	—	
9	—	$36n_1 \pm 8$	—	
		$36n_1 \pm 16$		
		$36n_1 \pm 4$		
10	$30n_1 \pm 9$	—	—	
	$30n_1 \pm 3$	—	—	
11	$33n_1 \pm 12$	$44n_1 \pm 12$	$66n_1 \pm 12$	
	$33n_1 \pm 9$	$44n_1 \pm 20$	$66n_1 \pm 24$	
	$33n_1 \pm 3$	$44n_1 \pm 8$	$66n_1 \pm 30$	
	$33n_1 \pm 15$	$44n_1 \pm 4$	$66n_1 \pm 18$	
	$33n_1 \pm 6$	$44n_1 \pm 16$	$66n_1 \pm 6$	

§ 20. The tapping of derivative machines for N phases.—In the cases given in the preceding section, there are no repetitions of E.M.F. and no points of equal potential to which equalizer connections can be attached, and as a consequence there is no possibility of employing parallel paths in each phase. For this to be possible the multipolar machine must contain t repetitions of E.M.F., so that one tapping can be connected to t points in the winding at distances of the equipotential pitch, y_{eq}, apart. But for the purpose of discovering whether there are accessible points in the winding for the tappings for N phases, only the parent machine containing S' slots and p' pole-pairs which serves as the original must be considered. It is not therefore sufficient in multipolar simplex lap-winding or multiplex wave-windings that S should be divisible by N; the stricter condition must be fulfilled that $S/t = S'$ or the slots included in one repetition should be divisible by N. The number of slots in the multipolar simplex lap-winding must therefore be $3pn_1$, $4pn_1$, or $6pn_1$ in the three cases, and in the multiplex wave-windings it must be t times the numbers given for $p' = p/t$ pole-pairs.

Lastly, in multiplex wave-windings it is only the symmetrical windings marked in heavier figures on the right-hand side of Table V which can be tapped when $a > 1$; for these are the only windings derived from parent machines with $a' = 1$, which alone can be tapped for N phases when S' is given its correct value.

The phase pitch is y_{ph} in slots $= S'/N$, or in coils $= cS'/N$, and

Phase I taps in sectors or coils,

$1,\quad 1+y_{eq},\quad 1+2y_{eq},\ \ldots\ldots\ldots\ 1+(f-1)y_{eq};$

Phase II,

$1+y_{ph},\quad 1+y_{ph}+y_{eq},\quad 1+y_{ph}+2y_{eq}\ \ldots\ldots\ 1+y_{ph}+(f-1)y_{eq};$

Phase III,

$1+2y_{ph},\quad 1+2y_{ph}+y_{eq},\quad 1+2y_{ph}+2y_{eq}\ \ldots\ldots\ 1+2y_{ph}+(f-1)y_{eq};$

and Phase N,

$1+(N-1)y_{ph},\quad 1+(N-1)y_{ph}+y_{eq},\quad 1+(N-1)y_{ph}+2y_{eq}\ \ldots\ 1+(N-1)y_{ph}+(f-1)y_{eq}$

NOTE TO CHAPTER XII

Armature Circulating Currents in Lap-wound Multipolars[1]

When a lap-wound drum armature is displaced eccentrically within the bore of a multipolar magnet system, the reaction from internal currents circulating at no load within the armature (or under load from unequal distribution of the actual armature currents in the different branches of the winding) tends to compensate for the inequalities in the pole strengths by which the circulating currents have themselves been originated. It is here assumed that the armatures considered are not fitted with equalizing rings, since the purpose of these is as far as possible to prevent the passage of circulating currents through the brushes, yet it is only when they do so pass that they develop to its full the above-mentioned property of automatic compensation. The exact mechanism, magnetic and electric, by which this desirable effect is produced in the 4-pole machine has been explained by Dr W. Lulofs,[2] and the explanation is here repeated in a slightly different form, which does not affect the principle.

I. The superposed fluxes representing the effect of, say, vertical displacement in line with a pole in a 4-pole field are indicated in Fig. 112 (a). It may be explained that it is here a matter of indifference whether the magnetic lines (shown open-ended) are closed *through* the horizontal poles and air-gaps or whether they are joined up to form a two-pole field. In the former case the direction of the lines in the two halves of a horizontal pole or air-gap being opposed, the net E.M.F. due to the horizontal poles is zero, or physically speaking, the increased and reduced fluxes will in nature distribute themselves almost uniformly over the horizontal poles and air-gaps, and if these fluxes are for the present purpose regarded as greater or less than the normal by the same amount, there is no superposed flux to be considered therein. From the E.M.F.'s shown by crosses and dots in Fig. 112 (b) there arise what may be called primary circulating currents, I_1, passing through each half of the armature winding and uniting to flow through the top and bottom brushes and the lead connecting them. If r = the resistance of one of the four branches of the armature winding, r_b = the resistance of one brush set, and R_c the resistance of the connecting lead, and e_1 be the extra E.M.F. induced in one branch of the winding under two poles,

$$I_1 = \frac{2e_1}{2r + 4r_b + 2R_c} = \frac{e_1}{r + 2r_b + R_c}$$

From the way in which the primary currents are distributed (Fig. 112 (c)) the magnetic effects of the side conductors cancel out, and the net magnetic effect is that due to the ampere-turns that are left as shown in Fig. 112 (d). By comparison with the direction of the currents in the exciting wires on the poles which are also shown in Fig. 112 (d), it will be seen that on *the actual magnetic circuits* of the machine (shown dotted) the general result is a strengthening or "forward" effect (F, F), increasing the normal fluxes on the one side of the vertical diameter (the left side with the assumed polarities and direction of rotation); and a weakening or "back" effect (B, B), reducing the fluxes on the other side of the vertical diameter. The line of the magnetic effect of the primary currents is thus displaced 90° from the line of the eccentricity.

The magnetic result of the primary currents is therefore precisely the same as if the armature were displaced horizontally (in our case to the left), and all the same phenomena of Figs. 112 (a)–(c) are repeated with secondary E.M.F.'s and secondary currents, e_2 and I_2, displaced through 90°. Since S. poles are now strengthened and weakened, the new superposed fluxes would be represented by the fluxes of Fig. 112 (a) displaced *forwards* 90°, the poles remaining stationary. In consequence the new secondary E.M.F.'s and currents are represented by Figs. 112 (b) and (c) similarly displaced 90° forwards,

[1] Abbreviated mainly from an article by the writer in *Electrician*, Vol. 72, p. 901.
[2] *Journ. I.E.E.*, Vol. 43, p. 166; and *Electrician*, Vol. 70, p. 303.

as shown in Fig. 112 (*e*) and (*f*). By analogy $I_2 = \frac{e_2}{r + 2r_b + R_e}$, where e_2 is the E.M.F. due to the magnetizing ampere-turns of Fig. 112 (*d*). The

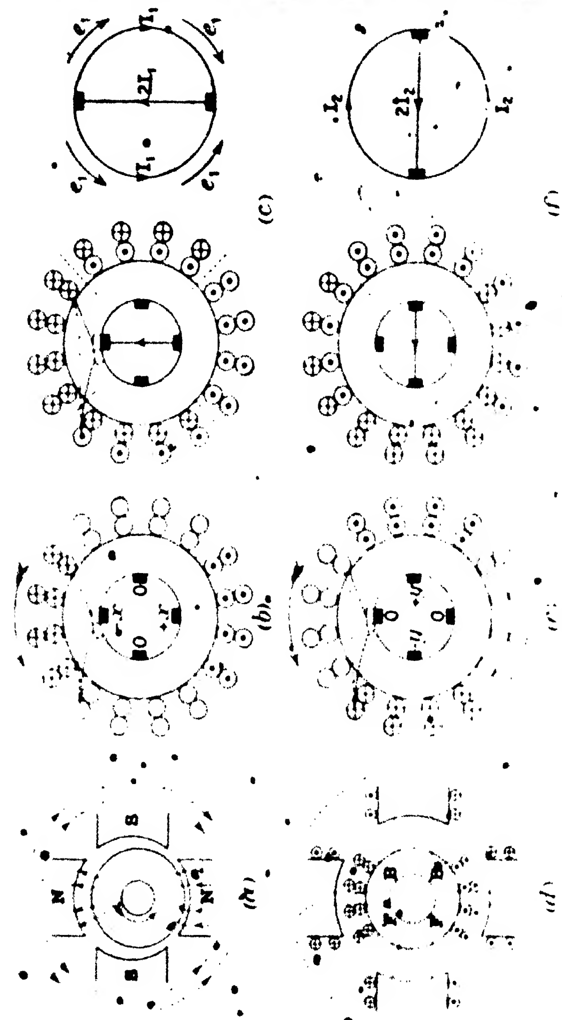


FIG. 112.—Vertical displacement in 4-pole field.

(*a*) The superposed flux representing effect of eccentricity. (*b*) E.M.F.'s due to (*a*). (*c*) Primary circulating currents due to (*b*). (*d*) The primary magnetising ampere conductors. (*e*) Secondary E.M.F.'s due to (*d*). (*f*) Secondary currents due to (*e*).

magnetic effect of the secondary currents on the actual magnetic circuits of the machine will be represented by Fig. 112 (*d*) with the armature rotated forwards through 90°. It is thus displaced 180° in relation to the fluxes of Fig. 112 (*a*), and is in direct opposition to them. The secondary currents, therefore, oppose the cause from which they originally arose, and partially counterbalance the effect of the eccentricity.

Instead of considering the effect of the primary and secondary currents on the actual magnetic circuits of the machine, it would be equally legitimate in the present case to credit each system with a two-pole field or a set of lines similar to that of Fig. 112 (a), but the former method of treatment is adopted since it will be found to give the key to the cases of machines with more than four poles.

The combination of the two sets of currents, i.e. of Figs. 112 (c) and (f), yields the result shown in Fig. 113 (a) and (b), with a resultant magnetic effect shown in Fig. 113 (c). If, therefore, e_1 and I_1 be the primary E.M.F.

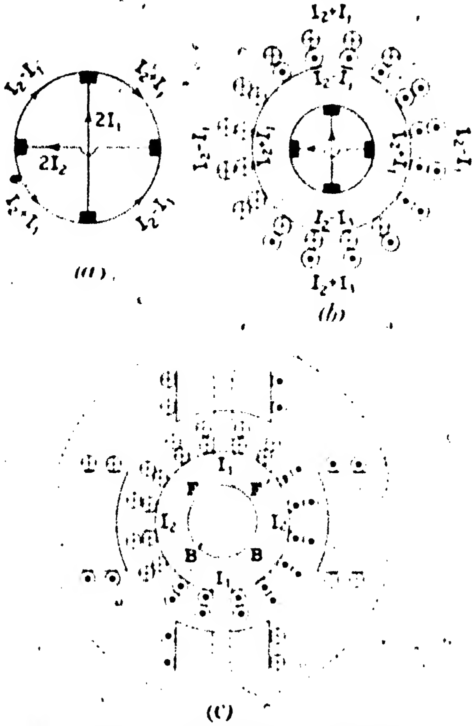
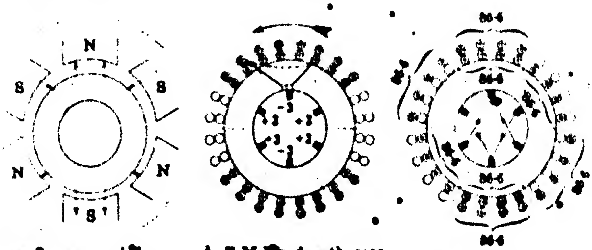


FIG. 113.—Combined primary and secondary effect in a 4-pole machine.

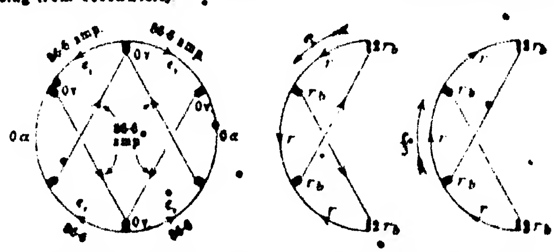
(a) and (b) The combination of (c) and (f) of Fig. 112. (c) The resultant magnetizing ampere conductors from (b).

and current due not simply to the superposed flux representing the eccentricity considered by itself, but due to this flux *as cut down by the ampere-turns due to* I_2, and e_2 and I_2 be the secondary E.M.F. and current due to the current I_1 as above defined, Fig. 113 (c) represents the actual result in the 4-pole machine. The "forward" ampere-turns, F, partially neutralizing the effect of eccentricity, are proportional in the left-hand upper quadrant to the sum $I_2 + I_1$; on the right-hand upper quadrant the forward ampere-turns, F', are less powerful, being proportional to the difference $I_2 - I_1$. Similarly the back ampere-turns, also neutralizing the effect of eccentricity in the two lower quadrants, are B proportional to $I_2 + I_1$, and B' proportional to $I_2 - I_1$. A very small current I_1 will suffice to excite a small E.M.F. which,

owing to the low resistance of the circuit upon which it acts, will give a comparatively large secondary current; further, owing to the ineffective way in which the secondary ampere-turns are linked with the original magnetic circuits, a comparatively large secondary current will be required to produce much effect in the way of cutting down the flux due to eccentricity. Consequently the current I_2 much exceeds I_1, and this conclusion as shown by Dr. Lulofs is borne out experimentally. If I_2 were negligible, not only would the effect of eccentricity be partially balanced, but also the fluxes in the two upper quadrants would be identical among themselves, and similarly the fluxes of the two lower quadrants. In the actual machine, however, the effect is always as if the eccentricity were not only reduced in amount but shifted in direction, being more or less inclined to the vertical, as if compounded of a vertical and a horizontal displacement.



a Superposed flux arising from eccentricity *b* E.M.F.'s due thereto *c* Currents due thereto



d System of circulating currents *e* Components of Fig. 104 *d*

FIG. 114.—Effect of eccentricity of armature in a 6-pole machine.

If the original displacement were along an interpolar line, it may be regarded as the result of a horizontal and a vertical displacement. In this case the currents I_1 and I_2 would retain the same relative magnitude for each component displacement, but the total currents through the horizontal and vertical brush sets would then become equal.

II. Turning to the 6-pole machine, in which the eccentricity of the armature due to wear of the bearings is in a direct line with a pole (Fig. 114 (*a*)), the reduction of the flux in the upper pole and its two adjacent neighbours due to the longer air-gaps may be represented by a counter flux precisely analogous to that of Fig. 109 (*a*); the increased flux in the lower pole is similarly represented by an additional local flux. These two fluxes are symmetrical, and if of equal amount, then with the same numerical quantities as were employed in connection with Fig. 109, they give the E.M.F.'s between brushes shown in Fig. 114 (*b*), and the primary circulating currents of Fig. 114 (*c*) and (*d*). The brush connecting rings are assumed to be complete circles round the commutator. One-third of each ring (shown dotted in Fig. 114 (*c*)) carries no primary current, since the brushes connected by these thirds are at the same potentials.

It will be seen that the E.M.F.'s in each of the branches of the armature

winding on the horizontal diameter cancel out, the local fluxes which they cut being in opposite directions and supposed equal in amount. There is, therefore, no E.M.F. in the horizontal branches, and no current therein. The final distribution of the primary circulating currents due to Fig. 114 (*b*) is simply obtained from Fig. 109 (*d*) by folding it over on a horizontal diameter and adding the currents algebraically; thus the currents in the upper and lower branches are 49 + 37·6 = 86·6 amperes (Fig. 114 (*d*)), and in the horizontal or side branches 11·4 − 11·4 = 0. The system is symmetrical on either side of the vertical diameter, and the circulating current I_1 = 86·6 amps. passes in series through *two* branches of the armature winding, two brush connecting leads, two brush sets and the *halves* of two brush sets. Let e_1 = the voltage induced in one of the upper or lower branches; let r = the resistance of a branch of the armature winding, r_b = the resistance of one brush set, and R_e = the resistance of one lead short-circuiting two adjacent sets of brushes of the same sign. Then

$$I_1 = \frac{2e_1}{2r + 6r_b + 2R_e} = \frac{e_1}{r + 3r_b + R_e}$$

and with the values assumed in Chapter XII, § 1, R_e being supposed to be practically negligible, $3r_b + R_e = 0{\cdot}3r$, and

$$I_1 = \frac{6}{0{\cdot}0533 \times 1{\cdot}3} = 86{\cdot}6 \text{ amperes.}$$

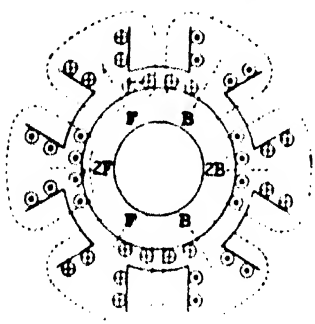


FIG. 114 (*f*). Magnetizing effect of primary ampere conductors in a 6-pole machine.

Not only are all brushes of the same sign reduced to the same potential, but from the symmetry of the arrangement all are reduced to zero. There is, therefore, unlike the case of Fig. 109 (*e*), no diminution of the terminal voltage, since the flux has been assumed to be as much strengthened in the lower half as it is weakened in the upper half.

In the unsymmetrical case of Fig. 109, the side branches of the armature winding each have an E.M.F. = $e_1/3$, and the currents of Fig. 109 (*d*) are obtained by superposing the two systems of Fig. 114 (*e*). The one gives a current

$$\frac{e_1}{r + 3r_b + R_e + \dfrac{r\,(r + 3r_b + R_e)}{2r + 3r_b + R_e}} = \frac{6}{0{\cdot}0533\left(1{\cdot}3 + \dfrac{1{\cdot}3}{2{\cdot}3}\right)} = 60{\cdot}4 \text{ amps.}$$

and the other

$$\frac{e_1/3}{r + \dfrac{r + 3r_b + R_e}{2}} = \frac{2}{0{\cdot}0533\,(1{\cdot}65)} = 22{\cdot}8 \text{ amps.}$$

Hence in an upper branch the current is 60·4 − 11·4 = 49, and in a lower branch 26·2 + 11·4 = 37·6; in the side branch, 34·2 − 22·8 = 11·4 amps.

The ampere-conductors of Fig. 114 (*c*) as acting on the magnetic circuits of the machine reduce themselves to the single layer of Fig. 114 (*f*), since any two wires carrying current in opposite directions and symmetrically situated on either side of an interpolar dotted line, so far as that magnetic circuit is concerned, annul one another. By comparison with the exciting wires on the poles it is seen that the ampere-conductors which are left have a forward (F) effect and a back (B) effect on the two upper and lower circuits, and twice as strong a forward ($2F$) and back ($2B$) effect on the side circuits. The total effect from Fig. 114 (*f*) is then to increase the fluxes on the left side of the vertical diameter and to reduce the fluxes on the right-hand side, *i.e.* to alter the pole-strengths on a line at right angles to the line of the eccentricity as in the 4-pole case. And this is in fact the universal law

in the lap-wound multipolar as regards the first effect from the primary circulating currents. As before, this effect would be reproduced by imagining the armature to be displaced horizontally to the left, giving rise to the superposed fluxes of Fig. 115 (*a*). The only difference is that the displacement

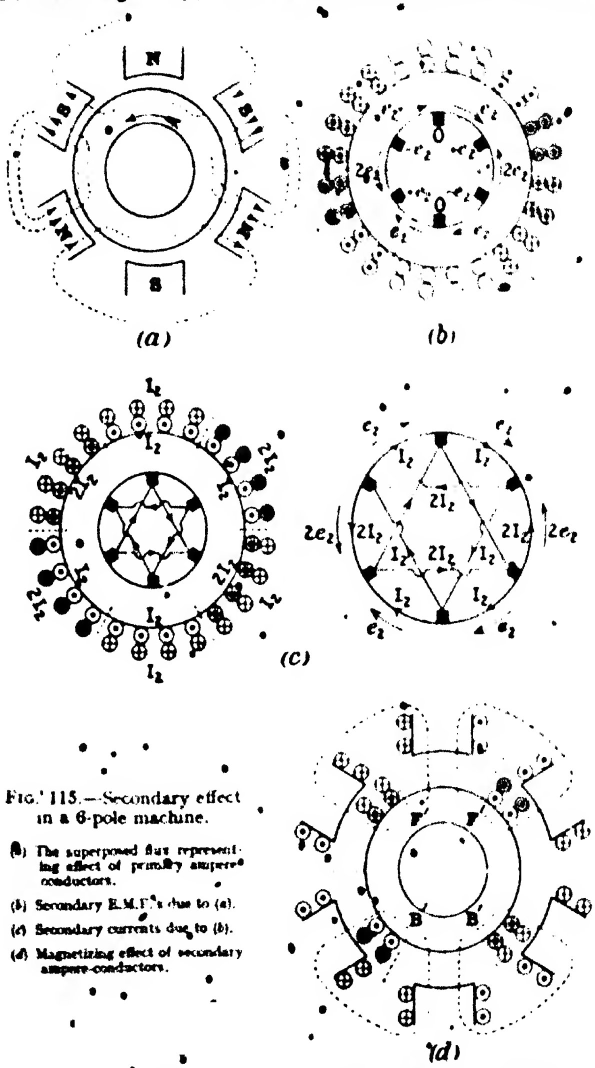


Fig. 115.—Secondary effect in a 6-pole machine.

(*a*) The superposed flux representing effect of primary ampere-conductors.

(*b*) Secondary E.M.F.'s due to (*a*).

(*c*) Secondary currents due to (*b*).

(*d*) Magnetizing effect of secondary ampere-conductors.

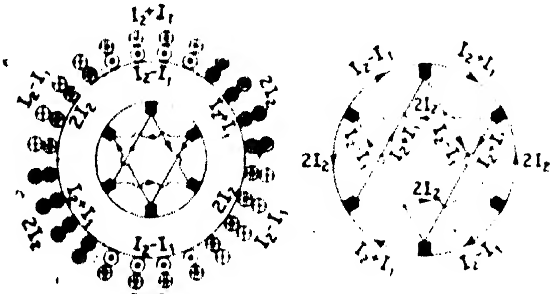
would now have to be made not in the direct line of a pole, but along an interpolar line between two poles. Consequently the distribution of the new superposed fluxes does not exactly reproduce that shown in Fig. 114 (*a*), but requires a new diagram as given in Fig. 115 (*a*). This difference in the

pictures of the two superposed fluxes is a characteristic of the machine with an uneven number of pairs of poles.

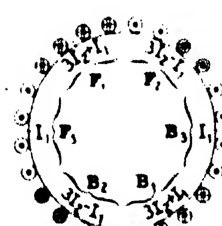
The system of secondary E.M.F.'s and currents is shown in Fig. 115 (*b*) and (*c*). The E.M.F. and the current in the circuits on the horizontal diameter must be double that of the other branches of the winding as indicated by the double crosses and double rings with central dot. The secondary current is

$$I_s = \frac{e_s}{r + 3r_b + R_c}$$

The current $3I_s$ crosses twice from one side of the armature to the other, $2I_s$ by a direct connecting lead and I_s via two connecting leads, touching at a brush but not re-entering the armature winding since the potentials of the points in contact are the same.



(*a*)



(*b*)

FIG. 116. Combined primary and secondary effects in a 6-pole field.

(*a*) The combination of Fig. 114 (*c*) and Fig. 115 (*c*).

(*b*) The resultant magnetising ampere conductors.

The magnetic effect of the secondary current-turns when grouped on the original magnetic circuits leave as their effective residue the groups shown in Fig. 115 (*d*), from which by comparison with the exciting wires on the poles, it will be seen that there is a forward effect (F') on the two upper circuits and a back (B') effect on the two lower, both tending to compensate for the originating inequalities in the pole-strengths. The fluxes due to Fig. 115 (*d*) would again reproduce Fig. 114 (*a*), but in exact opposition.

If the original displacement had been along an interpolar line, the primary circulating currents would have been given by Figs. 115 (*b*)-(*d*), and the secondary by Figs. 114 (*b*)-(*d*).

The combination of the primary and secondary systems in the present case, i.e. of Fig. 114 (*c*) and Fig. 115 (*c*) yields Fig. 116 (*c*). Their net effect on the original magnetic circuits may be obtained by adding Fig. 115 (*d*) to Fig. 114 (*f*), but it is equally and as easily obtained from Fig. 116 (*a*), by cancelling out pairs of wires which carry the same current in opposite directions and are symmetrically situated on either side of an interpolar line up to the

centre of a pole. Thus Fig. 116 (*b*) is equivalent to Figs. 115 (*d*) and 114 (*f*) combined, although this may not be evident at first sight. The apparently fragmentary nature of the groups of ampere-wires in Figs. 115 (*d*) and 114 (*f*), and their unsymmetrical distribution in relation to the magnetic circuits thus disappear when the complete result is reached, and form no real objection to the correctness of the method employed. It will be seen from Fig. 116 (*b*) that as before the final result is a partial equalization of the fluxes above and below the horizontal, and a virtual displacement of the armature along some line inclined to the vertical. On each magnetic circuit the effective ampere-turns due to the circulating currents are proportional either to $3I_2 + I_1$ or to $3I_2 - I_1$ or I_1, as marked for each sixth of the circumference. For quantitative calculations they would have to be multiplied in each case by some factor to take into account the fact that they do not all equally embrace the entire flux of the circuit.

In every case when once the system of circulating currents has been clearly established, the mental aid afforded by the paraphernalia of imaginary superposed fluxes due to these currents as intermediate steps in the argument may be dispensed with, and the attention riveted solely on the resultant ampere-turns as acting on each circuit.

III. The 8-pole case, as might be expected, more nearly resembles the 4-pole case in that the horizontal displacement which is virtually the equivalent of the "forward" or strengthening effect from the primary current turns on all the circuits to the left of the vertical and the corresponding back effect on all the circuits to the right of the vertical is once again in direct line with a pole. Consequently as in the 4-pole machine the additional superposed flux which would represent such a displacement of the armature repeats the original superposed flux from eccentricity in its distribution (although not in amount) if the latter be rotated through a right angle. But in contrast with a 4-pole machine, since the poles which are most weakened and strengthened by the horizontal displacement are of the same polarity as those which are weakened and strengthened by the vertical displacement, the primary superposed flux must be rotated *backwards* through 90°, instead of forwards, in order to give the secondary fluxes. Similarly the secondary E.M.F.'s and currents are also represented in direction by the primary E.M.F.'s and current rotated *backwards* through 90°.

The distribution of currents in the 8-pole case has been worked out by the writer in the above-mentioned article (*Electrician*, Vol. 72, p. 904), and need not here be repeated.

The general conclusion for all lap-wound multipolars is as follows:

Eccentricity of the armature within the bore sets up unbalanced E.M.F.'s and a system of primary circulating currents.

The magnetic effect of these strengthens the fluxes on one side of the line of eccentricity and weakens the fluxes on the other side. Hence it is virtually equivalent to a displacement of the armature along a line at right angles to the line of eccentricity.

The magnetic effect of this virtual displacement sets up a system of secondary E.M.F.'s and currents analogous to the former, and in the case of machines with an even number of pairs of poles exactly repeating them to some larger scale of values. With an odd number of pole pairs the virtual displacement is along an interpolar line if the real eccentricity is in line with a pole, or *vice versa*.

The magnetic effect of the secondary circulating currents is in exact opposition to the original effect of the eccentricity, and partially wipes it out.

The combination of the two sets of current in the actual machine means in reality an unequal distribution of current such that on each magnetic circuit there is more or less than the normal number of ampere-turns. The added or subtracted ampere-turns partially equalize the fluxes from the several poles, and the effect is the same as if the armature were displaced to a lesser degree and along some line inclined to the true line of the eccentricity. The larger the secondary currents as compared with the primary, the greater the equalization and the more nearly the remaining small virtual displacement is along the line of eccentricity.

CHAPTER XIII

CONTINUOUS-CURRENT ARMATURES

§ 1. Lamination of armature core to avoid eddy-currents. As mentioned in Chapter III, the use of an iron supporting core to carry the armature winding either necessitates its rotation together with the rotating wires, or, if the armature is stationary and the field-magnet revolves, implies that its surface is swept over by the moving field of lines. From this there follows the necessity for *laminating* the core. Whenever a metallic mass actively cuts lines of flux, it becomes the seat of induced E.M.F.'s, which, unless prevented from so doing, will set up electric currents; the direction of these in that part of their path where the E.M.F. is generated will be at right angles to the direction of the lines and to the direction of movement. If, therefore, the core whereon the active conductors are wound be formed of a solid metal mass, the latter will itself act inductively, and although the E.M.F.'s induced in it may be small, yet, owing to the low resistance of the numerous metallic paths open to them, they will cause large *eddy-currents* to circulate round the core; the latter will then become heated, and power will be wasted equal in amount to the product of the E.M.F. and the strength of the eddy-current multiplied by the cosine of the angle of lag of the latter. In a solid drum armature the eddy-currents will flow in sheets along the length of the iron core, in opposite directions under adjacent poles. The current-density will be greatest near the surface, where the rate of cutting lines and the induced E.M.F. is greatest; throughout the entire length of the core, and especially at its ends, the current-sheet under one pole will curve round on either side and again unite to complete its circuit under poles of opposite sign. In all cases the direction of the eddy-currents on the surface of the core will be similar to that of the useful currents in the active conductors, since these latter are expressly arranged to obtain the best inductive effect. So long as the metal core rotates, or is swept by the lines, it remains impossible to prevent the induction of E.M.F.'s in its surface, but it is possible to interpose in the paths which the eddy currents would follow such very high resistances as to prevent their attaining any appreciable strength. This is effected by laminating the core in planes at right angles to the active conductors, the edges of the laminations being presented to the lines of the field, and each lamination being separated from its neighbours by a thin layer or covering of some insulating material. The mass of metal is thus

broken up at the actual seat of the E.M.F. and transversely to its direction, and the subdivision must be especially perfect at the periphery, where the E.M.F. is greatest. Small E.M.F.'s will still be induced transversely across the surface of each lamination, but they will be prevented from acting summationally or in series by the intervening insulation; the currents that can be set up are thus reduced, and likewise the power that is absorbed (*cp.* Chapter XXI, § 17). The thinner the laminations, the less will be the amount of the eddies, but the greater the loss of space by the insulation for a given over-all length of core, and the limit to the advantage of further subdivision is reached when questions of the cost of manufacture or of the loss of space outweigh a possible slight gain in efficiency.

§ 2. Discs for drum armatures.—The above requirements are met by the use of thin *discs*, either threaded on to the arms of a central hub or keyed directly to the shaft or assembled inside an external case when the armature is the stationary member. Each core-disc is lightly insulated from its neighbours on either side; hence they make little or no contact with each other except through their connection with the shaft or case, and the passage of eddy-currents along the surface or through the mass of the core from one disc to another is very largely prevented.

The discs are stamped out of soft annealed sheets of mild steel,[1] and are afterwards notched with slots for toothed armatures. Their thickness does not exceed 0·025″ or No. 24 (sheet-iron gauge), and in general ranges[2] between 0·025″ and 0·016″ (No. 28) or, say, 0·6 and 0·4 mm.; below the latter limit the reduction in the eddy-current loss within the core is so slight that it does not compensate for the increased cost of the discs and the greater labour in assembling them. Up to 4 feet in diameter, discs may be procured as complete rings, each forming one stamping, but for armatures above 40 inches in diameter segments are preferably employed, as the larger discs buckle somewhat in the annealing process and are unwieldy to handle. The waste, too, is less when the width of the segments is chosen to fit the size of the sheets from which they are to be stamped. In slotted armatures each segment should, if possible, contain a whole number of teeth and slots. A small clearance of a few mils is allowed in the width of a segment as compared with the proportion of the circumference corresponding to the number of segments. The segmental discs are then assembled together into a ring with butt joints in any one layer, but so arranged that the joints of neighbouring layers do not coincide in position (Figs. 124

[1] See Chapter XIV, §§ 4, 5.

[2] The density being 7·8 grammes per c. cm., the weight per superficial square foot of the thicker dimension is 1 lb.; a thickness of 0·020 (No. 28) is widely adopted, its weight per sq. ft. being 0·81 lb.

and 127). Each disc is either coated with insulating varnish, paste, or paint, or is effectively separated from its neighbours by an interposed thin sheet of paper cut out to the required shape; or the oxide, which forms on the discs after heating, and which is practically a non-conductor, is relied on to insulate the discs. When varnish is employed, the discs are either painted by hand or are passed between rolls dipping into a bath of the varnish; if heated and put under pressure, they will then adhere so tightly that the whole core is formed into a solid and compact mass. After allowing for slight inequalities in the thickness of the discs, and for the two coats of varnish, one on either side, and measuring in all about 0·0006", from 90 to 92 per cent. of the total cross-section of the core may be counted on as the net area through which the magnetic flux passes. "Insulined" discs are largely supplied by Messrs. Joseph Sankey & Sons, in which one side is covered with a thin layer of an insulating paste. Paper, when handled as separate sheets to be laid between the discs, cannot be conveniently thinner than one or two mils, or 0·025 to 0·05 mm.; the net volume of iron in the core then amounts to 88 or 90 per cent. of the gross volume. In the case of toothed armatures, an extra allowance of 1 or 2 per cent. must be made for irregularities in the assembled discs, and a combination of the two methods is frequently employed; a thinner paper, less than 1 mil in thickness, is pasted by rolls on to the sheet iron with varnish and at the same time dried, before the discs are punched in the stamping press. The teeth then remain well insulated right up to their edges, and the work of clearing the paper from the slots of the assembled core is obviated.

Annealing reduces the loss by hysteresis, but it is not practically worth while to re-anneal the discs after the teeth have been stamped out round their periphery, if this process is carried out separately after the inside hole is punched. Compound dies may also be employed to stamp out the inside and the slots simultaneously, but only if the number of stampings in view warrants the first cost of the expensive die required. More usually the slots are punched out separately in a disc-notching press.

§ 3. Construction of continuous-current drum armature cores. — We now turn to the consideration of rotating armatures, which to a large extent cover the same ground as continuous-current machines, since in this class, as opposed to alternators, the armature is invariably the rotating portion.

In small 2-pole drum armatures the discs are usually threaded directly on the shaft, and held in position by a driving feather running along the length of the core. At the two ends of the core are cast-iron end-flanges, between which the discs are tightly compressed; one of these is first driven home against a collar on the shaft, the discs are then assembled, and after compression by

hydraulic or mechanical means the second end-flange is slid on and secured in its place by a tightening-nut screwing on to the shaft (Fig. 117). The end-flanges are best keyed to the shaft to avoid all risk of their working loose. For a given armature diameter the maximum radial depth of disc is thus obtained, but in larger machines, especially if multipolar, sufficient cross-section of iron

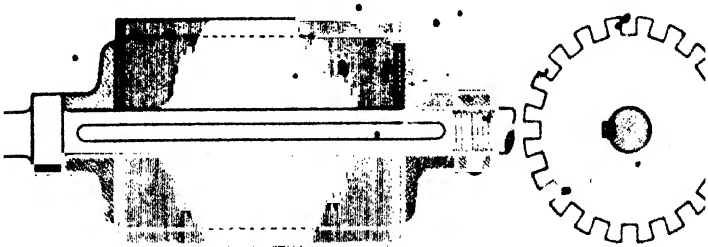


FIG. 117.—Small two-pole drum armature core.

is secured without using the full depth or difference between the radius of the disc and the radius of the shaft. The discs may then be pierced with holes towards their inner edge in order to lighten them and provide air canals (Figs. 118, 199, and 200), or they may be cut away on the inside, four projecting lugs being left, which fit

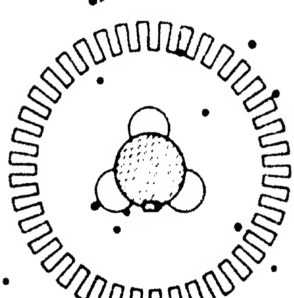


FIG. 118.—Drum armature disc with air spaces.

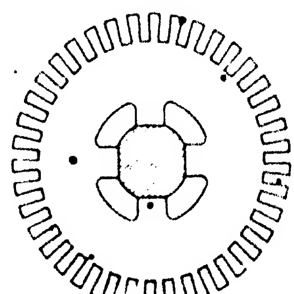


FIG. 119. Drum armature disc with internal lugs.

exactly on to four flats milled along the length of the shaft (Fig. 119). For armatures above 22 inches in diameter it is cheaper to employ discs of lesser depth supported on a cast-iron hub, which is itself pressed hydraulically on to the shaft over a strong key, and the use of such a hub is often extended to still smaller armatures. Keyways are stamped on the inner periphery of the discs to fit four or more arms or ribs which project radially from the central sleeve of the hub. The latter abuts against a collar on the shaft, and has

an end-flange either cast in one with it or fastened to it. The tips of the radial arms are turned in the lathe to the correct diameter to fit the inside of the keyways, and the sides of the arms are milled to the correct width to fit the notches; the discs are then slid over the radial arms and compressed by the second end-flange, which

FIG. 120.—Hub and end-flange of multipolar drum armature.

is screwed directly to the shaft, or, in larger machines, is fastened to the hub, making the core entirely self-contained. Fig. 120 shows a large hub for a multipolar machine, with one end-flange cast on, the second end-flange being shown at the side, and in Fig. 121 is seen the same hub with the discs partly assembled thereon; Fig. 199 shows a hub also with one end-flange cast on it, but at the opposite end to the coupling. In Fig. 122 both

end-flanges are separate from the hub, the actual combined hub and end-flanges being shown in Fig. 123 without the core-discs. A fixed end-flange has the disadvantage that it prevents the milling of the arms to their correct width at the extreme end, so that a central hub with two detached and similar end-flanges as in

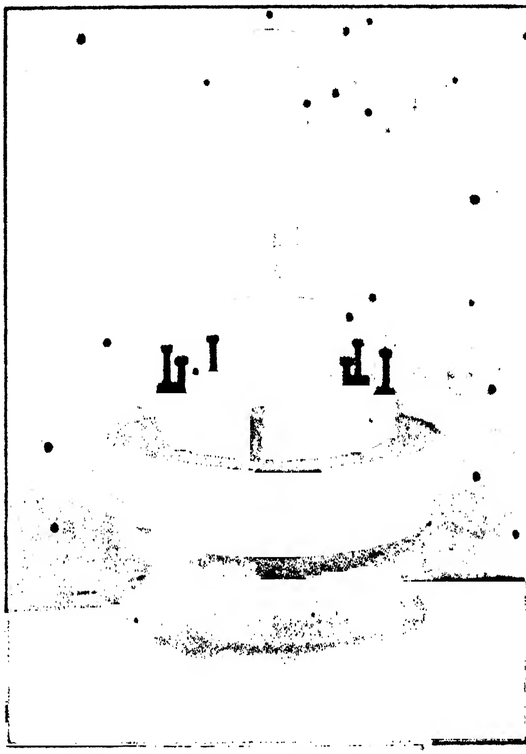


FIG. 124. — Core-discs being assembled on hub.

Figs. 122, 124–126 is to be recommended. At either end the compressing flange may be fixed to the hub by studs as in Fig. 122, or by bolts running from end to end of the core underneath the discs, as in Figs. 125 and 201. Other methods of fastening them are the use of a split ring of circular or rectangular section which is sprung into a recess in the hub where it is locked by the pressure of the discs, or a kind of bayonet joint, the flange being turned round through an angle after it is slid on to the hub; in both these cases

the discs are kept under compression during the process of fixing the locking arrangement.[1]

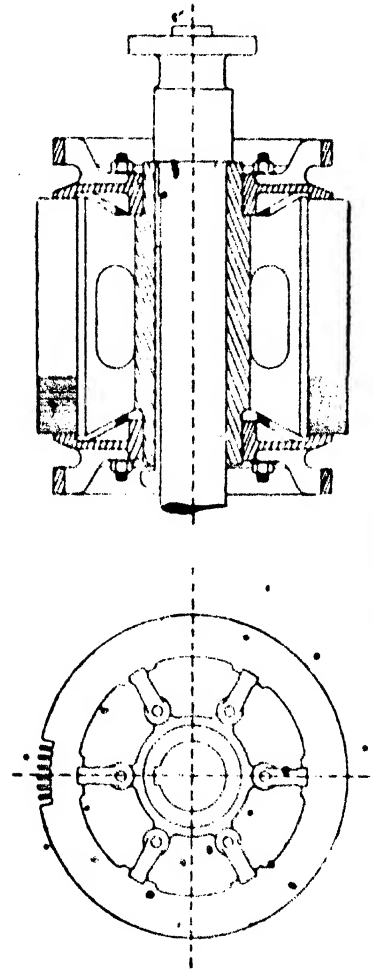


Fig. 123.—Multipolar drum armature core.

When the diameter of the armature is so large that the core must be built up out of segments, these may be held by bolts passing through holes (Fig. 124) near their inner edge, and clamping the

[1] Cp. R. Livingstone, *The Mechanical Design and Construction of Generators*, pp. 40–50.

segments to the end-flanges, or the discs may be keyed on wedge-shaped projections from the hub, with the compressing bolts lying

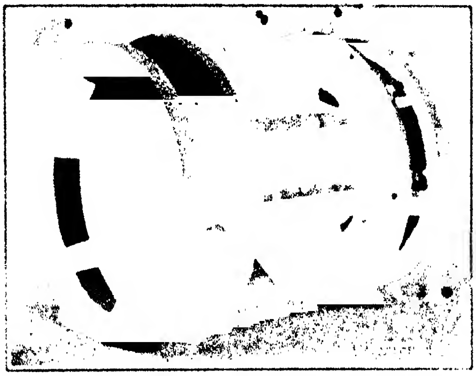


FIG. 123.—Hub and end-plates of armature.

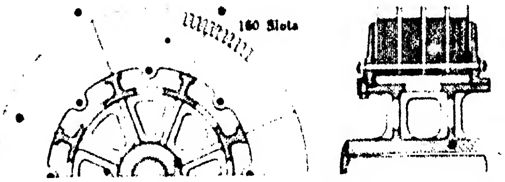


FIG. 124.—Segmental core-discs fastened by bolts to flanges.

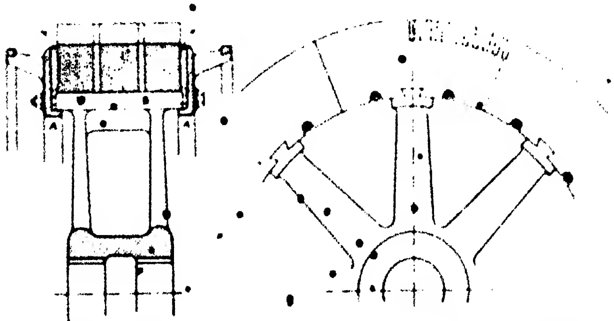


FIG. 125.—Segmental core-discs keyed on hub and compressed by bolts.

below the discs (Figs. 125 and 126). The end-flanges are conveniently made in several pieces, each being secured against centrifugal force by screws through a flange on the armature hub (Fig. 124), or by lugs engaging beneath a turned ring on the hub

(A, A, Fig. 125). Each segment must be held by at least two bolts or keyways. Or conversely to Fig. 126, the segments may be

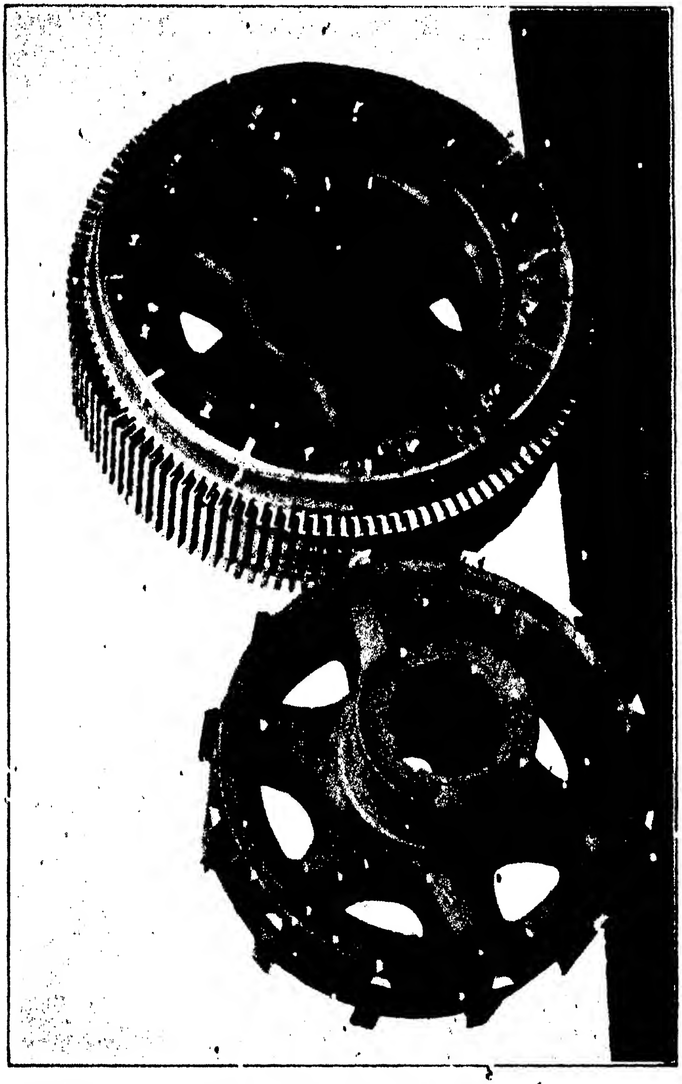


FIG. 126.—Multipolar armature cores in construction.

dovetailed to the arms of the hub by wedge-shaped lugs, so as to protect the discs from radial movement (Fig. 127) under the action of centrifugal force.

Large armature hubs usually have an extension at one end to carry the commutator sleeve, as in Figs. 128, 185, and 199; or the commutator is bolted up to the arms of the armature hub. Very large multipolar machines may have a double-armed hub, as in Fig. 125, with the nave split to avoid contraction strains, or they may be built up on the fly-wheel of the driving steam-engine.

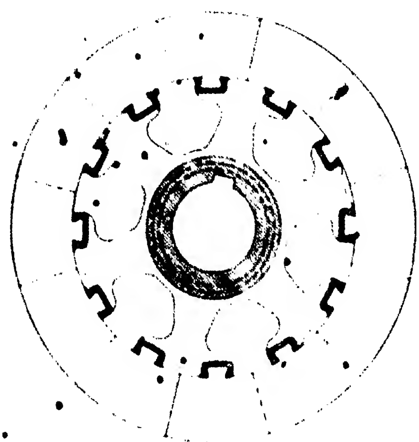


FIG. 127.—Segmental discs dovetailed to hub.

In small machines, if a tightening-up nut is employed to hold the armature core in place, it is advisable to lock it by a set screw, the point of which enters the shaft, since otherwise there is a possibility of the nut and discs slacking back through mechanical vibration. The nuts holding end-flanges in place may be fastened by split pins or by riveting over the heads of the studs or bolts.

At either end of the armature, especially when toothed, there are often placed one or more stout iron plates which serve to support the

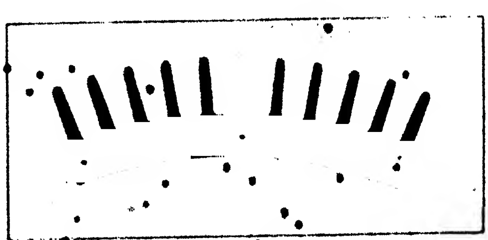


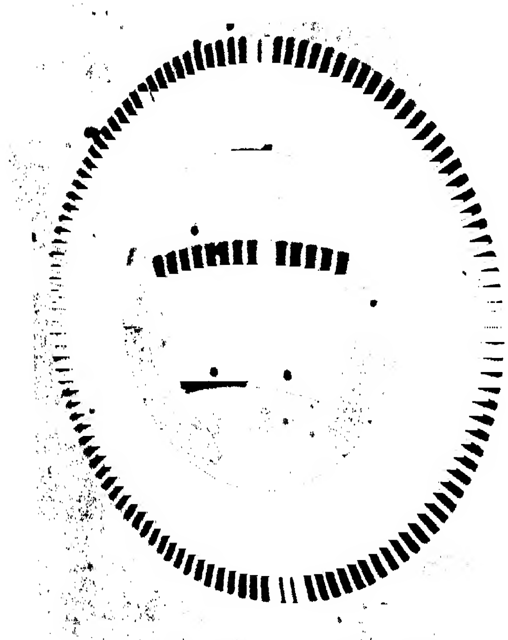
FIG. 128.—Pronged tooth support.

thinner discs where they are driven up against the end-flanges or end-arms of the hub. It must not be possible at any part of the surface to force the point of a thin knife blade between the discs.

In toothed cores, if the teeth are long and narrow, it becomes advisable to back them up at the ends of the core with prong supports of brass or malleable cast iron (Figs. 126 and 128), so that

not only the body of the core but also the teeth may be firmly compressed. Otherwise the distance-pieces forming the ventilating openings which are next to be described are not tightly held between the teeth, and if of iron they may under strong excitation vibrate slightly as they enter or leave a magnetic field.

At intervals of some 3 to 5 inches along the length of toothed cores (unless of small size), *air-ducts* or ventilating passages are formed by



The British Thomson-Houston Co. Ltd.

FIG. 129.—Spot welded ventilating disc (for armature with continuous ring stampings) and spacer (for segmental stampings).

distance-pieces between neighbouring discs. Such distance-pieces may be of thin iron or brass of channel or L section riveted or spot-welded to a core-plate to form an air-disc as in Fig. 129, or a number may be cast together in brass to form a segmental pronged strip, with wider lugs on its lower edge by which it is merely hooked under the core-plates as the armature is being built up (Fig. 130). In either case they must be mechanically strong, rigid, and well-secured. Free passages from $\frac{3}{8}''$ to $\frac{1}{2}''$ wide are thus obtained from the inner

air-chamber within the core to the outside, so that the air is drawn by centrifugal force through openings on the surface between the groups of coils in the slots, the numerous blades in the air-ducts forming a very effective fan. The air must have an unimpeded entrance into the centre of the core, in smaller machines through special holes in the end-flanges. It is especially important to keep the air-way through the narrow roots of the teeth as free as possible, so that the distance-pieces should there be thin; since without such precautions the usefulness of the ventilating discs is almost nullified.

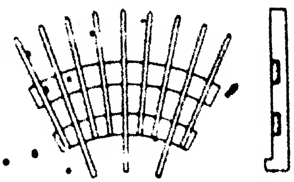


FIG. 130.—Ventilating distance piece.

Fig. 131 shows a completed core with three air-ducts, and similar air-ducts are also seen in Figs. 121 and 200. Ventilating passages at each end of the core between the end flanges and discs (as in Figs. 125 and 201) have considerable value, and do not take up any

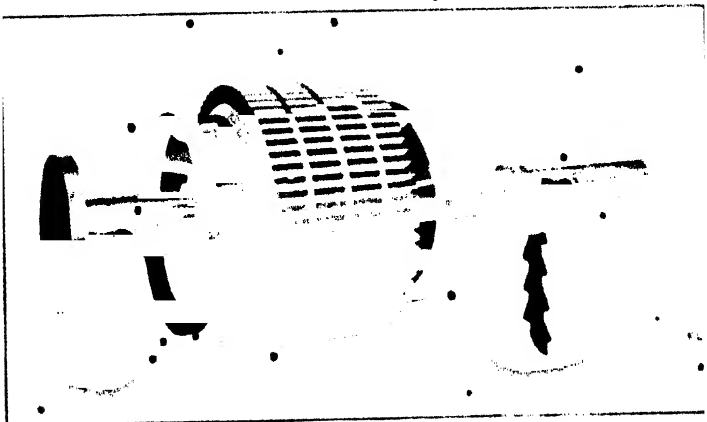


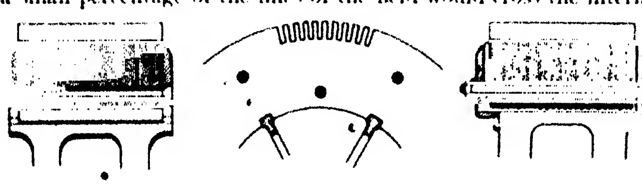
FIG. 131.—Completed drum toothed armature core.

part of the valuable core-length under the pole-face. Most of the cores illustrated above have outer rings cast on the end-flanges to carry the end-connectors of barrel winding, and the ribs which support these rings serve as the blades of a fan, or carry wings (Fig. 201) to dissipate the heat from the winding.

§ 4. Avoidance of eddy-currents.—All bolts used to hold the core in position and passing through the mass of the discs must either be so placed as to be cut by as few lines as possible (*i.e.* near to the inner circumference of the discs, well out of the path of the

magnetic flux, as in Figs. 124 and 125), or must be insulated from the core and end-plates by fibre tubes and fibre or mica washers (Fig. 132) in order to obstruct the paths that eddy-currents generated in them would follow (*vide* Chapter XXI, § 18).

In all cases care must be taken that the end-plates, end-flanges, supporting hubs, etc., are not the seat of eddy-currents; so far as possible they must be kept without the influence of the magnetic field, and hence in order to avoid eddy-currents set up in the thick end-plates by reason of lines curving round from the flanks of the poles into their outer surfaces, it is usual for the axial length of the armature core to exceed that of the pole-face by $\frac{1}{4}''$ to $\frac{1}{2}''$ at each end. In drum armatures with discs of considerable radial depth very few lines pass across from inner side to inner side of the discs through the air; yet if they were worked at a very high flux density, a small percentage of the lines of the field would cross the internal



I. II.

FIG. 132. Insulated bolts.

opening of the discs. Any such internal field would remain stationary, and would therefore be cut by the arms and sleeve of the central hub with consequent eddy-currents. The loss occasioned thereby would usually be small, but is one among other reasons which limit the permissible flux-density in the core. On general grounds, as well as to minimize any such loss, it is important to lighten the hub as far as may be consistent with the maintenance of sufficient mechanical strength.

Again at each stage of the building-up of the core the greatest care must be taken to ensure that the edges of the plates are not brought into contact; the dies for punching the slots of toothed armatures must be kept sharp, and any burring at the edges of the plates must be reduced to a minimum and afterwards removed. By such attention to details it is alone possible to secure the minimum core-loss from the eddy-currents that still persist.

§ 5. Mechanical strength of armature cores.—The centrifugal force of a particle of mass m at distance r from an axis about which it is rotated with an angular velocity of ω radians per second being $m\omega^2r$, the general expression for the single resultant centrifugal force acting on a body of mass $M = \frac{W}{g}$ in a direction radial from the axis and passing through its centre of gravity is

$$F = \frac{W}{g}\omega^2 r, \quad . \quad . \quad . \quad . \quad . \quad (57)$$

Since $\omega r = v$, we also have

$$F = \frac{W}{g} \cdot \frac{v_g^2}{r_g} \qquad (58)$$

Again, since $\omega = \frac{2\pi N}{60}$,

$$F = 0{\cdot}0109 \frac{W}{g} N^2 r_g \qquad (59)$$

Here r_g is the distance from the axis of the centre of gravity of the body considered, and v_g is the velocity per second corresponding thereto. By these formulae the centrifugal force from a separate mass, as *e.g.* a single pole on a rotating field-magnet, is found. Whichever form is used, r_g must be known either by approximate calculation or from the dimensions in the case of homogeneous bodies of regular shape, and g, the acceleration per sec. per sec. from gravity, must be expressed in a similar unit, *e.g.* in feet or inches; F is then found in the unit chosen for the weight W.

If the core-discs of an armature are complete annular rings, they form a homogeneous cylinder, thick or thin as the case may be, subjected to a radial force of uniform intensity over the whole cylindrical area of each concentric layer; the radial centrifugal force is similar for each elementary sector of the cylinder, and acting uniformly round the cylinder is exactly analogous to a fluid pressure within its interior, so that it causes a resultant bursting tension

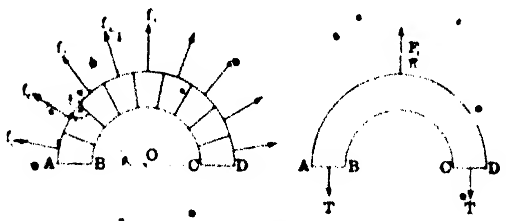


FIG. 133

across any radial section of the cylinder. The radial force of each small wedge-shaped element is balanced by the resultant of the circumferential or hoop tensions exerted by the neighbouring elements on each of its two radial side-surfaces, together with the difference of radial stress on the outer and inner faces of the element when the force is transmitted from layer to layer through a thick cylinder. In all such cases of a uniform radial force, whether external or internal, if f_c be its value per unit angle in circular measure, and if any diametral plane AD through the cylinder be taken (Fig. 133), the resultant force acting across the double section AB, CD is the sum of all the normal components $f_c \sin \alpha$ when the individual forces of all the similar sectors of unit angular width are resolved at right angles to the plane in question. The resultant bursting force acting on the double cross-section is therefore $f_c \cdot \int_0^\pi \sin \alpha \cdot d\alpha = 2f_c$, or on the single section AB on the one side is half this value, $= f_c$. The total centrifugal force summed up all round the periphery as acting independently in a radial direction on each small elementary sector being symbolized by F_c, the force per unit angle in circular measure is $f_c = \frac{F_c}{2\pi}$. If a be the radial depth of the cylinder and b its breadth in an axial direction, the single cross-section is ab. Hence the stress on the material averaged over the section is $\frac{f_c}{ab}$, or

$$\therefore \text{average } s_c = \frac{F_c}{\pi \cdot 2ab}$$

Or, if it be preferred to consider the intensity of the pressure per unit length of arc at any radius r, and per unit breadth in the axial direction, as is more usual in the case of a boiler shell, *i.e.* $p = \frac{F_e}{2\pi rb}$, it is evident that the resultant tension on unit breadth of the double cross-section is equal to the intensity of pressure p multiplied by the projection of the semicircle of radius r on the plane in question, *i.e.* by $D = 2r$; whence the resultant tension is pr per unit breadth along the cylinder, and the stress on the material is $s_e = \frac{pr}{a} = \frac{F_e}{\pi \cdot 2ab}$.

The total centrifugal force F_e as above defined will be given by the same expression as originally used, namely, $\frac{W}{g} \cdot \omega^2 r_g = F_c$, where, however, it must be borne in mind that, although W is the weight of the whole cylinder, r_g is in contrast to the previous case not the radius to the centre of gravity of the cylinder as a whole (since this is zero), but is the radius to the centre of gravity of a wedge shaped sector of indefinitely small width; the centre of gravity of the cylinder as a whole being at its centre, the centrifugal force acting on it as a whole is zero, yet the expression for the total sum of the elemental centrifugal forces acting independently and its symbol F_c are required in order to discover the intensity of the force acting per unit angle, or $f_c = \frac{F_c}{2\pi}$. The average stress on the material of the cylinder due to the centrifugal force is therefore

$$s_c = \frac{F_c}{2\pi \cdot ab} = \frac{W}{g} \cdot \frac{\omega^2 \cdot r_g}{2\pi \cdot ab}$$

where ab is the single section across the narrowest part of the core.

Since the centre of gravity of a wedge shaped sector of angular width 2θ is situated at a distance $\frac{2}{3}\left(\frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}\right)\frac{\sin\theta}{\theta}$ from the centre, and $\frac{\sin\theta}{\theta} = 1$ when θ is very small, the radius to the centre of gravity of a small wedge-shaped sector of indefinitely small width is $r_g = \frac{2}{3}\left(\frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}\right)$. We then have

$$F_e = \frac{W}{g} \cdot \omega^2 \cdot \frac{2}{3}\left(\frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}\right)$$

If w = the weight of unit volume, the weight W of the whole ring is $w \times \pi (R_o + R_i) \times ab$, thence

$$F_e = \frac{w}{g} \cdot \pi\omega^2 \cdot \frac{2}{3}(R_o^2 + R_o R_i + R_i^2) \times ab$$

The average stress on the material $= \frac{F_e}{\pi \cdot 2ab}$, or finally

$$\text{average } s_e = \frac{w}{g} \cdot \omega^2 \cdot \frac{1}{3}(R_o^2 + R_o R_i + R_i^2) \qquad . \qquad . \qquad (60)$$

The same result is reached if we consider the resultant centrifugal effect acting as a single force on the half of the ring, and tending to split it across any diameter. The centre of gravity of the semicircular annulus being situated at a distance from its axis, $r_g = \frac{4}{3\pi}\left(\frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}\right)$, and its weight being $w \cdot \pi\left(\frac{R_o + R_i}{2}\right) \times ab$, the resultant centrifugal force acting on it as a whole in a radial direction through its centre of gravity is, by formula (57), $\frac{w}{g} \cdot \omega^2 \cdot ab \times \frac{2}{3}(R_o^2 + R_o R_i + R_i^2)$. The average stress on the material of the ring, if it be complete, is equal to this resultant centrifugal force divided by the double section $2ab$; or

$$\text{average } s_e = \frac{w}{g} \cdot \omega^2 \cdot \frac{1}{3}(R_o^2 + R_o R_i + R_i^2) \qquad . \qquad . \qquad (60)$$

If R_o, R_i are expressed in feet, and g in feet per sec. per sec. $= 32·2$, with wrought iron or mild steel of specific gravity 7·8, w, the weight of a cubic foot, $= 486$ lbs., and

$$\text{average } s_c = 0·105 \times \omega^2 \times \frac{1}{3}(R_o^2 + R_o \cdot R_i + R_i^2) \text{ lb. per square inch}$$

$$= 0·00115\, N^2 \times \frac{1}{3}(R_o^2 + R_o \cdot R_i + R_i^2) \text{ lb. per square inch} \quad . \quad (61)$$

where N is the number of revolutions per minute. Or for cast iron of specific gravity 7·2, as in the end-flanges, $w = 448$, and

$$\text{average } s_c = 0·00106\, N^2 \times \frac{1}{3}(R_o^2 + R_o \cdot R_i + R_i^2)$$

If the core be smooth, with the winding held in place by binding wires, the above case of a homogeneous cylinder acted upon only by its own centrifugal force is reproduced. But while in the foregoing the average circumferential or hoop stress has alone been determined, in reality when the rim has appreciable thickness in comparison with its diameter the actual stress is not uniform over the whole radial depth. The radial force as we pass from the outside surface is transmitted from layer to layer to the interior with increasing intensity, if the cylinder is solid, but again decreases towards the inside circumference if the cylinder is perforated with a hole. In consequence, as mentioned at the outset, in the general case of a thick cylinder there is a difference of radial stress on the outer and inner faces of a small element forming part of any one component layer. Thus in a hollow thick cylinder the radial stress tending to divide one layer from another, and acting at right angles to the circumferential stress, is zero at the outer and inner circumferences and reaches a maximum at some depth within the rim. But even though the maximum radial stress tending to divide two layers may not be considerable, the important result follows that the hoop stress is no longer uniform but varies at different depths, and is a maximum at the inside edge. By reason of the minute expansion of the cylinder under the action of the centrifugal force, a true expression for the local stress must contain Poisson's ratio $\frac{1}{\sigma}$, which for metals may be taken as lying between $\frac{1}{3}$ and $\frac{1}{4}$. In order to render the problem capable of solution, in the case of a solid homogeneous cylinder the effect of its length must be ignored; the results are therefore only strictly true for a very thin flat disc, but this corresponds to the case of an armature core disc, and for such a disc with a hole at its centre it is found that the greatest value of the hoop stress, which occurs at the inner circumference, is given by the expression[1]

$$s_{c\,max} = \frac{w}{g} \cdot \frac{\omega^2}{4}\left\{\left(3 + \frac{1}{\sigma}\right)R_o^2 + \left(1 - \frac{1}{\sigma}\right)R_i^2\right\}$$

or, assuming $\frac{1}{\sigma} = \frac{1}{4}$, and with the radii expressed in feet, for wrought iron or mild steel,

$$s_{c\,max} = 0·00029\, N^2 (3·25\, R_o^2 + 0·75\, R_i^2) \text{ lb. per square inch} \quad . \quad (62)$$

Hence if the central hole be very small, the maximum stress is

$$s_{c\,max} = 0·00094\, N^2 R_o^2$$

while on the same assumption of a very small hole the average stress would only be

$$s_{c\,average} = 0·000383\, N^2 R_o^2$$

It will be seen that the maximum stress would be nearly $2\frac{1}{2}$ times the average in such an extreme case, and remains as much as 25 per cent. greater even

[1] Ewing, *Strength of Materials* (2nd edit.), p. 218. Cp. A. Jude, *The Theory of the Steam Turbine*, Chap. XIII. See also B. A. Behrend, "A New Large Generator for Niagara Falls," *Trans. Amer. I. E. E.*, Vol. 27, Part II, p. 1061.

when $R_i = \frac{1}{2} . R_o$. For solid cylinders of considerable axial length, even the above more accurate formulae can only be regarded as approximately true.

Thus it is only when the thickness of the ring is quite small as compared with its diameter that the radial stress which is zero at the outer and inner circumferences reaches so small a value within the thickness that it may be neglected; no difference between the radial stresses on the outer and inner faces of an elementary layer then arises, and the centrifugal force of each element is strictly balanced by the resultant of the hoop tensions on its two sides. The stress in a thin rim thus becomes sensibly uniform over the whole cross section, or the average and maximum values coincide.

Since in such a case R_o and R_i are each nearly equal to the mean radius $R_m = \frac{R_o + R_i}{2}$, the expression $(R_o^2 + R_o . R_i + R_i^2)$ in formulae (60) and (61) reduces to $3 R_m^2$, and we have

$$s_e = \frac{w}{g} . \omega^2 . R_m^2 = \frac{w}{g} . v_m^2$$

where v_m is the mean velocity corresponding to R_m.

Hence in the special case of a rim of small radial depth as compared with its diameter for wrought iron or mild steel, if R_m and v_m are expressed in feet and feet per second,

$$s_e = 0{\cdot}00115\ N^2 . R_m^2, \text{ or } 0{\cdot}105\ v_m^2 \text{ lb. per square inch} \qquad (63)$$

or for cast iron

$$s_e = 0{\cdot}00108\ N^2 . R_m^2, \text{ or } 0{\cdot}097\ v_m^2 \text{ lb. per square inch} \qquad (64)$$

The stress is then, as is well known, dependent only upon the square of the mean velocity, and in lb. per square inch is nearly $\frac{1}{10}$th of the square of the mean velocity in feet per second. Obviously since in the above expression the inner radius is *ex hypothesi* to differ but little from the outer radius, the outer radius or the peripheral velocity may also be used but with slightly less accuracy.

We have, however, in most cases to deal with rings of appreciable depth, and are therefore more interested in the maximum stress which holds in such cases, and upon which the bursting would in reality depend. From eq. (62) it will be seen that in the two extreme cases of a very thin ring and of a nearly solid ring, the maximum stress only varies in the proportion of 4 to 3·25; or with wrought iron as the material, from $0{\cdot}00115\ N^2R_o^2$ to $0{\cdot}00094\ N^2R_o^2$. There is therefore but little error in all practical cases in assuming the maximum stress as $0{\cdot}001\ N^2R_o^2$, or if we wish to be on the safe side in taking the maximum figure, $0{\cdot}00115\ N^2R_o^2$.

To this in the case of the toothed armature core must be added the stress from the centrifugal force of the iron teeth, and also from the copper winding within the slots when these are closed or locked by wooden wedges. Having calculated the total radial centrifugal force F_c^1 from the teeth and copper winding so far as the latter falls on the core-discs, this may be regarded as distributed uniformly over the whole cylinder below the bottom of the slots, so that if we retain R_o to signify the radius to the bottom of the slots, the radial tension on the outer surface is $\frac{F_c^1}{2\pi R_o . b}$. The hoop stress due thereto increases towards the inner surface, and there has the value $\frac{F_c^1}{2\pi R_o . b} \times \frac{2R_o^2}{R_o^2 - R_i^2} = \frac{F_c^1}{2\pi ab} \times \frac{2R_o}{R_o + R_i}$. The maximum stress at the inner surface is therefore

$$s_{e\,max} = 0{\cdot}001\ N^2R_o^2 + \frac{F_c^1}{2\pi ab} \times \frac{2R_o}{R_o + R_i} \qquad (65)$$

where R_o is the radius to the bottom of the slots, a is the radial depth below the slots, and b is the net length of the iron core with allowance for insulation between the discs and for air-ducts.

There are, however, but few parts of an armature which are only subjected to the direct stress from centrifugal force. Thus when the core is built up

out of segments fastened by bolts through the end-flanges on the hub, these latter must take some portion of the centrifugal force of the discs in different degrees according to the method of construction. Again, the core-discs when internal to the field-magnet are subjected not only to centrifugal force, but also to a nearly uniform external pull in a radial direction from the magnetic field. If the total value of this pull summed up all round the armature circumference, as will be further explained in Chapter XV, § 9, be P_m, the resultant tension as acting on a single cross section of a core composed of circular discs will be, just as in the case of the centrifugal force, $\frac{P_m}{2\pi}$.

Across a horizontal diameter there is a further additional stress due to the weight of the core itself; if as the worst condition the weight of the lower half is assumed to be entirely unsupported by the arms of the hub, the tension on the double cross section due to the weight would be $\frac{W}{2}$. These forces reinforce the simple bursting stress from centrifugal force, but the effect of the weight is usually negligible, so that the resultant maximum stress upon the material becomes approximately

$$s_a = s_{c\,max} + \frac{P_m}{\pi \cdot 2ab} \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

If the armature is eccentric in the bore, for $\frac{P_m}{\pi}$ in the above must be substituted the total resultant pull P_m' acting on a half of the ring in the direction of its eccentricity. If therefore the eccentricity is in the downward vertical direction, and it is supposed that the armature has simply sunk downwards while the magnet yoke-ring retains its true circular shape without deformation, the resultant pull P_m' may be approximately estimated on the lines given in Chapter XV, § 14.

Next, if it be supposed that the armature rim is rigidly anchored at each junction with an arm, there has further to be considered the bending moment acting on the core between each pair of arms, and reaching its maximum at the points of junction. The total force between a pair of arms is $\frac{F_c + P_m + W}{n_a}$ where n_a is the number of spokes or arms of the hub, and this force is uniformly distributed along a length $\frac{2\pi r_m}{n_a}$, so that the stress at the junction with an arm is

$$s_a' = \frac{F_c + P_m + W}{n_a} \times \frac{2\pi r_m}{n_a} \times \frac{1}{12Z} \quad . \quad . \quad . \quad . \quad (67)$$

on the analogy of a beam fixed at both ends and loaded uniformly, Z being the modulus of the section, which by the usual formula for a rectangle is $\frac{ba^2}{6}$. The core would thus become bowed outwards between each pair of arms, and the stresses in the core would be increased as compared with the case when the core suffers no deformation. Even the above estimate of the bending moment would be increased if there is any unequal magnetic pull, as will be described in Chapter XV, § 14. If the width of a pole-face exceeds the distance between two arms, and the maximum induction in the air gap is regarded as holding over the whole of the distance between two arms, the total pull which could occur between two arms would be $\frac{B_g^2{}_{max} \cdot 2\pi R_a \cdot l}{1{\cdot}735 \times 10^6 n_a}$, or since this is a maximum estimate, for the area of the periphery might be substituted the smaller area of the $2p$ pole-faces, *i.e.* $\frac{B_g^2{}_{max} \cdot A \cdot 2p}{1{\cdot}735 \times 10^6 \times n_a}$ lb., where A is the area of one pole-face in square inches. Hence in the above expression for s_a', instead of P_m would be substituted the imaginary value obtained on the assumption that the pull was uniform round the armature at its real maximum value, namely, $P_m'' = P_m \left(\frac{B_g \text{ max.}}{B_g \text{ average}}\right)^2$. The stress due to the bending moment is additive to s_a, but, as thus calculated, it is not strictly

correct and is somewhat over-estimated; a more detailed analysis would have to take into account the stretching of the arms, which, although considered above as inextensible, would in reality lengthen slightly.[1]

But if, as an approximation, the combined stress is taken as the sum of s_a and s_a', we must then have $s_a + s_a' \leq f_t$, where f_t is the safe permissible limit of tensile stress with the particular material employed. Such a maximum value, $s_a + s_a'$, probably errs on the high side, so that a high permissible tensile stress may be taken, and for wrought iron or mild-steel sheets f_t may be reckoned as 14,000 lb per square inch.

Owing to the abundant strength of the sheet steel used in armature cores, any such calculation is as a rule superfluous (even when allowance is made for any accidental excess of speed which might occur), except in the case of very high peripheral speeds such as with armatures driven by steam turbines, or of very large size; the above remarks are therefore only introduced in view of the increasing use of high speed machines, to indicate the nature of the stresses to which the armature core is subjected, and also on account of the general applicability of the methods and formulae obtained to many similar problems which occur in the calculation of the mechanical strength of various parts of the dynamo.[2]

§ 6. Mechanical strength of armature hubs.—If the armature makes N revs. per minute, and transmits HP horses' power, the torque or *twisting moment* acting on the armature is

$$T_m = \frac{HP \times 33{,}000 \times 12}{2\pi N} = 63{,}000\frac{HP}{N} \text{ inch pounds.}$$

For dynamo work this may be translated into a more convenient form. If the output of a dynamo be kW kilowatts, the actual power which passes into the shaft either through the pulley of a belt driven machine or through the coupling connecting it directly to the crank-shaft of an engine, is $KW' = \frac{kW}{\eta}$, where η is the efficiency, or the ratio between the useful power given out at the terminals and the total power absorbed in obtaining it.[3] By substitution of the quantity KW', reduced to horsepower in the above equation, we obtain as the normal torque

$$T_m = 85{,}000\frac{kW}{\eta N} \text{ inch-pounds} \qquad (68)$$

The commercial efficiency of dynamos varies considerably with their type and size, but in the greater number of cases it ranges from 70 in small to 90 per cent. or more in large machines; an average value of η is therefore about 0·85. Thus the coefficient for the conversion of the horse-power input into kilowatts of output may

[1] Cp. Unwin and Mellanby, *Elements of Machine Design*, Part II, p. 285.

[2] See Niethammer, *Berechnung und Konstruktion der Gleichstrommaschinen und Gleichstrommotoren*, Part II, p. 369 ff., from which the treatment in the latter part of the present section is derived in part, and to which the reader is referred for a fuller discussion of the whole subject.

[3] In the case of a separately excited machine, the exciter of which is not driven from the main dynamo shaft, the power absorbed in the field should strictly be taken into account in calculating the commercial efficiency, although in our present connection such loss in the field stands outside the calculation of the strength of shaft.

be taken on the average as $= \frac{\cdot 746}{1000} \times 0{\cdot}85 = 0{\cdot}63$, and any equation expressed in terms of horse-power must be divided by this coefficient to express it in terms of kilowatts, or $T_m = 100{,}000\frac{kW}{N}$ inch-pounds. If w = the output in watts,

$$T_m = 100\frac{w}{N} \text{ inch-pounds},$$

where $\frac{w}{N}$ = the watts per rev. per minute, a quantity the importance of which in dynamo designing has already been mentioned.

If then r_a be the radius of the arm in inches (Fig. 134), or the distance from the centre of the shaft to the tip of the arm, where the resistance to the motion is applied, and n_a is the number of arms or spokes, the force acting at the tip and tending to bend the spoke is $P_t = \frac{T_m}{r_a \times n_a}$ pounds. If any point be taken along the arm, distant l inches from its tip, the leverage is l, and the *bending moment* at

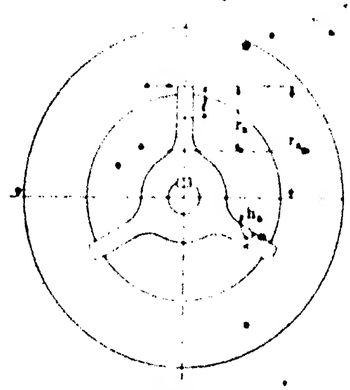


FIG. 134.

that point under normal full load is $B_m = \frac{T_m}{r_a \times n_a} \times l$. The bending moment divided by the modulus of the section Z gives the stress on the material at the point in question, *i.e.* $s_b = \frac{B_m}{Z} = \frac{P_t l}{Z}$ is the stress due to the transmitted torque across a transverse section taken at any point l inches from the tip of the arm. This reaches its maximum at the junction of the arm with the nave, so that if r_n be the radius to the nave, also in inches,

$$s_b = \frac{P_t(r_a - r_n)}{Z} = \frac{T_m}{r_a \times n_a}\left(\frac{r_a - r_n}{Z}\right) \text{ lb. per square inch} \quad . \quad (69)$$

Or if the arm be of rectangular cross-section, of thickness h_a inches, and of breadth b_a inches parallel to the length of the shaft, so that its section modulus is $\frac{b_a \times h_a^2}{6}$

$$s_b = \frac{T_m}{n_a} \times \frac{r_a - r_n}{r_a} \times \frac{6}{b_a \times h_a^2} \text{ lb. per square inch} \quad . \quad (70)$$

But while the above gives the normal bending stress on the arms, this may be very much increased by sudden retardation of the whole moving mass, as by a short-circuit occurring with a compound-wound dynamo. The inertia of a heavy fly-wheel as in a machine directly coupled to a steam-engine, would cause a greatly increased force to be transmitted through the arms of the hub. The design of an armature hub must then follow the lines of a large wheel or pulley, and the energy that may be delivered to the armature core from the actual fly-wheel of the steam-engine, as due to the greatest retardation that is likely to occur, say complete stoppage in three to five seconds, must be calculated; the torque that is at the same time being given by the steam pressure acting on the pistons would also have to be taken into account. Further, it has been above assumed that each arm takes its strict share of the total driving force, and so is equally effective. On all these grounds it is advisable to take from 4 to 5 times the normal value of P_t or T_m in the calculation of the bending stress.[1]

The stress must then be $\leqq f_t$, the safe permissible tensile stress. Since s_b is to be reckoned on a liberal basis, as at least four times that corresponding to normal full load, the permissible tensile stress in cast iron may be taken as high as 4,000 lb. per square inch. The arms of the hub must also be strong enough to withstand the shearing action; this is uniform throughout the length of the arm and is

[1] In addition to the bending stress at any section of the arm there is also the tension from its own centrifugal force, although this is of but slight importance in ordinary cases. If the cross-section of the arm be uniform throughout, its weight is $w \times (r_a - r_n) \times$ section, and the radius to its centre of gravity is $\frac{r_a + r_n}{2}$. Its centrifugal force is therefore $\frac{1}{2} \cdot \frac{w}{g} \cdot \omega^2 (r_a^2 - r_n^2) \times$ section, and the stress due thereto with cast iron as the material is from § 5,

$$\frac{1}{2} \times \frac{0.000106}{144} N^2 (r_a^2 - r_n^2) \text{ lb. per square inch,}$$

the two radii being now measured in inches, *i.e.*

$$s_c = 3.68\ N^2 (r_a^2 - r_n^2) \times 10^{-7} \text{ lb. per square inch}$$

Further, if the core is in segments dovetailed to the hub, there has to be added the tension due to magnetic pull; the total maximum pull that can act on one arm is, as in § 5, approximately $\frac{P_m'}{n_a} = \frac{B_{g\,max}^2\, A \cdot 2p}{1.735 \times 10^4 \times n_a}$. If W be the weight of the armature core, the proportionate share of its weight on one arm will be $\frac{W}{n_a}$, and in each of these two latter cases the stress is equal to the force divided by the section. The total combined stress at the root of the arm would therefore be

$$s_b + s_c + \frac{P_m' + W}{n_a \times \text{section of arm}}$$

If, as an extreme supposition, it be assumed that the arms of the hub will also be called upon to withstand the centrifugal force of the armature core, the second term must again be increased. Cp. Urwin and Mellanby, *Elements of Machine Design*, Part II, p. 285.

equal to the drag or force acting at the tip of each arm, *i.e.* to $P_t = \frac{T_m}{r_a \times n_a}$ transferred without change to any transverse section of the arm.[1] Then

$$\frac{P_t}{b_a h_a} \leq f_s$$

where f_s, or the safe working stress under shearing, may be given half the value of f_t, or 2,000 lb. per square inch.

It will be found that there is usually no difficulty in obtaining ample strength in the hub against either bending or shearing.

When the hub is pressed on to the shaft by screws or hydraulically, an excess allowance rising from 2 to 4 mils is made in the diameter of a shaft of 6 to 10 inches, in order to secure a sufficiently good driving fit.[2] In large machines the nave of the hub is split in order to avoid strains from contraction during cooling, and is subsequently drawn together by wrought-iron rings shrunk on or by bolts.[3]

In all cases care must be taken that sufficient surface is given to the tips and driving edges of the arms, and that the discs are accurately fitted or keyed to them. If this precaution be neglected, the driving stress will after several years of use end by causing the edges of the discs to cut into the arms, with the result that as soon as the least relative movement takes place, wear of the arms begins, and rapidly increases, until the core either in one solid mass or in certain sections becomes loose on the shaft.

§ 7. Mechanical strength of armature shafts.—Armature shafts are usually of mild steel, produced by the Bessemer or Siemens-Martin process. The determination of the diameter required at different parts of their length is largely a matter of experience, but nevertheless in the main it requires to be based on the theoretical rules appropriate to shafts which are subjected to the combined stresses of torsion and bending. The calculation of the torsional stress due to the continuous and steady transmission of horsepower through a rotating shaft is an easy matter, since, as in the preceding paragraph, the twisting moment about the axis of the shaft or

$T_m = 100{,}000 \frac{kW}{N}$ inch-pounds.

But torsion is by no means the only stress to which the armature shaft is subjected; even more important is the bending stress, due to several causes, chief among which is the weight of the armature itself. In ring or drum armatures the weight is more or less

[1] See Unwin, *Elements of Machine Design* (edit. 1909), part i, p. 66.

[2] Cp. R. Livingstone, *The Mechanical Design and Construction of Generators*, pp. 33–38.

[3] See Unwin and Mellanby, *Elements of Machine Design*, Part II, pp. 278–281, and Niethammer, *Berechnung und Konstruktion der Gleichstrommaschinen und Gleichstrommotoren*, pp. 378–383.

uniformly distributed over the length of the shaft between the bearings; in discoidal and disc armatures the weight is more nearly concentrated at the centre, although even then the axial length of the supporting hub is usually a considerable number of inches. If W be the weight of the complete armature in pounds, and l be the span in inches between the bearings on either side of the armature, then if we assume the whole of the weight to be concentrated at one point (Fig. 135), the reactions at the centres of the bearings Q, R are $w_1 = W\frac{l_2}{l_1 + l_2}$ and $w_2 = W\frac{l_1}{l_1 + l_2}$. The bending moment diagram is drawn as follows; upon the direction of R produced, mark off a length ab representing to some convenient scale the moment of w_1 about R, a negative or clockwise moment being measured downwards and a positive or counter-clockwise moment upwards. Join bc, intersecting the direction of W at d, and lastly join da. Then abc is the moment

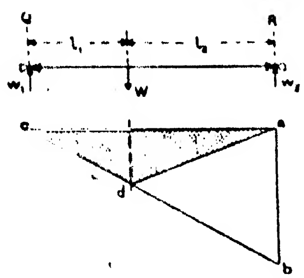


FIG. 135. Bending moment diagram due to weight of armature if concentrated.

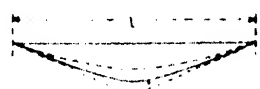


FIG. 136. Comparison of bending moment diagrams due to weight, concentrated and distributed.

area of w_1, and abd is the moment area of W; $W \cdot l_2$ being positive, its area above the horizontal line ac is equally well given by the area of the triangle, abd lying between the same parallels, when the base ba is measured upwards. Thus the positive moment area abd balances a portion of the negative moment area, and the remaining shaded triangle adc is the bending moment diagram, its vertical ordinates being proportional to the bending moment at corresponding points of the shaft. The greatest bending moment is $W\frac{l_1 l_2}{l_1 + l_2}$, and in the case of Fig. 135, if we take $\frac{l_1}{l_2} = \frac{0 \cdot 4}{0 \cdot 6}$, it is but little different from that due to a load concentrated at the centre which gives the dotted equilateral triangle of Fig. 136 and $B_m = \frac{Wl}{4}$. If, however, the weight is assumed to be uniformly distributed along the entire length between the bearings, the bending moment diagram is given by the dotted parabola with vertex at the centre (Fig. 136), the greatest bending moment being reduced

to $B_m = \frac{Wl}{8}$. Neither supposition is strictly true, and in default of an accurate setting-out of the bending moments a fair approximation is obtained if the intermediate full line of Fig. 136 with $B_m = \frac{Wl}{6}$ is adopted.

Comparison of a large number of dynamos of different sizes and types shows that the weight of the armature is largely independent of the type of field, and may be expressed to a fair degree of accuracy as proportional to the $\frac{2}{3}$rds power of the kilowatts per rev. per minute, or $W_a = c \cdot \left(\frac{kW}{N}\right)^{\frac{2}{3}}$ lb. The value of the constant c ranges from 6,000 in small to 10,000 in large machines, but more often approximates to the lower limit, so that its average value may be put at 7,200,[1] *i.e.*, $W_a = 7{,}200 \left(\frac{kW}{N}\right)^{\frac{2}{3}}$ lb., or $72 \left(\frac{W}{N}\right)^{\frac{2}{3}}$ lb.

The length of the span between the bearings in average cases is of such an order that we assume it to be $l = 96 \left(\frac{kW}{N}\right)^{\frac{1}{3}}$ inches. The bending moment due simply to the weight of the armature as given by the above formula will then be

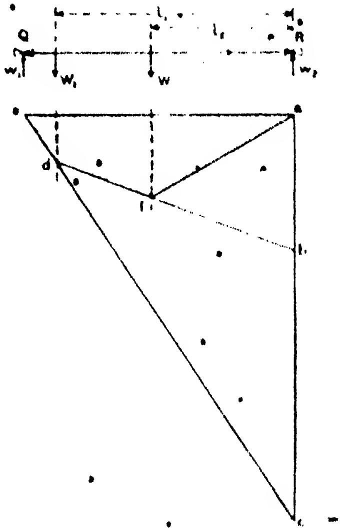


FIG. 137.—Bending moment diagram due to weight of armature and of fly-wheel.

$$B_m = \frac{7200 \left(\frac{kW}{N}\right)^{\frac{2}{3}} \times 96 \left(\frac{kW}{N}\right)^{\frac{1}{3}}}{6}$$

$$= 115{,}000 \frac{kW}{N}, \text{ or } 72{,}500 \frac{HP}{N} \text{ inch-pounds} \quad . \quad . \quad . \quad (71)$$

In engine-driven dynamos, where the heavy mass of the fly-wheel is supported between the inner bearing of the engine and the outer or commutator bearing of the dynamo, it will be necessary to take this further weight into account as concentrated at a certain distance

[1] In alternators in which the revolving portion is very usually the field-magnet, and this is combined with, or itself forms, the fly-wheel of the driving engine, the weight will be considerably higher than in a continuous-current dynamo with revolving armature; owing to the great variations of the conditions, c may then reach values between 10,000 and 20,000, and can hardly be said to be a constant, even in alternators of the same type.

from the centre of the armature. The bending moment diagram may then be obtained by adding the ordinates of the diagram due to each separate weight, or we may set off the moment of w_1 about R, *i.e.* $ac = Wl_2 + W_1l_1$ downwards, and join ce (Fig. 137) intersecting the direction of the fly-wheel weight W_1 produced at d; measure $cb = W_1l_1$ upwards from c, and join db and fa; then $afde$ is the combined bending moment diagram. The fly-wheel being supposed to be three times as heavy as the armature, and at the same distance from the commutator end that the pulley bearing formerly was, it will be seen that the bending moment is increased $1\frac{1}{2}$ times.

§ 8. Causes increasing the bending moment on shafts.—There are, still, however, a number of causes increasing the bending

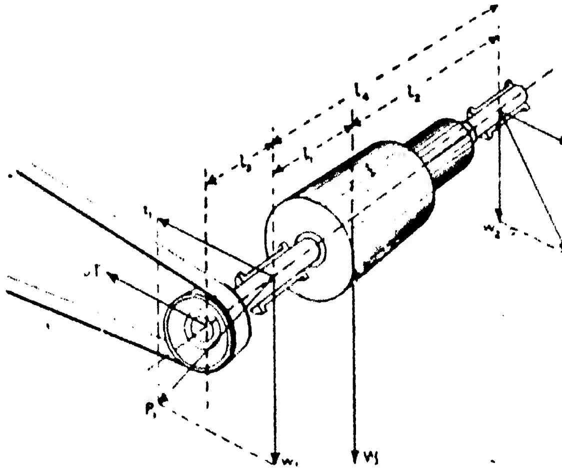


FIG. 138.—Forces acting on belt-driven dynamo.

moment which in the above equations are entirely unrecognized. First amongst these may be mentioned the lateral pull caused from the pulley end by the tension of the driving belt or ropes. The pulley in belt-driven machines is usually placed outside of, yet as close as possible to, one of the two bearings (cp. Fig. 6); and a certain amount of bending stress must be allowed for within the span of the shaft from the one bearing to the other, over and above that due to the weight of the armature. If $T = T_2 - T_1$ is the driving force at the periphery of the pulley in lb., or the difference in the tensions of the two sides of the belt, the total pull on the shaft at the centre of the pulley is $T_2 + T_1$. The arc embraced by the belt on the smaller driven pulley being taken as about 0·4 of its

circumference, and the coefficient of friction between the belt and the pulley as 0·3 (these assumptions being such as usually hold in practice[1]), $T_2 + T_1$ may be reckoned as $= 3T$. This pull causes reactions in the two bearings (Fig. 138) of values

$$t_1 = 3T\left(\frac{l_3 + l_4}{l_4}\right); \quad t_2 = 3T \cdot \frac{l_3}{l_4}$$

Their directions in the plane of the pull are as shown in Fig. 139, whence the bending moment diagram due to the overhung pulley is easily drawn by setting off $ab = 3T \quad (l_3 + l_4)$ on the direction of t_2 produced and joining bc. If the driving force and the weight were acting in the same plane, the two shaded areas *acd* of Figs. 139 and 135 could be combined by adding together their corresponding ordinates algebraically; but in general the pull of the belt acts horizontally or at right angles to the weight. In order to find the maximum bending moment the diagrams must then be combined by taking the square root of the sum of the squares of the ordinates at corresponding points. This has been done in Fig. 140, and from it the increase and altered position of the maximum bending moment within the bearings due to the belt is clear. To this has next to be added the possibility that there is a considerable unbalanced or one-sided magnetic pull, the nature of which will be described in Chapter XV, § 14; any such pull will tend to deflect the shaft, and will often coincide nearly in direction with the weight of the armature and increase the downward pressure by at least one-third. Further, its effect is cumulative, the pull increasing the eccentricity of the armature within the bore of the poles and exaggerating the initial effect.

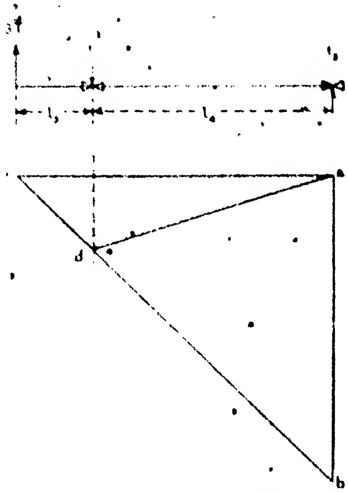


Fig. 139.—Bending moment diagram due to belt pull.

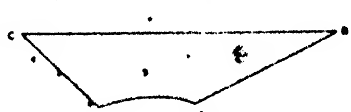


Fig. 140.—Combined bending moment diagram due to weight of armature and belt pull.

[1] See Unwin, *Elements of Machine Design* (1909 edit.), Part I, pp. 446–453.

Straining actions are also set up when the masses of the armature and fly-wheel are rapidly accelerated or retarded by reason of sudden changes in the armature load such as occur, *e.g.* in a generator supplying current to tramcar motors.

When the maximum bending moment has been determined, it must be combined with the twisting moment by the well-known rule through which the *equivalent bending moment* is obtained,[1] namely,

$$B_e = \tfrac{1}{2}B_m + \tfrac{1}{2}\sqrt{(B_m^2 + T_m^2)}$$

The *moment of resistance of a circular section to bending* is $\frac{\pi}{32}f_t d^3 = 0{\cdot}098 f_t d^3$, and equating this to the equivalent bending moment we obtain

$$d = \sqrt[3]{\frac{B_e}{0{\cdot}098 f_t}} \quad . \quad . \quad . \quad . \quad . \quad (72)$$

where d is the minimum diameter of a solid circular shaft for a given value of f_t, and f_t is the safe working stress of the material of the shaft. The value of f_t which is found by experience to give practical results for the case of a dynamo shaft ranges from 4,500 to 7,500 lb. per square inch, and B_e being reckoned in inch-lbs., the diameter is thence found in inches.

§ 9. Deflection of shafts.—There still remains a second point of view from which the diameter of the shaft as deduced from the combined twisting and bending moment must again be checked, namely, the deflection of the shaft regarded as a beam supported at either end by a bearing and carrying a more or less concentrated load. The amount of this deflection, although determined by the bending stresses, yet forms an independent consideration of great importance, especially in the case of shafts with a long span between the bearings. The shaft may then have sufficient strength to transmit the requisite horsepower torsionally and to resist the greatest bending moment, but still may be too weak from the deflection point of view, since any initial deflection itself increases the one-sided magnetic pull by decreasing the air-gap in one or other direction.

If a weight W be concentrated at a spot distant l_1 and l_2 units of length from the two bearings, the total length of span being $l = l_1 + l_2$, and if the shaft be throughout of uniform diameter d, the greatest deflection of such a beam under its concentrated load will be[2]

$$\delta = \frac{W \cdot l_1^2 \cdot l_2^2}{3l \cdot EI}$$

[1] Unwin, *Elements of Machine Design* (edit. 1909), Part I, p. 128.

[2] Unwin, *Elements of Machine Design*, (edit. 1909). Part I, pp. 79, 80.

where I is the moment of inertia of the section which for a circle is $\frac{\pi d^4}{64}$, and E is the modulus of elasticity. For steel E may be taken as 30,000,000 lb. per square inch, so that when W is reckoned in lb., and with the lengths and diameter reckoned in inches

$$\delta = 0{\cdot}227 \frac{W \, . \, l_1{}^2 \, . \, l_2{}^2}{l \, . \, d^4} \times 10^{-6} \text{ inches} \quad . \quad . \quad (73)$$

If $l_1 = l_2$, $\qquad \delta = \frac{W \, . \, l^3}{48 \, EI}$

and in the same units as above

$$= 0{\cdot}0141 \frac{W \, . \, l^3}{d^4} \times 10^{-6} \text{ inches.}$$

Strictly speaking, the calculation would require to be carried through for the deflection at each point due to each of the weights corresponding to the armature and commutator, to the fly-wheel, and to the unsymmetrical magnetic pull.[1] The resultant deflection would then be given by the geometrical sum of the component deflections at any point. It will, however, suffice to substitute for W the single weight W_e which, so far as the resultant bending moment is concerned, most nearly represents the combined effects of the various weights, W_e being located at the spot where the bending moment has been found to be greatest (as at d in Fig. 140); the assumption that each weight is concentrated will have led to an error on the safe side in the estimate of W_e.

The resultant deflection δ as deduced from W_e must then be limited to an amount not exceeding 10 per cent. of the normal air-gap as a maximum, no allowance being made for any additional stiffness added to the shaft by the close-fitting sleeve of the hub.

If all the weight were distributed perfectly uniformly along the shaft, there would be a great reduction in the deflection, the equation becoming

$$\delta = \frac{W_d \, . \, l^3}{76{\cdot}8 \, EI} = 0{\cdot}0088 \frac{W_d \, . \, l^3}{d^4} \times 10^{-6} \text{ inches} \quad . \quad . \quad (73a)$$

i.e. only $\frac{5}{8}$ths of the deflection for the same weight concentrated at the centre of the span.

In practice the weights are to some extent distributed along the shaft, although not uniformly so; on the other hand, the above expressions are only true for a shaft of uniform diameter. If an axial section of the shaft approximates to a cubic parabola (*i.e.*

[1] For the full procedure, see especially "The Bending, Vibrating and Whirling of Loaded Shafts," by Capt. J. Morris, No. 551 of *Reports and Memoranda of the Advisory Committee for Aeronautics* (H.M. Stationery Office); and R. Livingstone, *The Mechanical Design and Construction of Generators* (Electrician Publishing Co.), p. 75.

if $r = \sqrt[3]{a \cdot x}$, where r is the radius of the shaft at a distance x from the centre of the nearer bearing, and a is a constant, a condition which would give uniform strength under a load concentrated at the centre of the span), or to a truncated cone tapering from maximum d to $\frac{2}{3}d$ at the entrance to a bearing, the deflection would be increased as much as $1\frac{2}{3}$ times as compared with a shaft of uniform diameter. Either of the above cases more nearly resembles an actual shaft which is tapered towards the bearings, but in practice there should be sufficient strength to render it unnecessary to calculate the influence of the varying section of the shaft more exactly.[1]

§ 10. Stiffness of shaft to resist centrifugal whirling.—Great stiffness in the shaft is equally necessary in high-speed machines for another reason. Consider a rotor of mass M, and in order to eliminate any deflection due to gravity let its shaft be vertical; and in order still further to simplify the problem let this mass be so disposed that it can be imagined to be concentrated at one single point situated at a distance r_0 from the centre line of the shaft. No rotor built by human hands but is in some small degree out of true dynamic balance—it may be only from a want of homogeneity in the material—so that its centre of mass does not in practice fall absolutely on the centre line of the shaft, and it may when running require to be represented by two concentrated masses arranged at different points along the axial length between the bearings, but this additional complication is for our purpose neglected. Let the shaft be run up to any constant angular velocity ω. If the shaft be then imagined to be forcibly retained in its truly vertical line, it is subjected to a centrifugal force in the horizontal plane of the mass-centre and of magnitude $M\omega^2 r_0 = F_0$. When the restraint is removed, the shaft under the action of this force must be deflected through some distance radially; if c be the force in the horizontal plane of the mass-centre required to bend the shaft through unit distance, the first deflection of the centre-line of the shaft from the vertical axis through the centres of the bearings is $r_1 = F_0/c$, and if r_1 be expressed as a fraction q of r_0, $F_0/c = qr_0$. The increment to the displacement will give rise to a first additional increment to the centrifugal force of $M\omega^2 r_1 = F_1 = qF_0$. Assuming the deflection to be always in the same direction as the centrifugal force and the same proportionality between force and deflection to continue to hold for small deflections, there ensues a first increment to the deflection, viz. $r_2 = \frac{qF_0}{c} = q^2 r_0$, which again gives rise to a second increment to the centrifugal force, and so on.

Corresponding then to $F_0 + F_1 + F_2 + \ldots = F_0 + qF_0 + q^2F_0 + \ldots$ the final distance of the mass-centre from the vertical axis through the bearings is

$$\begin{aligned} r &= r_1 + r_2 + r_3 + \ldots \\ &= qr_0 + q^2r_0 + q^3r_0 + \ldots \end{aligned}$$

Thus at each stage in the process the displacement releases as it were an additional deflecting force, and the case is analogous as will appear later to that of two alternators in parallel with an initial divergence of phase between them which releases an additional synchronizing force magnifying the initial displacement.

[1] In shafts proportioned to give a uniform stress throughout, Mr. R. Livingstone (*Electrician*, Vol. 57, p. 570) gives, as the average of a number of cases,

$$\delta = 0{\cdot}0118 \frac{W \cdot l^3}{d^4} \times 10^{-6} \text{ inches}$$

where the weights of engine fly-wheel, armature and commutator, and the unbalanced magnetic pull are all simply added together to give W.

The geometric series will only have a finite sum if the first deflection qr_0 is less than the distance r_0 or $q < 1$, and its value is then

$$r = r_0 \frac{q}{1 - q}.$$

The force tending to straighten the shaft is cr and this is equal to the deflecting force $M\omega^2 (r_0 + r)$; if then the rotor is run up to such a speed that $M\omega^2 = c$, $q = F_0/cr_0 = M\omega^2/c = 1$, and it would appear either that r becomes infinite or that r_0 must be zero, which is ex hypothesi not the case. To put the same in a different form, since

$$cr = M\omega^2 (r_0 + r)$$

$$r = \frac{M\omega^2 r_0}{c - M\omega^2} = \frac{M\omega r_0}{c/\omega - M\omega}$$

If then $\omega = \sqrt{c/M}$ or $c = M\omega^2$ the denominator is zero, and r becomes infinite. If for any given values of c, M and r_0 the curve of r as connected with increasing values of ω is plotted, it rises very steeply as $M\omega^2$ approaches c, and finally disappears in positive infinity. If in the mathematical equation $M\omega^2$ is made to exceed c, the curve returns from negative infinity in a second branch.[1]

If we express ω as $k\omega_{crit}$ and ω_{crit} is the particular value of ω which is equal to $\sqrt{c/M}$, $r = r_0 \dfrac{Mk^2\omega^2_{crit}}{c - Mk^2\omega^2_{crit}} = r_0 \dfrac{k^2}{1 - k^2}$. The equation and curve are then exactly analogous to those for the displacement of an undamped oscillating system as in Chap. VI, § 23 (a). A negative r would indicate that the mass centre has passed to a position between the centre line of the shaft and the true vertical axis through the bearings, i.e. that the shaft is rotating outside the position of M, and that the mass centre is at a distance $r - r_0$ only from the vertical axis.

The case is, however, physically unreal as that of the undamped oscillating system; actually $M\omega^2$ may exceed c and must therefore have been equal to it at a certain critical speed which has been run through. The explanation is again to be sought in damping forces which do not appear in the simplified form of the equation.

When the unbalanced rotor has been run up to some constant speed less than the critical speed, let its shadow be imagined to be thrown by a lamp in front of it on to a vertical white wall behind it. The edge of the shadow will then oscillate to and fro, or the centre line of the rotor's shadow will describe a simple harmonic motion across the true vertical axis, with a frequency equal to the revolutions per second and with a certain amplitude. Let a pencil now be brought up in the horizontal plane of the mass centre and parallel to the wall, until it is just touched by the rotor once in each revolution. The question then is, Where will the mass centre be at that instant in the presence of some damping force, and, if not in line with the pencil, what will be the amount of lag or lead, i.e. what will be the angular phase displacement between radii through the pencil to the vertical axis and through the mass-centre respectively? The answer is supplied by the usual equations given in Chap. VI, § 23, for linear mechanical oscillations which are simple harmonic functions.

Assuming a damping force proportional and opposed to the apparent oscillating velocity, when the mass-centre is turning through a half revolution, say, from the front to the back on the right-hand side of the vertical axis through the bearings, the transverse component of the radial centrifugal force, growing and waning, is always directed towards the right and passes through a maximum when the radius to the mass-centre is parallel to the wall; the same also holds for the transverse acceleration which is proportional to the transverse force. A cyclic state of affairs being assumed to have been reached, the case is the same as the damped case of Chap. VI, § 23 (b), when velocity and displacement are interpreted in our case as the instantaneous projections on the wall, i.e. as the transverse components of the real velocity and real displacement.

[1] Cp. Fig. 4 in E. Rosenberg's paper, *Journ. I.E.E.*, vol. 42, p. 532.

As there explained, as soon as even the smallest damping force is introduced into the problem, at speeds below that for which $M\omega^2 = c$ or $\omega = \sqrt{c/M}$ the vector of the displacement *lags* behind that of the applied oscillating force (apparent only in this case) by some angle gradually increasing up to 90°, while the vector of the oscillating speed precedes that of the force. When $M\omega^2 = c$, i.e. at the resonant frequency, as it may be called by analogy, the vectors of speed and force coincide, and the displacement lags 90° behind. Above the resonant frequency, the lag of the displacement continues to increase and for small values of the damping factor p rapidly approaches 180°, but never reaches that value at any finite speed. At the speed ω_{crit} giving resonant frequency, the oscillating velocity always reaches its maximum for any degree of damping. But not so the displacement. This always occurs below resonant frequency, but while for small degrees of damping the frequency giving the maximum displacement is but very little lower than the resonant frequency, as the damping factor is increased, it occurs at lower and lower frequencies (see Figs. 47, 50).

Some proportion of the applied force **F** being expended in overcoming the damping force, it is only the component at right angles to the latter which is effective in causing displacement. This effective component is then not simply the sum of the geometric series $\mathbf{F}_0 (1 + q + q^2 + q^3 + \ldots)$; with each increment of displacement the angle η between the vectors of the oscillating force and displacement alters, so that for each of the steps into which the process has been mentally divided, the effective component is the value of the force at that stage multiplied by $\cos \eta$. The final value of the effective force is therefore $\mathbf{F} = \mathbf{F}_0 \cos \eta\, (1 + q + q^2 + \ldots)$, and the ratio of the amplitudes of the final and initial displacements is

$$a = \frac{\mathbf{F}}{\mathbf{F}_0} = \left(\frac{1 - q^n}{1 - q}\right) \cos \eta = \frac{r}{r_0}$$

At the resonant frequency, $q = 1$, but $\eta = 90°$, so that although $\frac{1 - q^n}{1 - q}$ becomes infinite, its product with $\cos \eta$ remains finite.

The supposed case so far described—but with the linear oscillations in a different plane—is closely reproduced practically in the method used for balancing high speed rotors.[1] The rotor is mounted between two bearings which are free to move in one plane (horizontally or vertically) although their motion is controlled by springs on either side of each plummer block. It is then driven either through a flexible coupling or by a vertical belt (so as to eliminate any horizontal component in the driving force), and brought up to speed. As the resonant frequency is approached, the amplitude of the oscillations increases notably, but since the damping friction even with the plummer blocks mounted on balls is considerable in relation to the centrifugal force, the maximum displacement occurs slightly below the true resonant frequency. The rotor is run at some speed below that for resonance in one direction; it is next run at *exactly the same speed* in the opposite direction, and in each case it is marked on some turned portion where it just touches a marking pointer. The bisector of the two marks then indicates the side of the shaft on which lies the mass centre.[2]

Returning to the vertical shaft, let the whole of the procedure first described in relation to one vertical plane be repeated in relation to a second vertical plane at right angles.[3] A duplicate set of projections or of apparent linear

[1] See E. Rosenberg, *Journ. I.E.E.*, Vol. 42, p. 551, Fig. 16, and especially J. J. King Salter, "The Balancing of Rotors," *Proc. Inst. Naval Architects*, Vol. 62, p. 156.

[2] Many variations are possible and are practised; thus each end may require to be dealt with separately, and one bearing may be locked fast. The rotor may be run up above the critical speed through a clutch by a motor, and then disengaged from the clutch and allowed to run down.

[3] The explanation of the centrifugal whirling of shafts by a theory based on linear oscillations in two planes at right angles to one another, was due originally to Prof. Miles Walker, and has now been described by him in *The Diagnosing of Troubles in Electrical Machines*, pp. 117–129, *q.v.*; in preparing the present article (written prior to the publication of the last-mentioned book) the writer has been indebted to Mr. S. Neville for much kind assistance.

oscillations and of oscillating forces is obtained in the second plane. The true movement and forces are then the vector sums of similar pairs in the two sets, and these being alike, their recombination leads to movement of the centre-line of the shaft in a circle round the vertical centre-line through the bearings. The phenomenon of centrifugal whirling thus finds a simple explanation, and the possibilities of running through the critical speed and of steady running at a speed considerably above it are fully accounted for.

As the speed is increased, the radii of maximum deflection or displacement and of the unbalanced centrifugal force gradually diverge, and when the divergence exceeds the 90 degrees corresponding to resonant frequency, the mass-centre has virtually crept round into a position between the centre line of the shaft (which is in agreement with the displacement) and the centre line through the bearings. The actual motion of the mass-centre is made up of two rotary motions, i.e. of the rotation of the centre line of the shaft round the centre line through the bearings and of the rotation of the mass-centre about the centre-line of the shaft. During the process of running up and of passing through the critical speed during the acceleration period, account would have to be taken of the actual components of the changing speed in the two planes, and the displacements above contemplated for the conditions of constant speed would not be reached even if the damping were very small. The value of the controlling force c per unit displacement is obtainable through an intermediary calculation of the gravity deflection of the rotor shaft under its own weight. But except in a simple case where the load is practically concentrated near the centre halfway between the bearings, there are several values of c, representing different forms of deflection of the shaft. Since the phase angle between mass-centre and maximum displacement is found during balancing to remain not far from 180° at speeds only slightly above the resonant speed, but to change very rapidly near the critical speed, it may be concluded that the damping factor b is less than 0·2, and more probably is only 0·1 of b_{crit} (cp. Fig. 48). The actual cause of the damping force is to be found in the air friction due to the rotation of the centre-line of the shaft about the centre line through the bearings, when resolved into two components at right angles, and in the viscosity of the oil in the bearings.

Next let a magnet be arranged at one side of the vertical shaft opposite the rotor, so as to cause a continued pull in one direction, deflecting the shaft laterally. At very low speeds, the shaft would revolve bent into its deflected position. But now the transverse component of the unbalanced centrifugal force would during one half revolution act in conjunction with the magnetic pull and during the other half in opposition to it. The bending of the fibres of the shaft would cause a loss by mechanical hysteresis, and this would set up a reaction equivalent to harmonic disturbing forces in the two planes at right angles. These components would have to be added to those already described for the simple centrifugal force, and the result would be movement of the centre-line of the shaft round the vertical axis in an elliptical orbit. The case supposed finds its exact counterpart in the ordinary rotor, horizontally supported and acted on by the unidirectional force of gravity.

The problem of centrifugal whirling has also been attacked from an entirely different point of view, and much discussion has been expended on the case of a shaft which is at one and the same time rotating and vibrating laterally. If a shaft supported between two bearings and at rest is imagined to be plucked to one side and released, it will vibrate with a certain frequency depending on its dimensions and elasticity. If the shaft is at the same time rotating, as its speed is increased, the frequency of vibration gradually diminishes. This is due to the action of the unbalanced centrifugal force opposing the resistance to deflection due to the elastic stress in the material of the shaft; the centrifugal force thence retards the vibration so that the shaft takes longer to pass from its position of maximum displacement inwards to the centre and thence outwards again. At some speed then there comes a critical point at which the lateral vibration vanishes, and the shaft "whirls," just as a skipping-rope held at its two extremities and swung round. At this point the unbalanced centrifugal force exactly annuls the righting force, and the periodic time of a revolution is equal to the original periodic time of a

lateral vibration when the shaft is at rest. The lateral motion of a vibrating shaft being resolved into a number of harmonics of different frequencies, let $k_1/2\pi$ be the lowest of these frequencies, then the critical angular velocity of whirling would be $\omega_{crit} = k_1$, and the determination of ω_{crit} is reduced to the problem of finding the lowest frequency of the transverse vibrations of the shaft when it is not rotating.[1] But while the truth of the facts last mentioned does not appear to be open to question, the difficulty remains that in the ordinary cases of practice there does not appear to be any sufficient cause to originate the transverse vibration which the theory presupposes. For this reason the simpler explanation first given appears preferable.

To apply it in order to determine the speed at which the *displacement* will be a maximum and which will be below the true resonant speed still remains a complicated problem. The resonant speed must first be found, and to find this, c, the proportionality factor of the controlling force, must be calculated. For a single concentrated mass $M = \frac{W}{g}$, on a shaft of uniform diameter, c would by § 9 be $\frac{3l\,EI}{l_1^{\,2}\,l_2^{\,2}}$, where l_1 and l_2 are the distances of the mass from the centres of the bearings. But when the diameter and moment of inertia of the several sections of the shaft and rotor, as in practice, vary greatly, the bending moment diagram must be reduced to that for an equivalent shaft of uniform diameter. Thence by a double integration a deflection diagram with maximum value δ can be obtained graphically, and the concentrated force F that would give a similar diagram can be approximated, from which $c = F/\delta$.

The whole problem assumes great importance in the case of dynamos and alternators driven at high angular speeds by steam turbines or waterwheels, but since these are not here under consideration, further treatment of it is postponed. It needs only to be added that the greater the diameter and stiffness of the shaft for a given length between its bearings, the less the deflection due to the initial lack of a perfect dynamic balance, and the greater the likelihood that the running speed will fall below the critical speed—a condition which, where possible, should always be aimed at.

§ 11. Diameter of armature shaft. It is manifest that the influence of all the causes which tend to stress the shaft is to a large extent indeterminate, and we must therefore in the absence of complete data fall back upon approximations which in practice have been found to give sufficient strength to withstand the working shocks and stresses and sufficient stiffness against vibration.

Experience shows that in most cases satisfactory working can only be effectually ensured by taking the maximum bending moment as about three times that which would be due to the simple weight of the armature—or in other words, on the approximate assumptions of § 7

$$B_m = 345{,}000\,\frac{kW}{N}, \text{ or } 345\,\frac{w}{N} \text{ inch-pounds}$$

$$= 217{,}000\,\frac{HP}{N} \text{ inch-pounds.}$$

The twisting moment being

$$T_m = 100{,}000\,\frac{kW}{N}, \text{ or } 100\,\frac{w}{N}$$

$$= 63{,}000\,\frac{HP}{N}$$

[1] As given in "The Bending, Vibrating, and Whirling of Loaded Shafts," by Capt. J. Morris (*Reports and Memoranda of the Advisory Committee for Aeronautics*, No. 551).

the ***equivalent bending moment*** is

$$B_e = \tfrac{1}{2}B_m + \tfrac{1}{2}\sqrt{(B_m^2 + T_m^2)}$$

$$= 352{,}500\,\frac{kW}{N}, \text{ or } 352\,\frac{w}{N} \text{ inch-pounds} \qquad (74)$$

$$= 222{,}000\,\frac{HP}{N} \text{ inch-pounds.}$$

It is evident that the bending action is of far greater importance than the simple torsion, and in consequence it is in the estimate of the bending moment that the apparently large factor of safety of 3 must be introduced. The absence of the belt pull in engine-driven dynamos is roughly counterbalanced by the greater span and the share which the dynamo shaft takes in carrying the weight of the fly-wheel.

Inserting the intermediate value of 6,000 lb. per square inch for f_t in equation (72) we have for the diameter of the shaft

$$d = \sqrt[3]{600\,\frac{kW}{N}} = 8{\cdot}5\,\sqrt[3]{\frac{kW}{N}} \text{ or } 0{\cdot}85\,\sqrt[3]{\frac{w}{N}} \qquad (75)$$

$$= 7{\cdot}25\,\sqrt[3]{\frac{HP}{N}}$$

By such equations the minimum diameter of the shaft at the centre of the armature or at any part beneath armature or commutator may practically be determined. Both the hub of the armature and the sleeve of the commutator add to the rigidity of the shaft, and if, as is often the case, there is a gap between these two, the bending is largely concentrated thereat, so that it is of chief importance to maintain the diameter to its full value at this spot. For very small armatures the shaft as given by the above general equations may not be sufficiently stiff, so that ⅛ to ¼ inch must be added to the diameter. In every case any special circumstances of the design must be considered; thus in dynamos with a very small air-gap or of large diameter with a comparatively small clearance, the magnetic pull due to inequalities in the strength of the several fields and any tendency to vibration must be specially guarded against by employing an exceptionally stiff shaft.

If a key-way is to be cut in the shaft for keying on the discs or their supporting hub, further allowance must be made; and in general to give stiffness at the centre, the shaft is usually there swelled out to a larger diameter than it has within the bearings. Any such alteration in the size of shaft requires, of course, to be effected with a fairly large radius in order to avoid opening of the fibres of the steel at the corner where the change of diameter is made.

Dynamos driven directly by the engine are bolted to the crank-shaft either by a solid half-coupling forged on the armature shaft or by a loose half-coupling of cast-iron keyed to the shaft. Sufficient space must be allowed on one or other side for the withdrawal of the coupling bolts.

In the case of traction generators which are subjected to great and sudden changes of load causing large and rapid interchanges of energy[1] between fly-wheel and armature, the actual dynamo shaft may be relieved of a large portion of the stress by prolonging the armature cast-iron hub to form a coupling by which it may be immediately bolted to the fly-wheel (Figs. 120 and 121). The shaft has then merely to take the bending moment due to the weight of the armature at the outer bearing, and the stress of driving is transmitted directly from the fly-wheel into the hub and its core-discs without passing through the key or keys by which the hub is fixed to the shaft.

Within the bearing nearest to the driving-point the bending moment is less than at the centre of the armature, and the diameter of shaft may be correspondingly reduced. The possible reduction is, however, much greater in small than in large machines, since in the former the diameter at the centre is proportionately larger in order to give sufficient stiffness to the shaft. Thus in machines giving less than 10 watts per rev. per minute the diameter within the bearing may be only 75 per cent. of the smallest diameter within the armature, and this proportion will rise to, say, 90 per cent. in machines giving over 100 watts per rev. per minute. The diameter of the shaft within the outer or commutator bearing may be still further reduced, since the horsepower is absorbed within the armature and the twisting moment becomes negligible; in many cases, however, for convenience of manufacture, the same diameter of journal is retained throughout, even though within the one bearing there is a surplus of strength, and in all cases a small number of different diameters along the shaft conduces to economy in its manufacture.

The above approximate figures require in every case to be checked by consideration of the strength of the journal. Thus in the case of the belt-driven dynamo of Fig. 138 the journal next to the pulley is subjected to a combination of a bending[1] moment $3T \cdot l_3$ and a twisting moment T_m.

The equivalent bending moment[2] is then

$$B_e = \tfrac{1}{2}(3T \cdot l_3) + \tfrac{1}{2}\sqrt{(3T \cdot l_3)^2 + (T_m)^2}$$

[1] The weight of the pulley itself is not here taken into account.

[2] Unwin, *Elements of Machine Design* (edit. 1909), Part I, pp. 254–5. More accurately, if l' is the length of the journal, and if t_1 = the reaction of the bearing against the tension of the belt as before, the bending moment due to this reaction if distributed uniformly over the whole length of the journal

Equating B_e to the moment of resistance of the circular shaft to bending, the minimum diameter of the shaft within the pulley bearing is given by the relation

$$B_e = 0{\cdot}098 f_1 (d')^3$$

where f_1 as before may be taken as 6,000 lb. per square inch.

In the outer bearing farthest from the pulley there is no twisting moment, but the bending moment is the combined result of the weight of the armature and the pull of the belt. The load on the bearing is $P_2 = \sqrt{w_2^2 + t_2^2}$, and the bending moment[1] of this at the inside end of the shaft nearest to the armature is $\frac{1}{2} P_2 l'$. The diameter is then deduced from

$$\tfrac{1}{2} P_2 l' = 0{\cdot}098 f_1 (d')^3 \quad . \quad . \quad . \quad . \quad (76)$$

Since l' is at present undetermined, the above relation can provisionally be solved by substituting $\frac{P_2}{p \cdot d'}$ for l', where p is the intensity of the pressure on the area $d' l'$ of the bearing projected on to the diametral plane, and must be given a normal value such as is found in practice (*see* § 13). Thence

$$d' = \sqrt{\frac{5 P_2^2}{p \cdot f_1}} \quad . \quad . \quad . \quad . \quad (77)$$

§ 12. Friction of bearings and their dimensions. With a copious supply of oil well introduced between the journal and bearing surface, and provided that the intensity of the pressure per square inch is not so great as to be on the point of squeezing out the lubricant, the coefficient of friction μ, so far from being a constant as in the case of solids, has been found to vary nearly inversely as the intensity of the pressure and inversely as the temperature within the usual range of these quantities. The relation of μ to the velocity is of a more complex character; up to 470 feet per minute for a given pressure and temperature it is nearly proportional to the square root of the velocity, between 470 and 790 feet per minute it is more nearly proportional to $\sqrt[5]{v}$, while for speeds above 2,000

$= t_1 \cdot \frac{l'}{8}$, and this amount must be deducted from the bending moment due to the belt pull. The true bending moment at the centre of the journal neglecting the weight of the armature which may or may not act in the same plane is thus

$$B_m = 3T \cdot l_2 - t_1 \frac{l'}{8} = 3T \cdot l_2 - 3T \cdot \frac{l'}{8}\left(\frac{l_2 + l_3}{l_4}\right)$$

$$= 3T\left\{l_2 - \frac{l'}{8}\left(\frac{l_2 + l_3}{l_4}\right)\right\}$$

which must be substituted for $3T \cdot l_2$ in the equation for the equivalent bending moment.

[1] Unwin, *Elements of Machine Design* (edit. 1909), Part I, § 165.

feet per minute it is practically independent of v. The watts lost in bearing friction are then proportional in the three cases to $v^{1.5}$, $v^{1.2}$ and v. For the lower speeds from 300 to 600 feet per minute, which are usually met with in dynamo practice (apart from turbo-generators), the square root proportionality may be assumed, and we then have $\mu = c \dfrac{\sqrt{v}}{p \cdot T}$, where c ranges from 1·32 to 1·6 when p is expressed in lb. per square inch of projected bearing surface, v in feet per minute, and the temperature of the bearing bush is reckoned in degrees Centigrade. The influence of different kinds of oil and of different combinations of metals in the journal and bearing bush respectively is but small, so that finally it may be said[1] that for a steel shaft running on a gun-metal or white-metal surface

$$\mu = 1{\cdot}49 \frac{\sqrt{v}}{p \cdot T}, \text{ approximately.}$$

Unless the bearing is artificially cooled, a high velocity is accompanied by a high temperature, and *vice versâ*, so that, as will be seen from Fig. 141, if an initial temperature of 20° C. be assumed for the air, $\sqrt{v}/T$ has almost a constant value under given conditions of natural cooling, and for bearings of similar type whatever the speed. It averages about 0·4, so that μ is then simply inversely proportional to the specific pressure, and $\mu = 0{\cdot}596/p$. Inserting the specific pressures of 170 and 65 lb. per square inch of projected bearing surface as the upper and lower limits which are likely to occur in practice, μ is found to vary between 0·0035 and 0·009, the higher value corresponding to the lower specific pressure. It must again be emphasised that the above practical values for μ only hold good if the natural relation between the velocity and the temperature of the bearing is not largely modified as would be the case if the running of the journal was entirely dependent upon artificial cooling.

Since the total load on the bearing $P = pd'l'$ lb. where d' and l' are in inches, the frictional resistance $R = \mu P = 1{\cdot}49 \dfrac{\sqrt{v}}{p \cdot T} \cdot pd'l'$ $= 1{\cdot}49 \dfrac{\sqrt{v}}{T} \cdot d'l'$ lb., is seen to be completely independent of the load for a given diameter and length; the work done and the heating

[1] See especially the results of the exhaustive experiments given by Lasche, "On Bearings for High Speeds" *Traction and Transmission*, Vol. 6, p. 33 ff., and the earlier experiments of G. Dettmar, "On Friction Losses in Dynamos," *E.T.Z.*, Vol. 20, pp. 380 and 397. The latter gives a much higher value for c, namely, 2·69, *i.e.* $\mu = 2{\cdot}69 \dfrac{\sqrt{v}}{pT}$, whence it results that for a temperature of 37·5° C. in the bearing bush the loss by friction is at the rate of $2{\cdot}16 \times 10^{-4}$ $d'l'(d'N)^{1.5}$ watts instead of the value which is given above. But in this is included windage or air friction as well as true bearing friction.

which thence results are due to the shearing of the viscous liquid in its given state of temperature, and though the application of the load may alter the distribution of the pressure round the circumference of the journal within the bearing, the mean thickness of the lubricating film remains practically constant, and so also the work done in shearing it. The power lost in the bearing is then

$$K = \frac{1{\cdot}49\, d'\, l'\, v^{1.5}}{33{,}000\, T}$$

$$= 45{\cdot}1 \times 10^{-6}\, \frac{v^{1.5}}{T}\, d'\, l' \text{ horsepower}$$

$$= 33{\cdot}7 \times 10^{-3}\, \frac{v^{1.5}}{T}\, d'\, l' \text{ watts} \quad . \quad . \quad . \quad (78)$$

It is therefore proportional to the bearing surface and to the 1·5th power of the velocity. Or again, since $v = \pi d' N / 12$ feet per minute, the friction loss is

$$6{\cdot}02 \times 10^{-6}\, \frac{(d' N)^{1.5}}{T}\, d'\, l' \text{ horsepower}$$

$$= 4{\cdot}5 \times 10^{-3}\, \frac{(d' N)^{1.5}}{T}\, d'\, l' \text{ watts} \quad . \quad . \quad . \quad (78a)$$

Assuming the temperature of the bearing bush to be 37·5° C. (100° F.), an average figure for the friction loss in a bearing would be

$$1{\cdot}61 \times 10^{-7}\, (d' N)^{1.5}\, d'\, l' \text{ horsepower}$$

$$= 1{\cdot}2 \times 10^{-4}\, (d' N)^{1.5}\, d'\, l' \text{ watts} \quad . \quad . \quad . \quad (78b)$$

As mentioned above, a reduction of the diameter, provided that the bearing is not artificially cooled but is dependent upon its own specific rate of radiation, reduces its rise of temperature. From the formula for the friction loss it is seen that the reduction of the rise of temperature has in itself the effect of increasing the friction loss; yet since the friction loss is proportional to $d'^{2.5}$, while the rise of temperature is less than proportional to d, and even then has to be added to a fixed initial temperature of the air, it is evident that a reduction in the diameter of the journal to the minimum required by considerations of strength is in every case to be recommended on the double score of increasing the efficiency of working and of reducing the temperature of the bearing which makes its running more reliable.

The diameter being thus fixed as low as is safe, $l' = \frac{P}{p d'}$, and p is to be taken as high as is found practicable, since the efficiency is increased by reducing the length as much as possible. Especially is this the case with the outer bearing farthest from the driving engine, since from § 11 the necessary diameter for this is itself

partly dependent upon the reciprocal of the intensity p; in consequence the higher the specific pressure, the less the diameter and the less the rise of temperature, which again assists by reason of the viscosity of the lubricant being better maintained in a state to withstand the tendency for it to be squeezed out. It is, however, evident that in any case, even if the bearing is artificially cooled, the intensity of the pressure must never approach the limit when the oil is liable to be squeezed out. The higher the temperature, the less the viscosity of the oil, the film which separates the two surfaces becomes thinner, and as soon as it is ruptured the journal seizes; the intensity of the pressure when the lubricant is squeezed out is thus partly dependent upon the temperature, but under ordinary conditions when the oil is not forced through the bearing under pressure it ranges from 400 to 500 lb. per square inch of projected bearing surface.

In the case of dynamo bearings in which the load is continuous and always in the same direction, as *e.g.* on the lower bush when the effect of any eccentricity of the armature within the bore is added to that of the weight, the permissible specific pressure must be very much reduced below the above value, especially when due regard is had to the very important question of the amount of wear of the bearings in prolonged work.

In order to ensure durability and reliability in working it is necessary in all high-speed machinery to make the journals of considerable length as compared with their diameter, and in belt-driven armatures of small size running at 1,000-1,200 revolutions per minute, the ratio most frequently observed is $l'/d' = 4$, where d' is the diameter of the journal. In engine-driven armatures and in large machines generally, running at about 300 revolutions per minute or less, the proportionate length may be reduced, say, to $2\frac{1}{2}$ diameters; or in general $l'/d' = \frac{1}{7}\sqrt{N}$ to $\frac{1}{8}\sqrt{N}$.

The increase of the ratio of length to diameter with higher speeds, such as is shown in the following table,

N revs. per min	200	300	400	500	700	900	1100
$\frac{l'}{d'}$	2	2·5	3	3·25	3·5	3·75	4

really amounts to a reduction of the pressure per unit of bearing surface as the velocity of the relative movement between journal and bearing increases. Experience therefore dictates that from considerations of wear the necessary length of bearing must be deduced from specific pressures which are reduced as the velocity is increased. We thus have in practice such limiting values as $p = 170$ lb. per

square inch of projected bearing surface with a velocity of 270 feet per min., decreasing to $p = 50$ lb. per square inch with a velocity of 600 feet per min. Thence $l = \frac{P}{p \cdot d'} = \frac{P \times \pi N}{p \times 12v} = 5{\cdot}7 \times 10^{-6}\, PN$ inches with the higher pressure and lower speed, and $= 8{\cdot}7 \times 10^{-6}\, PN$ inches with the lower pressure and higher speed; or on the average, say, $p = 70$ lb. and $v = 500$ feet per minute, whence approximately

$$l = 7{\cdot}5 \times 10^{-6}\, PN \text{ inches} \quad . \quad . \quad . \quad (79)$$

As a general rule for ordinary bearings pv does not exceed 60,000.

As an example of the calculation of P, the total load on the pulley bearing of Fig. 138 is $P_1 = \sqrt{w_1^2 + t_1^2}$, or if $l_1/l_2 = 4/6$, as was assumed before, and $l_3/l_4 = 1/4$,

$$P_1 = \sqrt{(W \times 0{\cdot}6)^2 + (3T \times 1{\cdot}25)^2};$$

that on the other bearing is

$$P_2 = \sqrt{w_2^2 + t_2^2} = \sqrt{(W \times 0{\cdot}4)^2 + (3T \times 0{\cdot}25)^2},$$

and the intensity of pressure in pounds per square inch of projected area is $p = \frac{P}{d' l}$. Since the driving tension $T = 126{,}000 \frac{HP}{dN}$ pounds $= 200{,}000 \frac{kW}{dN}$ pounds where $d =$ the diameter of the pulley in inches, or if V is the velocity of the belt in feet per minute since $T = \frac{33{,}000\, HP}{V} = \frac{52{,}000\, kW}{V}$ pounds, the above equations may also be expressed as follows:

$$P_1 = \sqrt{18{\cdot}6 \times 10^{6}\left(\frac{kW}{N}\right)^{\frac{4}{3}} + 3{\cdot}8 \times 10^{10}\left(\frac{kW}{V}\right)^2}$$

$$P_2 = \sqrt{8{\cdot}4 \times 10^{6}\left(\frac{kW}{N}\right)^{\frac{4}{3}} + 15{\cdot}2 \times 10^{8}\left(\frac{kW}{V}\right)^2}$$

It will be found that $P_1 = 4T$ is a convenient approximation which holds very closely in all ordinary cases, the effect of the weight acting at right angles to T raising the total load from $3{\cdot}75T$ to $4T$.

In drum machines the armature core centres itself longitudinally within the pole pieces by reason of the magnetic pull to which it is subjected when displaced axially.[1] A slight amount of end-play is not disadvantageous, since it secures a more uniform wear of the commutator surface, but it is seldom allowed to exceed $\frac{1}{16}$th of an inch to $\frac{1}{8}$th in large machines, and it must be limited by raised

[1] For the forces brought into action by a to-and-fro axial movement of the armature, see F. W. Carter, "Magnetic Centering of Dynamo-electric Machines," *Proc. Inst. C.E.*, Vol. 187, pp. 311–318.

shoulders on the shaft. The centre of the armature core must be in line with the centre of the pole-faces, so that there may be no magnetic pressure against the collar causing it to heat up.

§ 13. Temperature of bearings.—A second and independent question is the temperature which the bearing will attain in working, *i.e.* whether with a given diameter as fixed by considerations of strength and with a given number of revs. per minute it can be counted upon to be self-cooling. The rate at which heat is generated in the work of overcoming the frictional resistance of the bearing must not be greater than the rate at which it can be dissipated conveniently without any undue increase of its temperature. Two cases then have to be distinguished according as the bearing is dependent only upon its own radiation of heat from its outside surface assisted by any convection currents due to rotating masses in close proximity, or is artificially cooled by oil or water being circulated through it.

The rate at which heat can be dissipated from a bearing without artificial cooling depends upon a number of complex conditions, chief among which are the difference of temperature between the bearing bush and the surrounding air, and the area of the radiating surface of the metal plummer-block. To these must be added the effect of the artificial ventilation which, especially in dynamos, may arise from the proximity of the rotating commutator or armature winding. But for a given rise of temperature of the bearing bush which causes the heat to be conducted to the outside of the plummer-block the rate of dissipation may be taken as proportional to the surface of the journal or to its projected area $d'l'$, so that for each value of the rise of temperature there corresponds a certain specific radiation of horsepower per square inch of projected bearing surface which is similar in bearings of similar type. According to the experiments above-cited of Herr O. Lasche, the specific radiation from the outside of the plummer-block (the oil remaining unchanged as in ring lubrication) increases faster than the difference between the temperature of the bearing bush and that of the surrounding air, and, assuming an initial air temperature of 20° C., is approximately proportional to the 1·5th power of the rise of temperature of the bearing bush, or $h = k_a(\Delta t)^{1 \cdot 5}$. In still air, with h reckoned in horsepower per square inch of projected bearing surface, k was found by experiment to be about $22 \cdot 5 \times 10^{-6}$ in the case of ordinary bearings, and 42×10^{-6} in the case of specially massive bearings, when t is in degrees Centigrade. In dynamos, partly perhaps by reason of the ventilation, which is always more or less present from the commutator or armature winding, these values appear to be exceeded in practice. The specific rate of radiation in horsepower per square inch probably ranges from 0·008 to 0·016, and assuming that in these two limiting cases the bearing temperatures are 46°·2 C.

and 61°·5 C., or the rises of temperature 26°·2 C. and 41°·5 C., k then becomes as much as 60×10^{-6}.

To the quantity $hd'l'$ must be added, in the case of artificially cooled bearings, the heat that is abstracted per minute by water circulated round the bearing or by the oil which is passed through the bearing and withdrawn as it becomes heated. Thus in the latter case it is found that the oil leaves the bearing at a temperature about 15° C. below that of the bush, or if the latter is to be maintained at say 75° C., the oil leaves at 60° C. If it is then cooled down to 45° C. and supplied anew to the bearing, Q gallons being passed through the bearing per minute, the rate at which heat has been withdrawn is proportional to $(60 - 45) \times$ specific heat per gallon $\times Q$. When expressed in horsepower, this quantity is to be added to the rate of radiation $hd'l'$, and then sum must then be equal to the rate at which heat is generated, namely,

$$\frac{\mu P\pi d'N}{12 \times 33{,}000} = \frac{\mu P d'N}{126{,}000} \text{ horsepower.}$$

In the more ordinary case of a bearing without artificial cooling it is only the specific rate of radiation which must be equated to the rate at which heat is generated, both being expressed, *e.g.* in horsepower per square inch of projected bearing surface. Assuming, then, the above expressions for h and μ (eq. 78) fairly to represent the facts under ordinary conditions of medium speeds, pressure, and temperature, the two rates of generation and radiation of heat in horsepower per square inch of projected bearing surface when equated give

$$\frac{45.1 \times 10^{-6}}{T} \times v^{1.5} = k \times (T - t_0)^{1.5}$$

From this the important result follows that with bearings of similar type naturally cooled, to every linear speed of journal there corresponds a particular temperature rise independently of its length and pressure. An initial air temperature of 20° C. may in all cases be assumed, so that

$$T^{1.5}(T - 20°) = \frac{45.1 \times 10^{-6}}{k} v^{1.5} \quad . \quad . \quad . \quad (80)$$

or with $k = 60 \times 10^{-6}$

$$= 0.75\, v^{1.5}.$$

Fig. 141 shows three curves obtained from (80) with values of $k = 37 \times 10^{-6}$, or 60×10^{-6}, or 90×10^{-6}, the first being the highest for the case of no auxiliary ventilation, and the intermediate corresponding to the more usual case of dynamo bearings with some natural windage. The general shape of the curves is closely borne out by experiment, although the value of the constant varies

considerably; it is practically independent of the pressure, and so of the length of the bearing, although there is a tendency for the rise to be slightly greater with higher pressure.

From a curve such as those of Fig. 141 it can at once be determined whether it will be necessary to resort to artificial cooling in any given case, a maximum temperature which should not be exceeded in the bush being fixed at 70° C., and the highest curve being taken in order to be on the safe side.

It is thus evident that h is only of importance as determining the question whether a bearing of given diameter and at a fixed speed

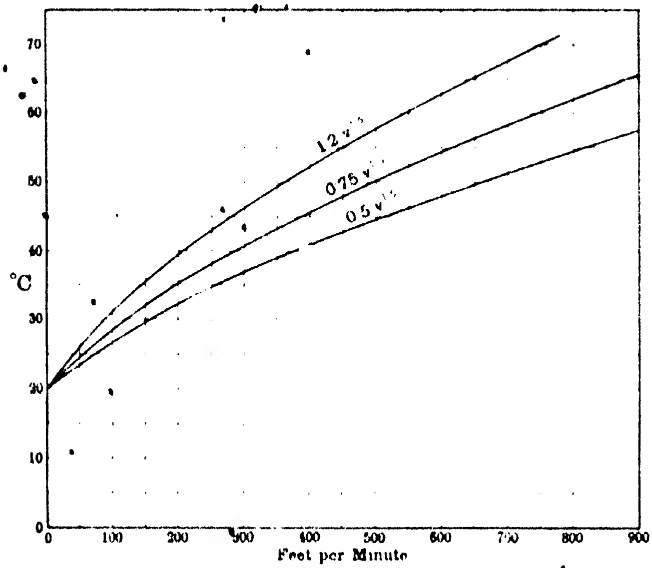


FIG. 141. Temperature of bearing bush and speed of journal.

can be self cooling, and its exact value in ordinary cases is of no great interest, since there is usually a good margin in reserve. The necessary length of the bearing can be made to appear dependent upon h, by the equality $\frac{\mu P d' N}{126{,}000} = h d' l'$, whence $l' = \frac{\mu P N}{126{,}000\, h}$. But such dependence is more apparent than real, since μ is itself increased by an increase in h, and it will be found that by substitution of the full expression for μ and of h in terms of the temperature and velocity we simply return to our original equation $l' = P/pd'$.

While in the above a medium speed of journal surface has been contemplated, the different law which holds for very high peripheral speeds must be taken into account in the case of turbo-dynamos.

When the peripheral speed of the journal exceeds 900 feet per minute, up to, say, 4,000 feet per minute, the friction is practically independent of the velocity, and between such limits as from 20 to 200 lb. pressure per square inch, and consequent temperatures in the rubbing surfaces of 30° to 100° C., Lasche found it to be approximately true that $\mu p T$ = a constant = 28·45, where p is in lb. per square inch and T in degrees Centigrade. The horse-power generated in the bearing, if v is in feet per minute, is

$$\frac{\mu P v}{33{,}000} = \frac{\mu p d l' v}{33{,}000}$$

and, since $\mu p = 28\cdot45/T$, the rate of generation of heat per square inch of projected area will be

$$\frac{v}{T} \times \frac{28\cdot45}{33{,}000} = k(t - t_0)^{1\cdot3}$$

or if the initial temperature is 20° C.

$$(t - 20)^{1\cdot3}\,t = \frac{862 \times 10^{-6}}{k}\,v \qquad (80a)$$

and if $k = 60 \times 10^{-6}$

$$= 14\cdot35\,v.$$

Above 125° C. (257° F.) Lasche found that the lubricating properties of oil rapidly fell off.

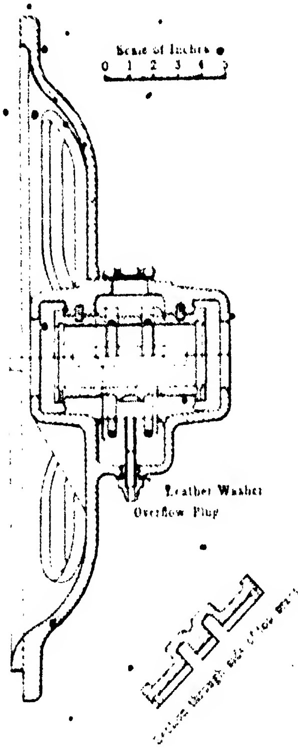


Fig. 142.—Bearing in end-shield casting of small machine.

§ 14. Pedestals and plummer-blocks.—The journals are supported either by 2, 3, or 4 arms from the end-shield castings of small multipolar machines (Fig. 142), or by pedestals which either form part of the base-plate casting or are separate plummer-blocks (Figs. 143 and 144). They are fitted with a gun metal or phosphor-bronze "brass," or with a cast-iron shell or bush lined with white-metal. The white-metal is locked in the shell by being run into recesses or grooves in it which have a dovetailed section or overhanging edges. In all except small machines the white-metal bearing is to be preferred, since if the bearing becomes overheated the white-metal melts and runs out; indication is thus given of the overheating without so much injury to the running surface of

the journal as always results from the "seizing" of a solid gun-metal bush. The brass or liner is prevented from turning either by snugs which engage in corresponding recesses in the pedestal or by

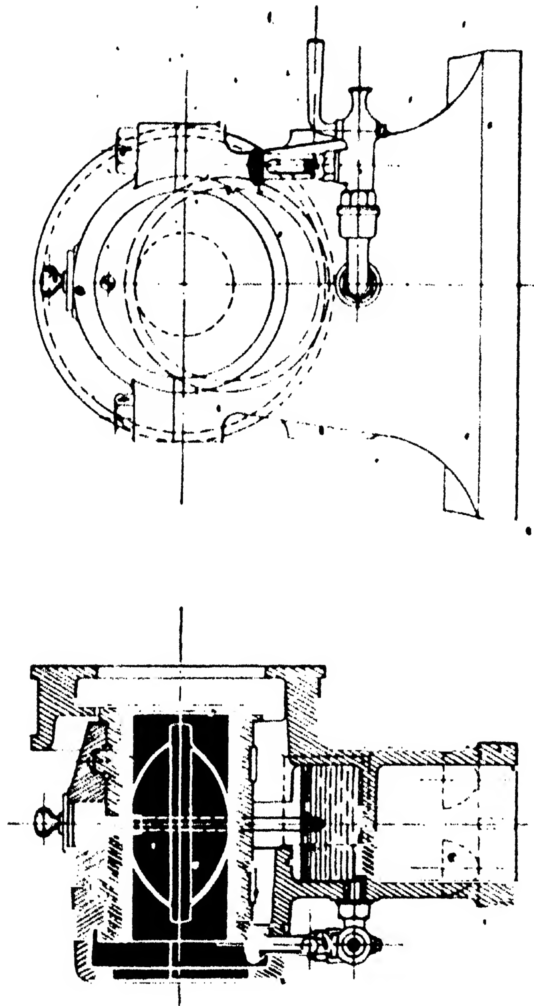


Fig. 11.—Self-oiling bearing for commutator end.

a screw in the upper part of the pedestal. The bush may be solid when it can be slipped over the end of the shaft, and then has a shoulder at one end to prevent axial movement in one direction. But preferably it is split, and can then have a shoulder at either

end (Figs. 142 and 143); the pedestal is provided with a separate cap fastened by two or four screws, and the arrangement has the advantage that the two halves can be closed together to take up wear, while the solid bush when appreciably worn requires to be entirely relined with white-metal. The division of the cap from the pedestal should be approximately at right angles to the line of maximum resultant pressure upon the bearing; the joint should therefore, in large dynamos driven by horizontal belting or by a horizontal steam-engine, be inclined at about 45° to the vertical.

In order to facilitate the self-alignment of the bearing on the shaft,

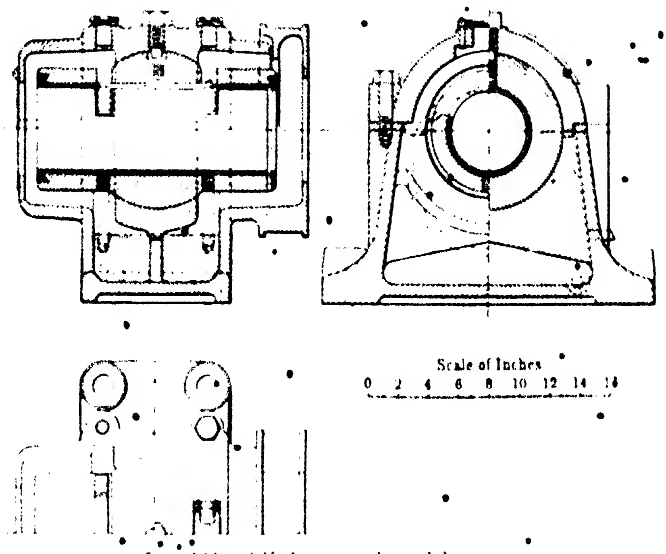


FIG. 144.—Self-aligning spherical bearing.

the sleeve or bush in the commutator bearing of the directly driven dynamo is frequently so made as to allow of a certain amount of *swivelling* movement. The outside of the bush at the middle of its length forms part of a sphere, and is given a correspondingly spherical seating (Fig. 144), or if the pressure be small the seating may be cylindrical when the bush may be slipped in from the end within a solid pedestal.

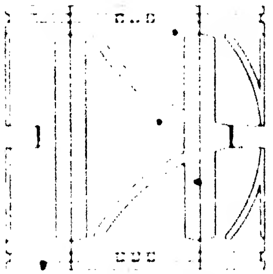
At each end of the bearing are hooded chambers in which the oil which exudes from the ends of the journal is caught and returned by grooves cut at the bottom of the bearing into a reservoir. These grooves should be of ample section and with sufficient fall to return the oil quickly to the reservoir. At the end nearest to the armature the shaft must be provided with an *oil-thrower* to prevent the oil from

creeping along the shaft on to the armature or commutator or brush-carrier and there destroying the insulation. Either an upstanding annular ridge is formed on the shaft when it is turned, or a collar with a thin pointed edge of larger diameter is shrunk on to the shaft (Figs. 200 and 184); in either case the oil is drawn to the outer edge of the ring and thence is flung off by centrifugal force and caught in the hollowed chamber within the bearing. The curvature of the hood must be such as to cause the oil to be returned on the bearing side rather than on the armature side, whence it may drip on to the shaft past the oil-thrower. At the outer or commutator end a plate over the opening of the bearing is of convenience to prevent the entrance of dust, and in other cases, if the rotating armature winding, or the proximity of a fan causes a draught of air along the bearing, a brass ring in halves should be fitted so as closely to embrace the shaft at the end of the bearing, in order to prevent the oil being drawn along the shaft.

In small machines the white metal is sometimes run directly into the cast-iron pedestal, an oiled mandril or the armature shaft itself being previously inserted and held centrally in place so that the metal may run round it. The whole is heated up, and passages are left to allow the heated air to be expelled. As the metal cools the mandril is turned round in guides by hand, and a hard smooth surface is obtained which does not require to be subsequently machined.

§ 16. Lubrication of bearings. Lubrication is usually effected in dynamo bearings by means of brass or cast white-metal or aluminium rings (Figs. 143 and 144) which rest on the journal and dip into an oil reservoir, formed in the hollow pedestal and of sufficient size to allow any sediment in the oil to settle. The section of the rings is slightly tapered, being broader at the base; when the shaft rotates they are carried slowly round by the friction of their contact with it, so that the journal becomes self-lubricating as soon as the armature rotates. In the upper half of the bush or liner are as many gaps as there are rings, while the bottom half remains whole; or the gaps in the upper half may be joined by bridge pieces at the sides so as not to interfere with inspection of the working of the rings through openings at the top (Fig. 142). The oil is drawn up by the motion of the ring and at the top of the journal enters a recess cut longitudinally along the brass, whence it is distributed by grooves cut spirally along the inner surface. All such grooves and recesses must be sealed at their farther ends so as to confine the oil, and it is important that the oil should be introduced at the point of least pressure so that it may be swept forwards by the rotating shaft; hence the grooves should follow the direction of rotation, and if the pressure between journal and bearing should prevent the entry of the oil on this side, the grooves must be made

to feed a second recess at some point where the pressure is relieved. Fig. 145 shows a bearing opened at the top and unrolled; with the given direction of rotation the oil is fed into the right-hand longitudinal recess, and thence distributed by the grooves. The edges of the grooves must not be sharp, but must be slightly rounded off so as to allow the oil to be drawn over the shaft by its rotation. Care must be taken that the actual bearing surface is not too much reduced by the grooves. At each end of the bush are circumferential grooves to collect the oil and return it to the reservoir through the vertical holes in the grooves before it spreads outwards along the shaft. Clearance[1] between the journal and bush reduces the friction, but has to be kept small to avoid vibration, especially at the commutator end; the diameter of the bearing exceeds the diameter of the shaft by 2 to 4 mils, according to the size of shaft. The rings and the surface at the sides against which they may touch must be quite smooth. Where it is inconvenient or impossible to thread them over the shaft they must be in two halves, hinged together, and fastened at the opposite side by a screw or spring clip as in Fig. 146. The diameter of the rings should be about one and a half times that of the journal, and large enough to pass over the ends of the brass. On bearings exceeding 8 inches in length two or more rings may be employed. Chains are also sometimes employed in place of rings, but although they lift more oil they are not so suitable, since they necessitate more of the brass being cut away, and at high speeds cause frothing of the oil owing to the admixture of air. The oil should not stand so high in the reservoir as to clog the free movement of the rings, or to rise into the groove at the bottom of the bush, and its height is usually indicated by an oil gauge with which may be combined a draining tap for use when at intervals the oil has to be changed (Fig. 142 or Fig. 143).



Brass cut through from top and opened out

FIG. 145. Plan of oil grooves in bearing brass.

[1] See Lasche, "Bearings for High Speeds," *Traction and Transmission*, Vol. 6, pp. 44–49, where experiments are also given as to the amount of oil delivered by ring lubrication, and "Steam Turbines and Turbo Generators" (W. J. A. London), *Journ. I.E.E.*, Vol. 35, pp. 188–190.

The covers of the inspection openings, if circular plugs, should be fastened to the bearings by a small chain to prevent their being lost, but preferably are either sliding or hinged with a spring to keep them closed. There must be good clearance between the ring in its working position and the surrounding walls so as to lessen the amount of oil thrown on to the joint whence it will creep to the outside; a stepped or registered joint is for the same reason of assistance, and should be employed in large bearings. In any case the joint of the split bearing should be carefully surfaced so as to prevent the oil working its way through it. In large machines the bearings are sometimes arranged with a water jacket through which a stream of water can be passed in the emergency of a hot bearing.

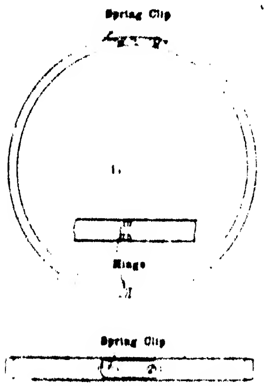


FIG. 146. Hinged oil ring.

§ 16. Insulating materials. No one substance combines all the various qualities that go to make up the ideal insulating material, and among a number of materials a judicious selection has to be made to obtain that which is best suited to the exact end in view. For insulating purposes in dynamo work a high specific resistance, as expressed by megohms per centimetre cube of the material, is by no means so important as a high "*disruptive*" or *dielectric strength* to resist breaking down under the stress of a high voltage. The one property is not necessarily accompanied by the other, and in fact no very direct relation between the two can be traced. Comparison of the value of different materials must therefore primarily be based upon the voltage (continuous or maximum alternating volts) which on an average they can be relied upon to withstand per mil of thickness without being pierced. For the same maximum volts the alternating current is the more likely to cause puncture, probably chiefly owing to the heating of the dielectric which results from the alternating electrostatic stress. The energy loss in a dielectric under such stress is for the same temperature proportional to the square of the voltage, and also increases with the frequency. A high frequency is therefore more trying, since under many circumstances the heat cannot be dissipated as quickly as it is generated; the temperature then rises, perhaps only locally, but this again increases the heating, and so on cumulatively until breakdown occurs, especially if the initial temperature is high. Fibrous materials usually break down from burning rather than from dielectric rupture, unless the applied voltage is much above

the material's disruptive strength, when the rupture is instantaneous. For the same reason the severity of the test increases with the length of time for which the pressure is applied, although not very markedly, as much depends upon the circumstances. A duration of 1 to 5 minutes is sufficiently long to give reliable results; between 1 and 10 minutes there is usually a reduction of some 15 per cent. in the disruptive strength.[1]

The electrolytic action which might be feared with continuous current is rendered innocuous in practice owing to the fact that the heating of the dynamo when at work drives off the moisture without which electrolysis cannot be set up; it is only of importance in the case of newly wound coils or dynamos that have been standing for some time in a damp place.

At the head of the list of insulating materials, as combining many of the most valuable qualities, stands *mica*. Besides being incombustible, it is non-hygroscopic (so that it does not absorb moisture if the dynamo is exposed to damp, and does not in consequence deteriorate), and mechanically strong to withstand great pressure. Its specific disruptive strength varies considerably in different qualities, that of pure white mica being very high. But although it must be free from iron oxide in patches of any thickness, the mica for commercial work need not be entirely transparent; streaked or green spotted mica may be counted on to withstand a pressure of 1,000 maximum volts for every thousandth of an inch in thickness. Mica in its natural state cannot, however, be obtained in large sheets; and further, it is extremely inflexible when an appreciable thickness has to be bent to a curved surface of small radius. On the score of price and of the need for insulating curved or broken surfaces, it therefore becomes necessary to employ the artificial forms of mica known as *micanite* and *megomit*.

These artificial forms are made of small and thin laminae of mica re-assembled into a sheet and cemented together with an insulating gum under great pressure and high temperature. Owing to their greater homogeneity their disruptive strength is if anything more uniform than that of pure mica, and may also be taken as 1,000 maximum volts per mil of thickness at ordinary temperatures. The specific strength of composite mica products is but little affected by rise of temperature, and their heating under a rapidly alternating potential is but small. In order to apply the micanite to the

[1] See Turner and Hobart, *The Insulation of Electric Machines* (Pitman & Sons); Fleming and Johnson, *The Insulation and Design of Electrical Windings* (Longmans); Miles Walker, *Specification and Design of Dynamo-electric Machinery*, pp. 174–187; Rayner, "Report on Temperature Experiments carried out at the National Physical Laboratory," *Journ. I.E.E.*, Vol. 34, p. 613, and "High-voltage Tests, etc.," *Journ. I.E.E.*, Vol. 49, p. 3; Miles Walker, *Journ. I.E.E.*, Vol. 47, p. 553; S. Evershed, "The Characteristics of Insulation Resistance," *Journ. I.E.E.*, Vol. 52, p. 51; and A. S. Langsdorf, "The Fatigue of Insulation," *Electr. World*, Vol. 52, p. 942.

armature core, it is heated to about 200° F., the kind employed being such that it then becomes soft and pliable, and can be easily moulded on to curved surfaces. ***Flexible micanite*** is also made which has a slightly lower disruptive strength, 750 maximum volts per mil when cold, and lastly ***micanite cloth***, which has a backing of fine muslin or linen, or ***micanite paper*** in which the backing is of paper on one or both sides. The layers of mica rise from 2 to 3 as the thickness increases from 0·008″ to 0·014″ in the case of cloth, or from 0·005″ to 0·011″ in the case of paper. The piercing voltage of the two latter is less than in the case of the sheet; it averages from 450 to 850 maximum volts per mil for paper, and from 430 to 700 for linen, but depends to some extent on the method and carefulness of the manufacture.

In general the curves for disruptive strength in relation to thickness gradually bend over, showing that the strength per mil in thin sheets is not maintained with greater thickness. With mica products this is not very marked, and when present it is probably due to want of complete homogeneity, as, *e.g.* through the inclusion of a minute air bubble in natural mica, or of varying amounts of cement in the artificial forms, either of which causes may alter the gradient of the potential fall through the material.

In the case of organic fibrous materials in sheets, whether untreated or impregnated with insulating varnishes, the decreasing strength per layer is quite marked, so that it has been said that the disruptive strength is proportional to the $\frac{2}{3}$rds power of the thickness. But the reason is not clear, and it is probably to be ascribed either to films of air between the layers unless these are very tightly compressed, or to irregularity of impregnation by the oil or varnish, or to the presence of moisture still remaining in the inner layers.

The specific insulation resistance of organic materials, such as paper, linen, cotton, or vulcanized fibre, is much improved by such moderate heating as will expel all moisture, and so also is their disruptive strength. But as the temperature is increased above 80° C. there sets in a real deterioration due to partial disintegration of the material, if the heating is maintained for long periods, as is the case with dynamos after many years of work. The effect of temperature, and also the allied question of brittleness after frequent heating thus become all important when we pass from the inorganic mica series.

Ebonite and vulcanite are unsuitable for coil or core insulation from their brittleness when cold, and from the fact that they soften at a temperature of 65° C., which is within the working range of the dynamo, while rubber products are inadmissible from their deterioration under the action of light and air.

Red or grey vulcanized fibre in thin sheets up to 0·080″ has a disruptive strength of 400–200 maximum volts per mil of thickness;

it is mechanically strong, tough, and durable, but it is very hygroscopic, when dry quickly absorbing moisture and swelling, and shrinking again when heated; further, it becomes brittle when continually heated, so that it has practically been superseded as a material for insulating dynamo coils or cores.

Passing these by, therefore, we come to *press-spahn*, which in thin sheets is largely used for low voltages, and possesses many recommendations. It is flexible, homogeneous, and uniform in thickness, with a smooth glossy surface; it is, however, somewhat hygroscopic, especially if creased, which destroys its glazed surface, and must always be carefully dried before use. It varies greatly in quality, but on an average when thoroughly dried it has in sheets 0·010" thick, a disruptive strength of 400 maximum volts per mil of thickness, decreasing in sheets of 0·040" thickness to 300 volts per mil. Much the same reduction is found when several thin sheets are used to give the same total thickness, and in either case if creased it cannot be relied upon to withstand more than 250 maximum volts per mil. If dried, soaked in hot linseed oil for several hours, and then again thoroughly dried, a thin sheet of 0·010" which will have increased in thickness to 0·016" has its disruptive strength increased to 520 maximum volts per mil of its increased thickness, but this improvement as compared with untreated press-spahn is not maintained when several sheets are superposed on one another. Plain varnishing of the surface with a good insulating coating is equally effective, and can raise the disruptive strength to 600 volts or even more, although, strictly speaking, the effect can hardly be stated as dependent on the mils of total thickness.

Linens, cambric, muslin, and other stuffs impregnated with twice-boiled linseed oil or other varnishes, *i.e.* "oiled linen," etc., which are sold under various trade names, are from their flexibility especially useful for the insulation of the groups of wires in former-wound coils. In general for a thickness of 0·006" they may be relied upon to have a disruptive strength of 1,000–500 volts per mil of thickness at ordinary temperatures of the atmosphere, decreasing to 500–300 volts per mil in greater thicknesses up to 0·040", or in superposed layers. In many cases the effect of temperature upon such fabrics is marked, the disruptive strength being reduced at 60° C. to 350 maximum volts per mil for the thinner, and to 270 volts for the thicker sheets, while at higher temperatures the reduction may be still greater, the figures approaching to 300 and 200 maximum volts per mil respectively for the thin and thick limits given above. Canvas varnished is not sufficiently good to rely upon as armature insulation.

Thin *cotton tape* 0·006" thick has a disruptive strength of about 150–100 maximum volts per mil, and is not greatly improved by

varnishing, although this precaution is very necessary in order to check its absorption of moisture.

Lastly, we come to *papers* of various kinds, which must always be coated with insulating varnish on account of their hygroscopic nature, and which are in fact chiefly to be regarded as carriers of the varnish, so that they should possess mechanical strength. On these grounds Manilla, Willesden, and bond papers are valuable, since, besides being mechanically strong, they show a fairly uniform disruptive strength under different conditions of thickness, temperature, and duration of test. When untreated they break down with a thickness of 0·006" approximately between 350–250 maximum volts per mil, and when built up to 0·040" between 235–150 volts per mil, but with a good varnish or impregnated with hot linseed oil for some time (for which their fibrous nature renders them well suited) and thoroughly dried, these figures are raised to amounts varying from 1,000–600 in thin sheets, and from 600–390 volts in thicker sheets respectively per mil of the thickness after treatment. Very carefully prepared papers reach a strength of even 1,250 to 1,500 maximum volts per mil. On the whole the papers are better electrically than the linens and impregnated fabrics, but are mechanically not so strong.

Asbestos paper is to a considerable extent incombustible, yet is of but little use owing to its being strongly hygroscopic; its disruptive strength is only about 100 maximum volts per mil, but can be increased by soaking in insulating oil or paraffin wax.

All papers and linens should be carefully examined for pinholes or metal particles adhering to their surface.

Of recent years, cellulose acetate has been introduced as a new insulating material which is non-inflammable and a good insulator. It can be moulded into tubes, and its ultimate field will depend on its durability which a longer experience can alone show.

As a plastic insulator which can be moulded, a condensation product from the chemical action of phenol on formaldehyde, known as "Bakelite" from its discoverer, Dr. L. H. Backland, has also met with considerable success[1] in slabs or as packing-pieces.

The function of *insulating varnishes* is, in the first place, to reinforce the disruptive strength of fibrous materials by filling up their interstices and covering their surface with a layer of highly insulating material; in the second place, to prevent the re-entrance of moisture into their pores after they have once been well dried; and finally, on exterior surfaces to give and maintain a smooth hard finish. The varnish should be quick drying, without requiring the use of a large amount of an expensive and highly inflammable solvent to thin it and to prevent it from drying up in the dipping bath. It

[1] See "Moulded Insulating Compositions," by R. T. Fleming. *Journ. I.E.E.* Supplement to Vol. 57, 1919, p. 323.

should be chemically stable, and should not contain free resinous acids which attack copper and produce green salts of very low resistance; it should not soften under heat, and should be waterproof and unaffected by lubricating oil. Especially is it desirable that varnishes should not become brittle, crack or peel off, but remain tough, flexible, and elastic after prolonged heating; under the combined effects of heat and long-continued vibration they must show no tendency to disintegrate into powder.

Shellac dissolved in methylated spirits dries quickly and sticks well, but becomes brittle from age and vibration, it is therefore unsuitable for general use, and a similar liability to powdering forbids the use of copal and asphaltum varnishes.

The action of lubricating oil on varnishes is that it unites with part only of the constituents of the varnish, and thence liberates free acids which attack copper, as evidenced by green colorations, and finally reduce the resistance so much that break-down follows. The paraffin non-acid varnishes are free from the liability to chemical resolution, since they unite wholly with the oil without losing their insulating properties; they are, however, difficult to dry and to handle, do not give a smooth hard surface, and from the very fact that they remain plastic after prolonged heating are liable to be thrown out by the action of centrifugal force in the case of quickly-rotating coils.

There remains linseed oil with or without admixtures of vegetable resins and gums. Linseed oil requires to be oxidized in order to dry it, so that the advantage of the vacuum chamber in drying without the application of great heat is largely nullified, since air has to be admitted at intervals and stoving is as effective and more rapid; further, the oxidation process continued by the rapid rotation of well-ventilated armatures in time renders the surface brittle. The well oxidized substance has been called "linoxyn," and is a stable body, solid, but slightly flexible.[1] Under severe conditions as when ozone is produced by silent electrostatic discharge between insulated wires and the slot insulation in extra high-voltage alternators, the oxidation process is carried too far and as the result of this "super-oxidation" the insulation becomes soft and pasty. Chemical products are then formed which are soluble in water, corrode the insulation, and attack the copper. Lastly, there is some risk of oil combining with linseed-oil varnishes under the action of gentle heat. Yet on the whole for impregnating purposes the balance of advantage may perhaps be said to rest with linseed-oil compounds, since they can be given various degrees of quickness in drying, of toughness and of smoothness of surface, according to the purpose which they are to serve, and it is better in practice to prevent lubricating oil from reaching the varnished surface.

[1] C. J. Beaver, *Journ. I.E.E.*, Vol. 47, p. 552.

Cotton-covered coils should be dried before the varnish is applied since acid moisture in the covering may with the linseed oil produce deterioration. Coils varnished with linseed-oil frequently show a greenish discoloration of the cotton insulation due to the action of weak organic acids on the copper wire, but such discoloration is practically harmless, and must not be confounded with the corrosion due to high-voltage discharges.[1] Linseed oil expands in drying, which is of advantage in closely filling all pores.

To withstand the acid fumes in the proximity of accumulators, special exterior enamels are required.

§ 17. The insulation of armature wires.—The armature conductors are either round *wires* (in special cases made up into stranded cables) or solid *bars* of rectangular (or less usually of square) cross-section with slightly rounded corners. In all cases soft annealed copper is employed both for the active conductors and also for their connectors, and such copper is now commercially obtained in accordance with a standard of conductivity defined by international agreement, viz., that a metre length of Standard Annealed Copper Wire 1 sq. mm. in sectional area, has a resistance of $\frac{1}{58}$th of an ohm at 20° C., so that its resistivity at that temperature is 1·724 microhms per cm. cube. So long as the current to be carried by each active conductor does not exceed about 50 amperes, solid wires of circular section can be used; their diameter will not exceed 0·150″, and they can be readily bent or shaped. Single-cotton covering does not give sufficiently good insulation for armature wires, and *double cotton covering* by itself is only suitable for armatures of low voltage and on small wires, since it is apt to open out when the wire is bent. Double-cotton covering adds 10 mils to the diameter of a wire measuring 0·050″ or less, 12 mils to the diameter of a wire measuring from 0·055″ to 0·075″, and 14 mils to the diameter of wires above that size, the thickness of the cotton thread employed being increased on the larger wires. A finer thread may also be obtained at slightly increased cost by the use of which the increase of diameter is reduced to 6 mils for wires from 0·028″ to 0·036″ diameter, or 8 mils from 0·040″ to 0·050″ diameter, and above to 10 mils. Triple-cotton covering is too thick and wastes space, so that the most usual insulation is a *braided cotton covering*. With fine cotton this may be taken as adding 13 mils on a diameter of 0·060″ or less, 15 mils on to 0·100″ diameter, and 17 mils if the bare diameter be 0·160″.

The percentage of the total slot area which is filled with copper in a toothed armature turns upon two entirely independent questions. The first, which is for the present postponed, is the relation between the thickness of the insulating lining of the slot or envelope of a composite coil to the width and depth of the slot, which will depend

[1] A. P. M. Fleming and R. Johnson, *Journ. I.E.E.*, Vol. 47, p. 550.

solely upon the voltage which the insulation has to stand. The second question, with which alone we are here immediately concerned

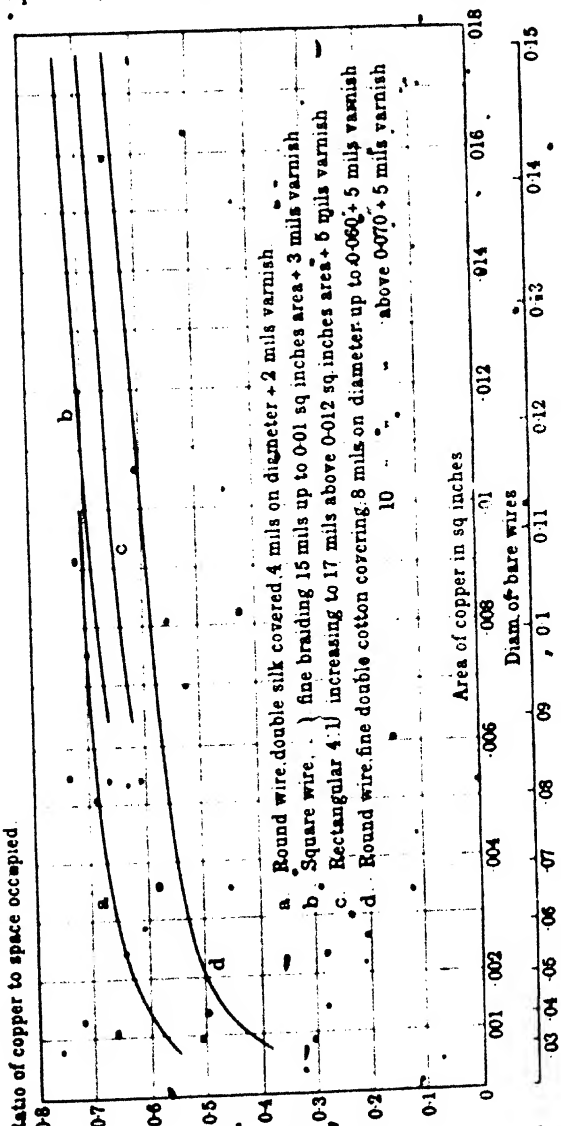


Fig. 147.—Space-factor of insulated wires.

is the ratio of the copper area to the area of the space which the insulated wires take up, and this will depend primarily upon the

area of each wire and also upon its shape. The smaller the wire, the greater the space which the thickness of its insulation takes up in proportion to its area, so that small outputs, high voltages, and low speeds all combine to demand more room for insulation than their opposites; further, small round wires utilize the space much less efficiently than conductors of the same area with square or rectangular cross-section. The ratio of copper to space occupied for small and large conductors respectively, or their "space-factor," is plotted in Figs. 147, 148 in relation to their copper area; from these figures a number of conclusions are evident to the eye, and it is at once seen how far it is possible to go in making use of the available space to be occupied by the conductors.

Double-silk covering lightly varnished only adds from 3 to 5 mils to the diameter, and since very small round wires below 0·040″

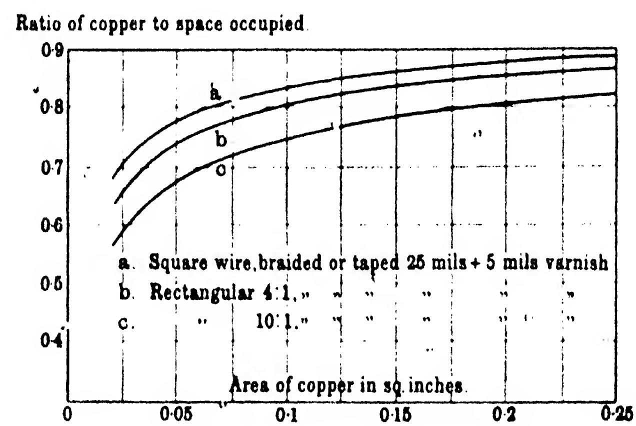


Fig. 148. Space factor of rectangular insulated bars.

diameter, as used in small motors, are, when silk-covered, less than twice as expensive as cotton-braided wires, the saving in the space which they take up as shown in Fig. 147 (on the supposition that there is no bedding of the round wires) may more than compensate for their increased cost, while for still smaller wires the use of silk becomes a necessity. Wires less than 0·040″ in diameter, however, hardly enter into ordinary dynamo practice, and above this size silk covering is prohibited by the fact that it more than doubles the cost of the insulated wire. A fine cotton-braiding will add 14 to 15 mils to each dimension of a small rectangular conductor, say, 0·100″ × 0·050″ bare, and such conductors, even though of small area, soon show nearly as good a result as silk-covered round wires. Square conductors which would utilize the space best are in practice to be avoided, as their tendency to turn on edge renders them very

difficult to wind and to bend without danger of their cutting into the insulation of neighbouring wires in the same coil. There is, however, no very great loss of space with rectangular braided wires with 4 : 1 ratio of depth to width, and still less with 2 : 1 ratio, so that for any area above, say, 0·11" × 0·055" a rectangular strip is to be recommended.

We thus pass gradually to "bar-winding," the dimensions of large rectangular bars being increased by 0·020" with double-cotton covering and by 0·025" with braiding, which is preferable. The curves for the extreme cases of a square section and a rectangular section with a ratio 10 : 1 for the dimensions of its sides, and both with 25 mils of insulation on their thickness, are given in Fig. 148. Where the conductor is thin and deep, so that braiding would not lie closely against its deep side, and further, when there is only one turn per section, the copper strip will first be bent to its required shape and will then be wrapped round with a half lapped covering of thin cotton tape ($\frac{1}{2}$" or $\frac{3}{4}$" wide × 0·006" thick) in a taping machine.[1] The four thicknesses of tape are then equivalent to the braiding, and add 25 mils to the dimensions. The same also applies to large bars, each of which forms one half of a loop, the two halves being subsequently united by a soldered joint in the process of winding the armature.

The corners of all conductors of rectangular section are slightly rounded off in the manufacture. In all cases the insulated bars or coils forming a composite group corresponding to a slot are dipped in an insulating varnish, so as to become thoroughly impregnated, and afterwards dried in a vacuum chamber[2] or stove. In Fig. 148 an allowance of 5 mils on the dimensions has been made for the varnish which thus soaks into the braiding or taping.

Under prolonged heating, even if the temperature does not much exceed 100° C., the cotton covering of wires and bars begins in course of time to deteriorate; it turns brown and becomes carbonized, so that, although its insulation resistance may still remain very high, it is friable and mechanically weak, and from this fact an important limitation is set to the temperature which the dynamo should in ordinary working be allowed to reach.

§ 18. The toothed drum armature. In addition to its perfect system of driving the conductors, shielded as they are from the greater part of the magnetic drag (Chapter IV, § 7), the slotted or toothed drum armature has the additional advantage that it allows of the use of solid bars of much greater width than are permissible on the smooth-surface core. Since by far the greater proportion of the flux passes through the teeth, the density of the lines within the slots is but a small fraction of the average density in the air-gap

[1] See Turner and Hobart, *The Insulation of Electric Machines*, Chap. XIX.
[2] *Ibid.*, Chap. XX.

of a similar armature with smooth surface; the possible range of density within any one slot is therefore very greatly reduced. Consequently at any moment the difference in the density of the flux cut by the two sides of even a fairly wide bar never reaches the magnitude that it would have in a bar of the same width on a smooth-surface armature as it emerges from under the pole-edge where the air gap flux-density changes very rapidly. The so-called eddy-current loss in the copper bars due to want of uniformity in the current-distribution over their section is thus greatly reduced.

Lastly, the slotted armature is well fitted with its projecting iron teeth for the rapid dissipation by radiation of the heat generated within it, and can be much better ventilated by air canals than is possible in the smooth-surface core which is entirely covered with a heat-retaining layer of insulation. For all these reasons, the toothed armature has displaced the smooth-surface armature formerly in vogue.

The shape of the slots in the toothed core permits of many variations, but if it is open at the top the width of opening usually does not exceed twice the length of the single air-gap.[1] Otherwise the unequal distribution of the flux caused by the alternating slots and teeth becomes extended in a marked degree to the bored face of the pole-piece (cp. Fig. 28); the passage of the lines as they sweep over the pole-faces will then set up eddy-currents in the solid mass of the pole, and the loss of energy and heating due thereto may be so great as to necessitate the lamination of the pole-shoes.

The liability to eddy-currents in solid poles is largely reduced by employing half-closed slots (Figs. 149*b* and 150), and is entirely obviated by *tunnel* armatures, in which a number of holes are stamped in the discs close to the periphery, and the wires are threaded through these holes after they have been lined with tubes of micanite. Moulded tubes or troughs of micanite or megohmit can be procured in every variety of shape, and can be slid into tunnels or half-closed slots; their disruptive strength will reach 750 maximum volts per mil of thickness of the wall. Such half-closed slots or tunnels are, however, attended with the disadvantage that the inductance of the active conductors is much increased, and the difficulty of commutation and of sparkless collection of the current is correspondingly greater; hence in practice their possible use is largely limited. Further, the insertion of the wires or bars is more troublesome than with open slots, so that the form shown in Fig. 149*a* which is adapted to receive a wooden wedge or key is that most generally used for continuous-current dynamos and motors.

The slot-pitch may vary from ⅝in. in small machines to 1½-in.

[1] See Chap. XXI, § 27.

in large machines, with such proportions of slot and tooth-width at the top as below—

Slot pitch	*Slot-width* w_s	*Tooth-width at top,* w_{t1}	w_s/w_{t1}	$\frac{w_s}{\text{Slot pitch}}$
¾"	⅜"	⅜"	1	0·5
1"	[illegible]	9/16"	0·78	0·437
1½"	½"	1"	0·5	0·33
Average values—				
1¼"	0·45"	0·8"	0·56	0·36

The slot-pitch × 0·36 gives therefore a preliminary starting-point in design.

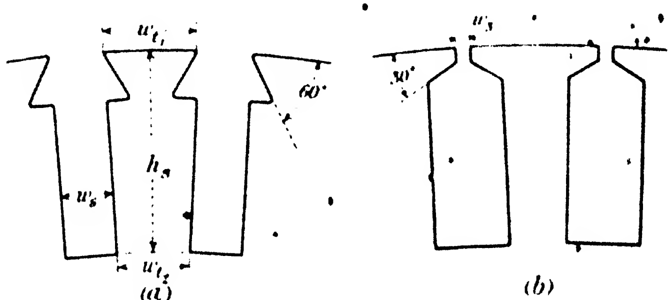


Fig. 149.—Open and half-closed slots.

The total depth of the slot ranges from 2 to 4 times the slot-width, and averages perhaps 3 times. It is, however, closely limited by the diameter of the armature. With slots having parallel sides, the smaller the armature, the greater the proportionate reduction in the width at the root of the tooth w_{t2} for a slot of the same depth, as will be obvious from Fig. 149. Hence with a normal density in the air-gap the flux-density at the roots of the teeth and the saturation thereat will be unduly great unless the depth of the slot is limited to such values as the following (*see* Fig. 202):

Diam. of armature.	*Slot depth,* h_s
5"	0·6"
10"	1·0"
15"	1·3"
22"	1·5"
50"	2·0"

§ 19. Insulation of armature core and coils. Returning to the process of constructing an ordinary drum armature, after the toothed discs have been built up into a cylindrical core, the next step will be to smooth off all sharp edges against which any portion of the winding is liable to be pressed, especially at the ends of the slots whence the winding will project. Any roughness along the sides of the

slots will be removed by filing, and this operation must be cleanly and sharply done so as to minimize any chance of burring over the edges of the discs; in fact, the file must be used as sparingly as possible, so as not to bring the discs into contact on their surface, whereby the advantage of the lamination would be largely nullified, and paths for eddy-currents would be formed. The need for great care in this respect and for good workmanship to secure a low core loss has already been emphasized.

At the ends of the core the rings forming the seatings for the end-connectors of the coils will be insulated with layers of paper, press-spahn, or in high tension machines of micanite, projecting well over any metallic surface in close proximity to the coils. The insulation can here be made thicker than on the active surfaces of the core, since space is not so valuable. All joints or seams in the insulating covering require particular attention, so that there may be no likelihood of the winding making contact with the core. The whole is then finally varnished and dried in order to rid it of all moisture preparatory to winding.

For a test pressure of 2,000 R.M.S. volts for one minute at 20° C., or ordinary atmospheric temperatures, a thickness of insulation of 0·035″ from copper to iron would give a factor of safety of at least 3 with the usual materials employed, and for a working pressure of 500 volts this thickness would be ample. With 250 volts working pressure or less, mechanical considerations demand much the same thickness of insulation, although the material may be of less electrical strength. The percentage of the total slot area which is taken up by the lining and insulation between the two layers of coils will obviously vary greatly with the dimensions of the slots, and the difference or the portion of the slot area which is available for the winding will range from 55 per cent. with a slot ¾″ × ¼″ to 83·5 per cent. with a slot 2¼″ × ¾″, while for very high voltages the percentages may fall much lower. It is not therefore advisable to employ a great number of very small slots, owing to the waste of space in their insulation, and two or more sections of the armature winding are usually grouped in one slot. The product of the above available percentages with the percentage deduced from Figs. 147 and 148 will give the ratio which the copper area bears to the total slot area and evidently this may vary very greatly, a high voltage not only demanding a thick wall-lining, but also being usually accompanied with a small sectional area of wire, especially if the output be small and the speed low.

With open slots either the slots may be lined for the reception of the conductors, or the conductors may themselves be encased with the wrapping which is to insulate them from the iron core, the latter being the preferable plan. In either case the wrapping or slot lining will be built up of practically the same materials and in

the same way. Two or more thin layers are better than a single thick one, as giving greater flexibility, and one at least of the layers must possess sufficient mechanical strength and hardness to prevent

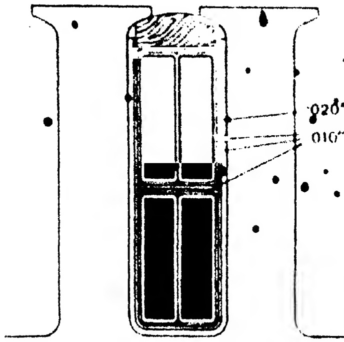


FIG. 150.—Half-closed slot insulated for 500 volts

any danger of its being cut when pressed into its place within the slot. Thus for 500 volts the insulation of Fig. 150 may be used with a half-closed slot. With an open slot, micanite or megohmit

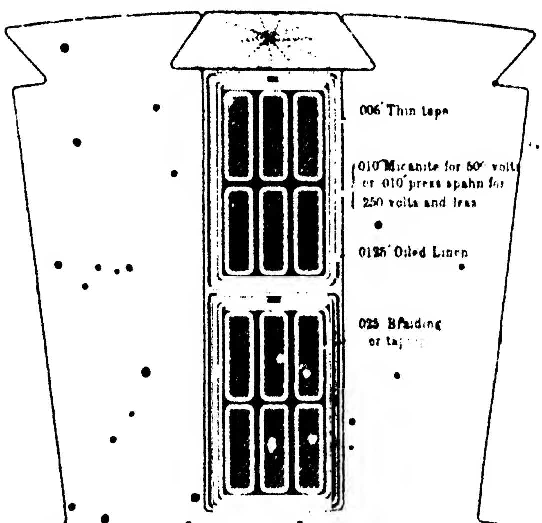


FIG. 151.—Insulation of coil in slot with wooden wedge

plate is readily moulded in place, a strip of the required dimensions being heated on a plate by a gas jet, laid in a slot, and pressed down tightly into its corners by a wood or iron bar having the exact

internal shape of the slot when insulated. It is, however, even better to insulate the composite coils first and afterwards to press them into the slots; a wrapping of oiled linen, 0·0125″ thick and over-lapping at the top, is placed next to the wires, and over this a nearly closed but not overlapping channel of micanite 0·010″ thick, the whole being then bound round with a half-lapped spiral of tape 0·006″ thick and ½″ wide. The tape is not reckoned to add much to the insulation, but is chiefly to retain fast the composite group during the processes of dipping them into varnish and forcing them into place, as in Fig. 151, which shows the coils afterwards locked with a wooden key. The thickness of a single side is thus 0·0125″ + 0·010″ + 0·012″, or say 0·035″, and of the double insulation at the top and bottom of an element is 0·037″ + 0·020″ + 0·024″

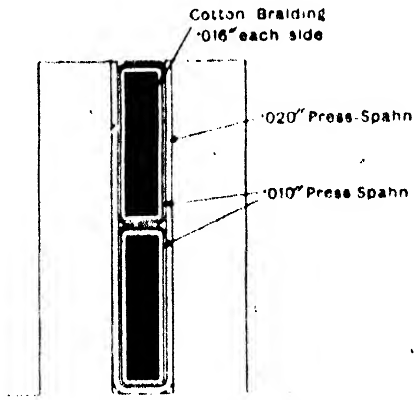


FIG. 152.—Insulated slot of toothed armature for 250 volts.

or say 0·080″. For 250 volts the same will again hold good, save that for the micanite will be substituted a channel of press-spahn, also 0·010″ thick, to protect the oiled linen, or the insulation may be as in Fig. 152, which shows the bars coming close up to the top of the slot in the less usual case of their retention by binding wire. In all cases the slot insulation must project ½″ beyond the actual length of the core at either end, and when the coils are not wrapped round as a whole with the main insulation a strip of micanite or press-spahn will be inserted to separate the two layers of conductors between which the full difference of potential of the machine exists, and in some cases a corresponding strip at the top of the slot.

Thus in designing, if 30 mils are allowed for the double thickness of the cotton braiding or taping on each wire or bar after the whole coil has been varnished, and a small play of 5 mils be added to each bar to allow for slight irregularities; further, if 0·075″ be allowed for the double thickness of the wall lining or coil wrapping, a total

deduction of 0·075″ + (0·035″ × number of conductors abreast) has to be made from the width of the slot, and the remainder divided by the number of conductors abreast gives the permissible thickness of the bar or diameter of wire. Or, if n_c = number of conductors abreast in one layer in a slot, and the copper thickness of each is t inches,

$$t = \frac{u_s - 0{\cdot}075''}{n_c} - 0{\cdot}035'' \quad . \quad . \quad . \quad (81)$$

If the bars are heavy and stiff, say larger than 0·4″ × 0·06″, an extra allowance of some 5 to 6 mils per bar should be made in the width of the slot. From the depth of the slot h_s there must be deducted 0·160″ + (0·030″ × number of layers), the wires being tightly held down, and the remainder divided by n_l, the number of layers, will give the permissible depth of conductor h, or with wooden wedge 0·2″ thick

$$h = \frac{h_s - 0{\cdot}360''}{n_l} - 0{\cdot}030'' \quad . \quad . \quad . \quad (82)$$

while as above if the bars are deeper than ½″, an additional allowance of a few mils will be required. If a specially fine braiding is employed, the bar allowance may be reduced to 0·025″ and 0·020″ in the width and depth respectively.

§ 20. Drum armature winding. I. (*a*) **Hand-wound coils.**—Although the various kinds of drum winding pass by natural transitions from one into another without sharp distinctions, they may broadly speaking be grouped into two classes, according as the armature is (I.) *coil wound, i.e.* wound with round wire or strips of comparatively small rectangular section, in coils usually of two or more complete turns, the wire being wound or shaped in its insulated state, or (II.) *bar-wound* with conductors of massive rectangular section—insulated after shaping.

In group I. which coincides practically with machines of small or medium size and output, the coils may be either (*a*) *wound by hand* directly on to the armature core, or (*b*) *shaped on formers* prior to assemblage on the core.

In hand-wound armatures where the loops overlap each other as they pass round the shaft at either end, the difference of potential at the crossing-points, since it amounts to the full E.M.F. of the machine, is apt to destroy the intervening insulation and lead to short circuits; further, the repair of any one loop almost invariably necessitates the complete unwinding and rewinding of the armature. Owing to these objections and its greater expense, hand-winding except in very small 2-pole machines has been superseded by *former-wound* coils.

§ 21. Drum armature winding. I. (*b*) **Former-wound coils.**—These are shaped on "formers" prior to being assembled on the

armature core; they are therefore perfectly symmetrical and interchangeable, and have the further advantages that they can be well insulated by wrappings of linen, micanite, paper or tape, can be tested before they are placed in position on the core, and are themselves inexpensive to wind in the first instance. The guiding principles on which the formed coils are shaped are as follows: either the coil is lozenge-shaped with its centre wider than the two parallel and straight inducing sides, or the two halves of a complete coil are of different width; in either case one side of a coil being of smaller width can be passed through the wider sides or centres of other coils. If C be the total number of coils, the first $C/2p$ coils are placed in succession on the core, one side of each being in its final position and the other side being temporarily and loosely held in place. The winding now proceeds, both sides of each coil being fixed in their final position, until there only remain $C/2p$ coils to be wound. The narrow side of these has to be passed through the sides of the first $C/2p$ coils, so that these latter are now lifted up a little to allow of the introduction of the remaining coils, and as each of these is placed in position the first $C/2p$ coils can be successively closed down until the whole armature is finished. It may here at the outset be mentioned that a trapezoidal loop or coil, in which every end-connection passes across the pitch in one straight line, is seldom or never used in practice; as, a coil on a former-wound armature, its side of less width could be passed through the greater sides of other coils, but whether on a former-wound or on a bar-wound armature, the use of one long and one short straight conductor joined by slanting end-connectors is vetoed by the fact that the axial length required for the whole mass of end-connectors is nearly twice as great as that required when the end-connectors, changing their direction of slant at the centre, first recede from and then draw inwards towards the core. It is therefore invariably the case in all except hand-wound drums that the end-connectors at their centre are given a twist, or are so formed by bending that they are as it were enabled to pass by one another in regular succession; they then fall either into two layers coaxial with the shaft (*barrel winding*), or into two whorls in planes at right angles to the shaft (*butterfly* or *involute* end-connectors, which are very frequently but less accurately called "evolute").

Thus the distinction between barrel-winding and winding with involute end-connectors is that in the former the end-connectors lie on the circumference of a cylinder practically of equal diameter with the core, while in the latter the end-connectors are bent down into planes at right angles to the shaft. This distinction occurs both with "former-wound" armatures and with the "bar-wound" armatures to be described later, and applies to both cases equally. In barrel-winding, either every alternate element must be cranked

down at each end so as to bring its end-connections into a layer below those of the other elements, or the elements must themselves be in two layers, and between the elements of any one layer there must be intermediate gaps not filled with conductors. The latter condition is at once given by the slotted armature, since in it one inducing side of each coil falls into the upper layer and the other side into the lower layer, while the intervening iron teeth supply the necessary spaces between the elements of each layer. Barrel winding is therefore by far the most widely used method for multipolar machines with slotted armatures; further, although former-wound coils may be used on both smooth and slotted armatures, they are specially suited to the latter when barrel wound and for four or more poles, and it is chiefly on this account that the multipolar machine now finds favour even for comparatively small outputs.

The shapers employed for forming the coils, whether for barrel or involute winding, are very various, and a coil may be barrel wound

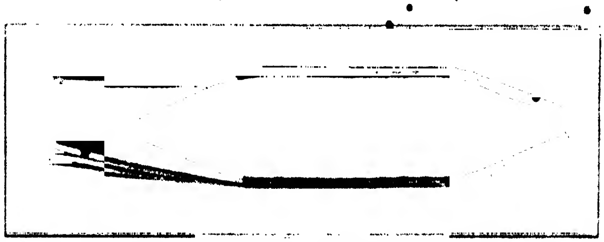


FIG. 153. Composite coil, three sections abreast.

at one end and be of the involute type in order to economize space at the other end.

Two different methods of manufacture may, however, be clearly distinguished; either the wires are wound in the first instance into a simple coil which is afterwards shaped by hand or machine into the exact form in which they are applied to the armature, or the coil is immediately wound to the required shape in a grooved wooden or cast-iron mould, the wires being held in clips as the coil is formed.

In the first case a generally adopted procedure for *barrel winding* may be described as follows. According as there are to be two, three or four commutator sectors per slot, two, three, or four conductors are wound abreast in a lathe from as many drums of wire arranged in tandem on to a wooden channelled frame, so as to form a composite lozenge-shaped coil, as in Fig. 153. The nose of the coil is then fixed in a vice or in a special machine, and the two sides of the V are pushed apart from one another; the same process is repeated at the second nose, and by this means the twist is given at each end of the coil which causes it to fall into an upper and a lower portion,

corresponding to the two layers of the finished armature. After being roughly shaped by hand the coil is then hooked over the shaper of Fig. 154, and by turning the handle its two portions are made to recede from one another, so that the coil is forcibly drawn out into its finished shape. The same result may also be obtained

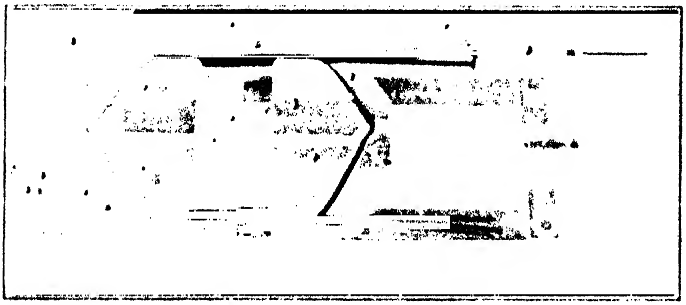


Fig. 154. Shaping tool for drum coils.

by transferring the coil of Fig. 153 to a machine having two halves hinged together, of radius equal to the radius of the armature, and each half holding firmly in a slot one side of the coil; the two halves are then rotated apart about the hinged joint, and the coil is opened out through an arc corresponding to the winding pitch (cp. Brit. Pat. 7373, 1900, Langdon-Davies and Soames). Or the former on

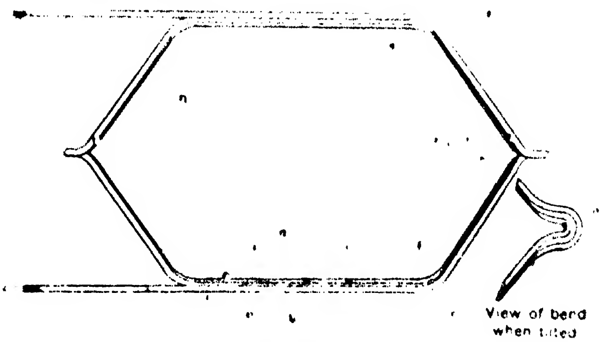


Fig. 155.—Shaped coil.

which the coil is wound may itself be in halves attached to a machine, by which the one half is pushed apart from the other (cp. also *Street Railway Journal*, vol. 18, pp. 2 and 3), the necessary curvature of the ends being imparted afterwards.

From Fig. 155, which shows a single separate section when formed, it will be seen that a wire which is at the top of the upper

layer and forms the beginning of a section leads on to a wire which is at the bottom of the lower layer, so that finally the free ends of a section come out respectively at the tops of the upper and lower layers. If the required number of sections per slot is such that a normal width of slot leads to an inefficient section of conductor, the

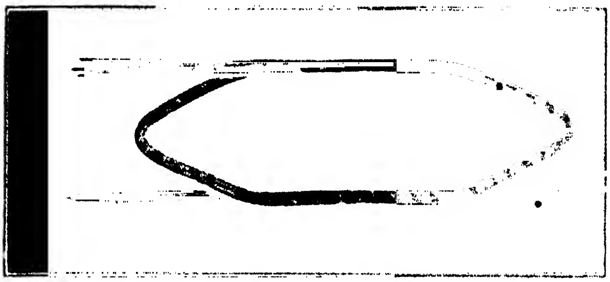


FIG. 156.—Composite coil, three sections deep.

sections must be wound on the top of one another, as in Fig. 156, which shows a composite coil of three sections wound on the top of one another, the width of the coil being only that of two conductors; but this method is only possible with lap-wound armatures. If the armature is lap-wound, the free ends of the section are brought

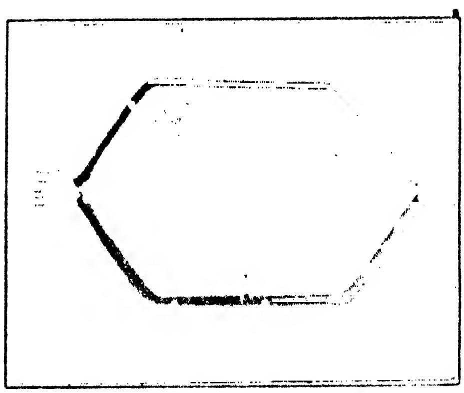


FIG. 157.—Finished coil, three sections abreast, for lap-wound armature.

along the ends of the coil to the centre (Fig. 157 corresponding to Fig. 153 and Fig. 158 to Fig. 156), ready to be afterwards united at the commutator lugs in their proper sequence after they have been placed on the armature.

If the armature is wave-wound, the ends of the sections are led away in opposite directions, since they are to be soldered to

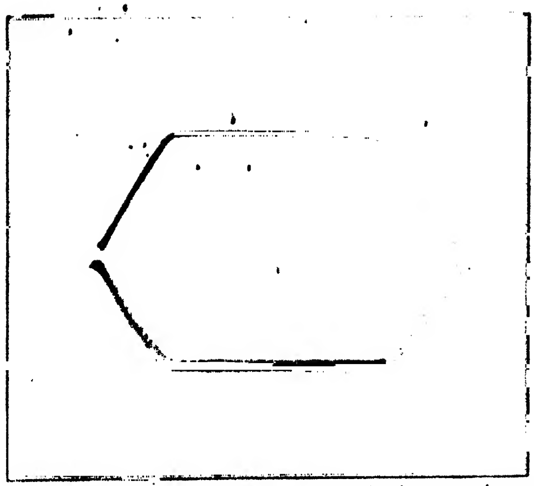


FIG. 158. Finished coil, three sections deep, for lap-wound armature.

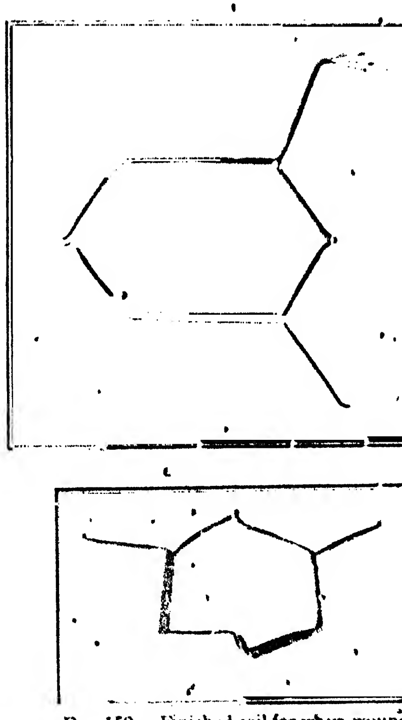


FIG. 159.—Finished coil for wave-wound armature.

commutator sectors which are at some distance apart from each other. (Cp. Figs. 93 and 177, the latter illustrating the connections for a single wave-wound section, consisting of two turns or loops). It is evident from this latter figure that the layers of commutator connections proceed in the opposite direction to the end of the coil from which they spring, and that therefore those free ends which would naturally fall at the top of the bottom layer must at some part of their course be bent down so as to clear the bottom half of the coil and come out at the extreme bottom below any other part of the coil. The passage of the free ends from the top of the lower layer to the bottom may be effected at the commutator end close to the armature core between the slots; but if the width of the teeth at their root does not give sufficient room between the slots it becomes preferable so to wind the coil that the passage is made at the farther end between the noses of the coils. How the winding must be arranged in order to effect this is seen from the two views of a wave-wound composite coil in Fig. 159, where the lower layer of the upper half is caused to pass to the lower layer of the lower half at the end farther from the commutator.

After forming the coils, lap- or wave-wound as the case may be, their slanting ends are taped, and the straight active portions are wrapped round with the insulation as described in § 19. The composite coil is then varnished or impregnated after drying to expel all moisture under a vacuum in a small chamber communicating with a bath of the insulating medium employed. It is next finally dried in a stove or drying chamber, and then forms a compact and easily handled element, the sides of which exactly fit within the slots. Or the coil may be soaked in varnish and pressed into accurate shape in a steam-heated mould. Prior to assemblage, each coil can be tested for short-circuits between its several turns by a small transformer, over the central core of which the coil is slipped so as to act as a secondary, an alternating current being passed through the primary; when a short-circuit exists, the ammeter shows a greatly increased secondary current,[1] and the coil becomes heated.

The winding of a coil immediately to the required shape is hardly feasible with rectangular wire of considerable depth, but is common with round wire. The shaper is then in two halves fastened together by a central bolt, and is so formed as to leave a deep groove between its two halves; it is mounted either vertically in a lathe or horizontally, and is turned round in quarter steps when released by a foot-lever, so as to allow the winder time to arrange the wire in its place between each quarter revolution. Fig. 160 shows one such coil for a barrel-wound armature, and a number may be

[1] See Turner and Hobart, *The Insulation of Electric Machines*, pp. 255 ff., for such testing transformers. Cp. also Miles Walker, *The Diagnosing of Troubles in Electrical Machines*, pp. 31-33.

arranged side by side and bound together to form a composite coil. In order to bring the free ends into the top and bottom layers respectively in the case of wave-wound armatures, the coil may be

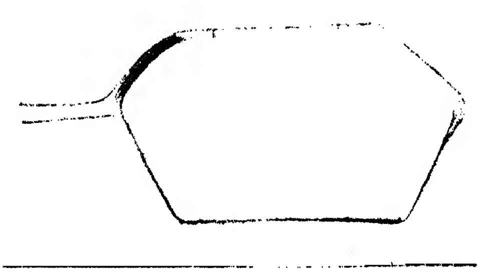


FIG. 160.

begun at its centre, one half of the wire being loosely coiled up at the side during the winding of the first half, and the coil being finished by winding the second half in the opposite direction. Or

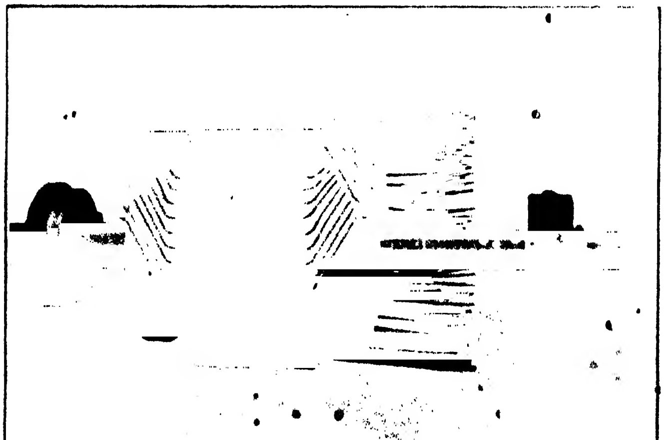


FIG. 161.—Winding of barrel armature.

two separate halves may be wound, and afterwards united by a thin sleeve of copper soldered to a pair of free ends on the side farther from the commutator.

The assembling of the composite coils on the barrel-wound armature is shown in Fig. 161. One side of each coil is pressed

down into its place at the bottom of a slot ; the upper side of the first $C/2p$ coils are at first only loosely arranged, but after these are

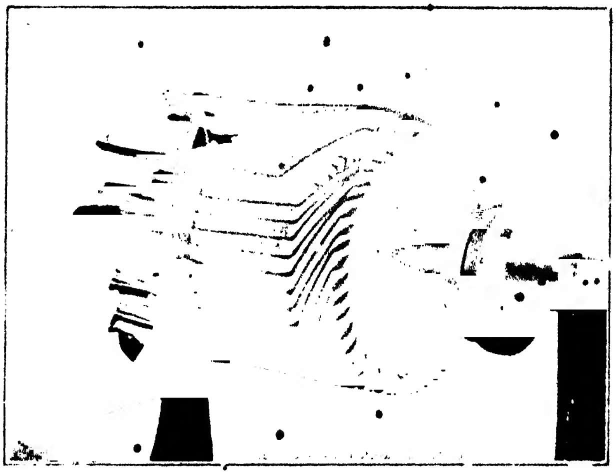


FIG. 162.—Insertion of last coils of barrel armature.

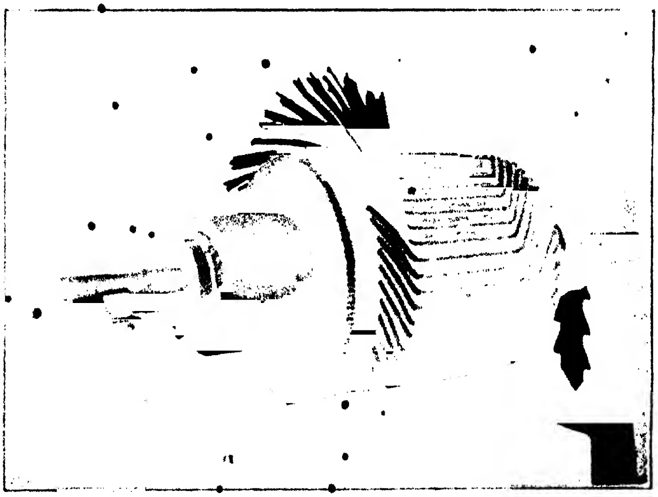


FIG. 163.—Connecting up to commutator.

assembled both sides of each coil are pressed down, the one into the bottom layer and the other into the top layer. As the winding approaches completion the upper sides of the first coils are lifted

up, to allow of the insertion of the last $C/2p$ coils, as seen in Fig. 162, where a coil is shown in the process of being threaded under the first $C/2p$ coils. The commutator is next placed on the shaft, and each of the free ends of the coils is inserted into its appropriate commutator lug, the lower layer being treated first, and afterwards the upper layer (Fig. 163). After soldering the pair of connections forming the beginning and end of two neighbouring coils into each lug, the armature winding is finished (Fig. 164).

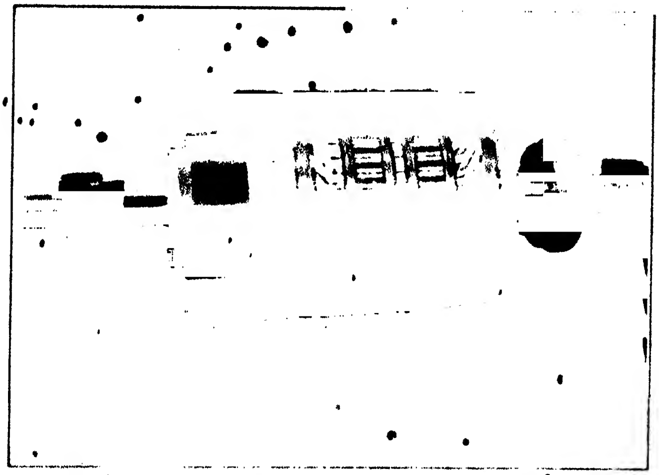


FIG. 164. Completed lap wound barrel armature.

§ 22. Axial projection of end-connections in barrel-wound armature. The total axial length l_e by which the coils of a barrel armature project from the core at either end is calculated as follows. The angle α, which the sloping portion of the coil-side makes with the edge of the armature core (Fig. 165), depends essentially upon the ratio which the width of a composite coil bears to the gap between itself and its next neighbour where they are straight; the former width may be identified with the width of a slot w_s, and the latter with the width of the tooth w_t. Let d = the clearance between neighbouring composite coils, at right angles to their direction of slant; for purposes of ventilation this will usually be made on the under-surface about $\frac{1}{16}''$ to $\frac{1}{8}''$. The hypotenuse AC of the right-angled triangle ABC on the development of Fig. 165, since it corresponds to one composite coil-side and its clearance, must evidently be equal to the slot-pitch $= w_s + w_{t2}$ measured at the root of the tooth where the coils lie nearest to one another and are most crowded, while $BC = w_s + d$. Thus $\sin \alpha = \dfrac{w_s + d}{w_s + w_{t2}}$, and since

w_s and w_{t2} in similar designs retain much the same ratio, α varies but little, and in the development on the flat averages from 30° to 40°. Let m = the distance measured on the circumference at the underside of the lowest layer of the coil, or PQ, through which a coil-side is bent, *i.e.* in the usual arrangement half the winding pitch in inches; then the axial length l of the sloping portion of the coils will be

$$QR = m \tan \alpha = \frac{m\,(w_s + d)}{\sqrt{(w_s + w_{t2})^2 - (w_s + d)^2}}$$

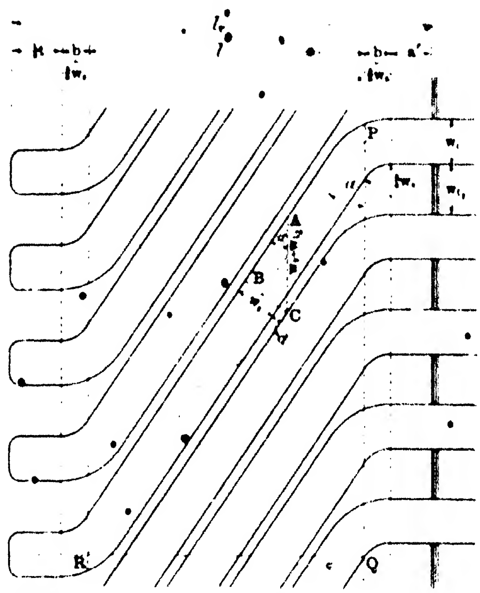


FIG. 165.—Axial projection of end winding in barrel armature.

Since $m = \frac{y_s'}{2}(w_s + w_{t2})$, where $\frac{y_s'}{2}$ is the half pitch in slots, this may also be put in the form

$$l = \frac{y_s'\,(w_s + w_{t2})(w_s + d)}{2\sqrt{(w_s + w_{t2})^2 - (w_s + d)^2}}$$

Thus the number of connectors cut through along a longitudinal section at one end as at RQ is $\frac{y_s'}{2}$; their joint thickness at right angles to the slant is $\frac{y_s'}{2}(w_s + d)$, and this thickness must be increased in the ratio of $\frac{1}{\cos \alpha} = \frac{w_s + w_{t2}}{\sqrt{(w_s + w_{t2})^2 - (w_s + d)^2}}$ in order to find the axial width.

Let a'' = the length of the straight projection from the slot before the slant commences, say, $\frac{1}{2}''$; thence an easy bend is given at a radius of $\frac{5}{4}\,w_s$ on the inside, and similarly at the outer or far end. The axial length of the bend or of the soldered joint by which the lower layer passes round into the upper layer is approximately equal to half the total depth h_s of the slot. Thus by Fig. 165 the total axial length at either end is

$$l_e = l + a'' + 1{\cdot}25\,w_s + h_s/2$$

$$= \frac{y_s'\,(w_s + w_{t2})\,(w_s + d)}{2\sqrt{(w_s + w_{t2})^2 - (w_s + d)^2}} + a'' + 1{\cdot}25\,w_s + h_s/2 \quad . \quad (83)$$

The length of the copper path along the entire end-connector is required in order to calculate the electrical resistance of the armature winding. If m_1 be the distance of the half-pitch measured on the *mean* circumference at the centre of the slot, $= \frac{y_s'}{2}\,(w_s + w_t)$ where

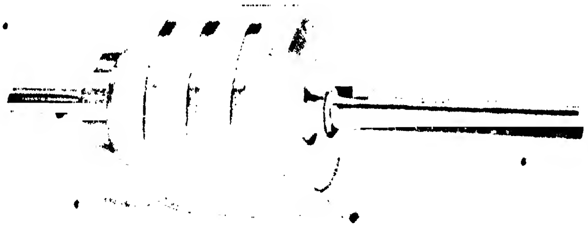


FIG. 166.

w_t is the mean width of tooth, the curved length of the sloping portion of a half loop will be the hypotenuse of the right-angled triangle having for its two sides the circumferential distance m_1 and the axial length l, or $\sqrt{m_1^2 + l^2}$. The total length from slot to slot, neglecting the slightly shorter pitch at the commutator end, is therefore

$$l' = 2\left\{\sqrt{m_1^2 + l^2} + a'' + 1{\cdot}25\,w_s + h_s/2\right\} \quad . \quad (84)$$

If the ends of the coils pass from slot to slot across a chord instead of lying on the circumference, the winding at the ends of the armature tapers inwards towards the shaft, and a form is obtained intermediate between the barrel and involute types (Fig. 166).

§ 23. Former-wound coils with involute end-connections.—Passing to coils with involute end-connectors, there are again the two methods according to which the coil is either first wound and then shaped, or is wound directly to the required shape. For the first method the coil may be first wound round pegs on a flat base (Fig. 167, i), the pegs being gradually shifted as the turns are wound in succession, or it is wound on a stepped shaper; the coil is next forcibly opened out approximately into a rectangle of which the sides are of different width, and the ends are finally bent by a press in two halves to the

required curvature, so that the coils fit into one another. The shapers for winding such coils immediately to the required shape have been worked out in detail by Eickemeyer,[1] and adapted to a variety of cases; with a number of wires per coil-side, these may be arranged one above the other along the straight inducing sides of the coil, and side by side at the ends so as to obtain enough room towards the shaft, or any re-arrangement of the component wires may be used giving a cross section to the ends different from that on the surface of the core. The nature of the involute winding is seen in Fig. 168, which shows a 4-pole armature with several coils in place, and with a number having their wider sides lifted up to allow of the introduction beneath and through them of the narrower sides of the last coils as indicated by a single coil at the top of the diagram.

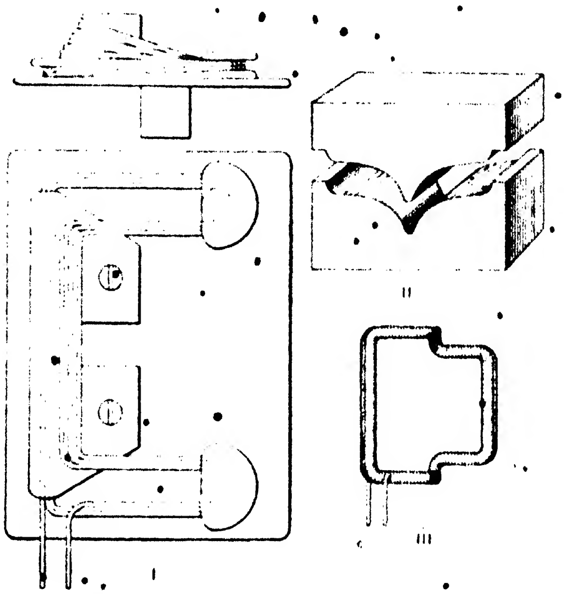


FIG. 167.—Former-winding for involute coils.

When finished the involute end-connectors lie flat against the insulated ends of the core with a gap of from [illegible] to [illegible] between the adjacent sides of the two whorls at each end. The axial projection from the core is now a minimum, and is equal at each end to twice the width of the connector plus the gap between the whorls and the thickness of the end-plate with its insulating check. The problem of the shortest path across the pitch for the two whorls whose planes are at right angles to the shaft is, *mutatis mutandis*, very similar to the same question in the barrel-wound armature. If the angle of the butterfly be chosen too acute, the length of the copper is unnecessarily increased, and the depth may approach the shaft too closely to allow of room for the central portions; if chosen too obtuse, the connectors cannot be forced into the room assigned for them. In order that they may lie evenly upon one another, and also take the shortest path, their curves must be portions of an involute of a circle, whose radius R_b is $= \frac{S \times w}{2\pi}$ where S is

[1] Brit. Patent 1888, 2246.

the total number of slots, and w is the insulated thickness of one composite end-connector corresponding to a slot measured normally to its length. The dimension w may often be identified with the width of a slot, but is not necessarily so, if the cross-section of the coil is altered when it emerges from the slot, and in any case a small margin must be allowed on the insulated copper for clearance when the thick coils are not easily compressible. The

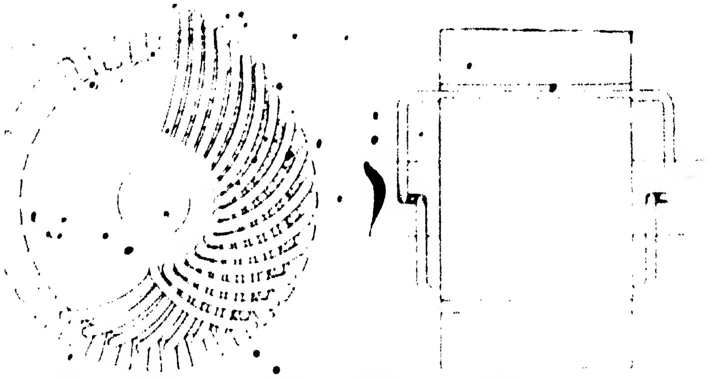


FIG. 168.—Four-pole armature with former-shaped coils, and involute end-windi[ng]

base-circle BD (Fig. 169), is that circle whose circumference is exactly filled by the end-connectors (and their clearances) as they spring normally outwards from its periphery. Taking any point A on the outer circle, and drawing a tangent from it to the base circle, as AB, the complete involute is that which would be traced if we imagine a string of length AB and fixed at B to be

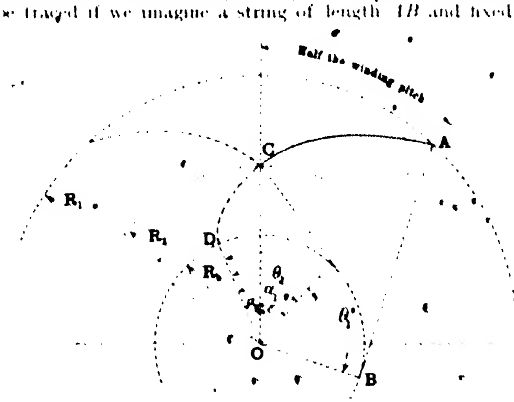


FIG. 169.—Involute end-winding.

gradually wrapped round the base circle while continuously stretched taut, and of this curve so much is used as suffices to span half the winding pitch, *e.g.* the length AC out of the total ACD, the length used diminishing as the number of poles is increased and the angular winding pitch is reduced. The double end-connector may therefore be set out as follows (Fig. 170). Upon a circle APQ of radius less than the radius to the slots by a distance of say $1\frac{1}{2}w$ (in order to allow the connectors to curve upwards and over into the slots), mark off a number of points A corresponding to the centre lines of the

slots; from each point draw the tangents 1, 2, 3, ... to the base circle, and upon these tangents which cut all the connectors normally mark off a number of widths each equal to w, beginning from the outer circumference as at A. The same process is repeated on both sides of the base circle, until by joining a succession of points A, M, N, etc., in the two directions of curvature, the two involute curves join as at C. To this depth must then be added an amount CF equal to the width of the connector parallel to the shaft ($= g$ which will in many cases be equal to $h_s/2$) in order to allow for the central portion of the end-connector curving round from one plane of whorls to the other. The radius to this inner edge F must evidently still fall outside or on the circumference of the base circle; otherwise connectors of the given thickness and spanning the required winding pitch cannot fall into the available room—a point which must be carefully verified.) The necessary radial depth may also

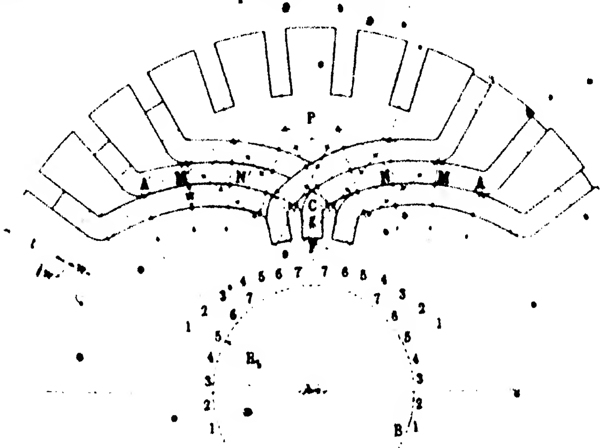


FIG. 170.—Setting out of involute end-winding.

be calculated mathematically as follows. If α = the angle subtended by an involute curve ACD, and θ = the angle subtended by the arc BD which is the evolute of the involute (Fig. 169), both being expressed in circular measure, AB which is equal to the arc BD is equal to $R_b \cdot \theta$, and

$$\tan(\theta - \alpha) = \frac{AB}{OB} = \frac{R_b \cdot \theta}{R_b} = \theta, \text{ whence } \alpha = \theta - \tan^{-1}\theta,$$

and by giving various values to θ, a curve connecting α and θ can be plotted applicable to all cases. Further, from the right-angled triangle AOB, $AB = \sqrt{AO^2 - OB^2} = \sqrt{R^2 - R_b^2} = R_b \cdot \theta$, so that universally $\theta = \frac{\sqrt{R^2 - R_b^2}}{R_b}$. The outer radius R to the circle APQ of Fig. 170, and also the radius R_b being known, θ_1 of the complete involute $= \frac{\sqrt{R_1^2 - R_b^2}}{R_b}$, and the corresponding value of α_1 follows from the relation given above. The angle ϵ corresponding to half the winding pitch being also known, a new value $\alpha_2 = \alpha_1 - \epsilon$ is found corresponding to the portion QD of the involute which is not used, and thence θ_2 from the general curve connecting α and θ. The value R_2 is then obtained from $\theta_2 = \frac{\sqrt{R_2^2 - R_b^2}}{R_b}$, and the radial depth of the portion AC is the difference $R_1 - R_2$. To this

must be added the constant $1\frac{3}{4}\,w$ reckoned from the bottom of the slots plus the central portion g.

The length of the involute ACD is $\frac{1}{2}\,R_b\,.\,\theta^2$; hence the length of the copper path in the curved portion of the end-connector between R_1 and $R_2 = \frac{1}{2}\,R_b\,(\theta_1^2 - \theta_2^2) = \dfrac{R_1^2 - R_2^2}{2\,R_b}$, and taking into account the various bends as the connector changes its plane, if $a'' =$ the length of projection outwards from the slot before it bends downwards into the inner whorl nearest to the core, the total length of path in the end-connector from slot to slot is

$$l' = 2\left\{\frac{R_1^2 - R_2^2}{2\,R_b} + a'' + 1{\cdot}75\,w + \frac{h_s}{2} + 2{\cdot}5g\right\}$$

§ 24. II. Bar-wound drum armatures.—From the former-wound barrel armature it is easy to pass to the ***bar-wound barrel armature***, the transitional stage being a single complete loop consisting of two active conductors and a pair of end-connectors, the whole made of copper strip of rectangular section. The required length of copper is cut off and bent round at its centre, so that it falls into two levels with a little space between them (Fig. 171); the bending tool is shown in Fig. 172, with a copper strip in process of being bent round on itself. Or, if the copper strip is thin and deep, it may be folded over on itself so as to form the junction of the upper and lower layers (Fig. 171). Each end is then bent through the correct pitch,

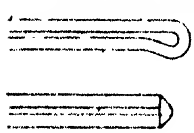


FIG. 171.

FIG. 172.—Tool for forming loop in copper strip

and by means of a shaper is given the proper curvature so that it may lie on a cylindrical surface in two levels. Fig. 173 illustrates a lap-wound loop, and Fig. 174 the shaper on which it is formed. The loops after being taped are placed in succession on the armature, the lower half being pressed down into the bottom

of the slot and insulation being inserted along the slot between the two layers where they are in close contiguity. The sloping gap between the two layers beyond the core is advantageous as assisting

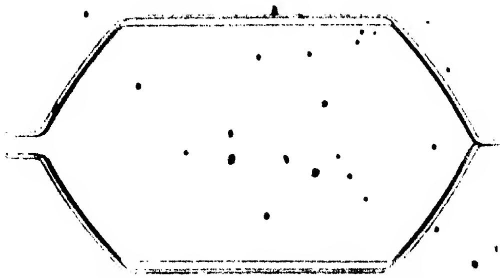


FIG. 173.—Lap-wound loop of copper strip

the ventilation. Finally, the upper halves of the first-wound loops are lifted up to allow of the lower sides of the last loops being introduced beneath them. Fig. 175 shows a lap-wound armature with coils thus made. The connections between the loops are

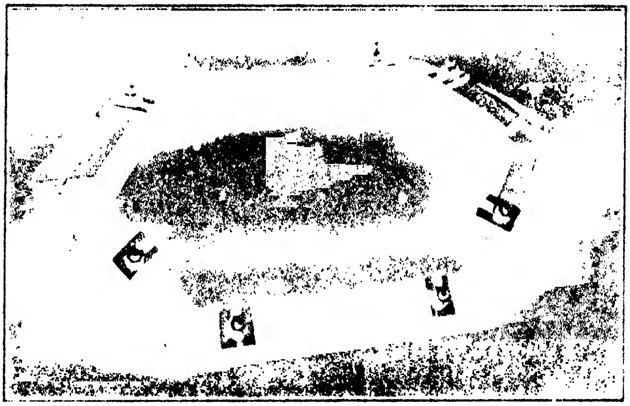


FIG. 174.—Shaper for lap-wound loop of copper strip.

made at their junctions in the forks of the commutator connectors. Fig. 176 gives the shaper required for a wave-wound armature in which there are two loops per section divided between different slots, the method of connection with every other loop reversed being indicated in Fig. 177.

The last form of the barrel armature is at once the simplest and

commonest for all machines having conductors of large cross-section; by it each loop is composed of two separate bars which are first placed in position and subsequently soldered together in their

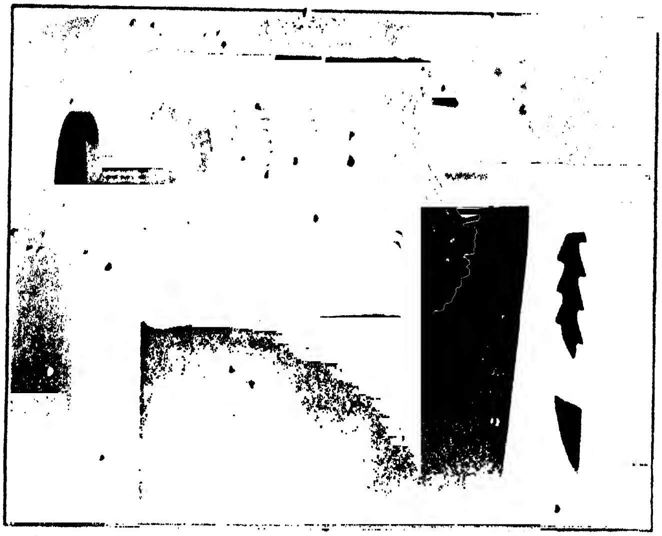


FIG. 175.—Barrel winding of armature with formed coils of copper strip.

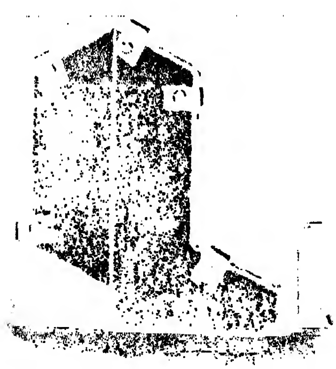


FIG. 176.—Shaper for wave wound loop with two loops per section.

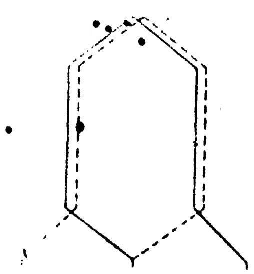


FIG. 177.—Connection of wave-wound loops, two per section.

due sequence when both layers have been arranged in the slots. Each end of the bar is bent through half the pitch, and is then curved to suit the armature circumference; Fig. 178 shows the

shaper used for this second operation, together with two bars in place for shaping. A group of two, three, or four bars corresponding to one slot and taped together are usually bent as a whole. The

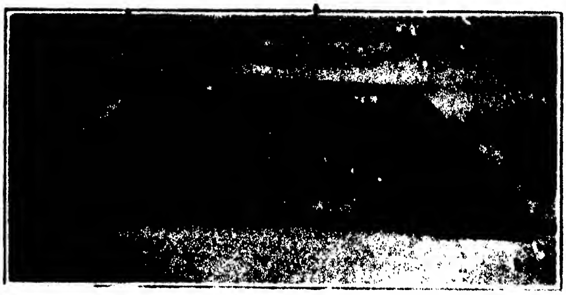


FIG. 178.—Shaper for half loops of lap wound bar armature.

shaped bars of the lower layer are then laid in position, one or more at the bottom of each slot, with their ends projecting so as to form a complete cylinder. Fig. 179 shows the commencement of the

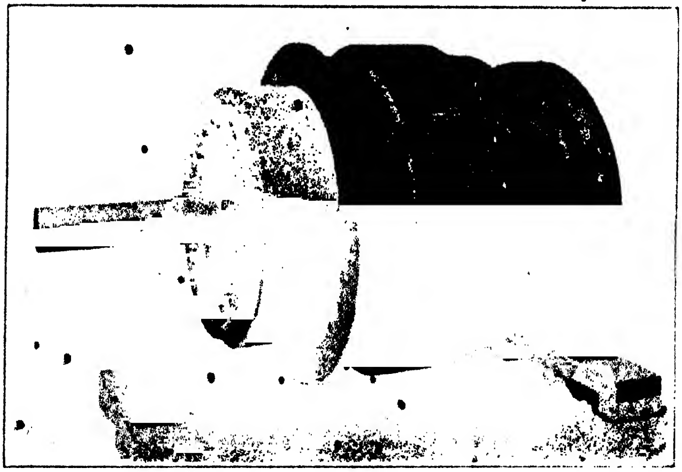


FIG. 179.—Lap-winding of bar armature with half-loops; lower layer.

winding of the armature of Fig. 124. After the insertion of insulation above the lower layer the remaining bars are placed at the tops of the slots, their ends having previously been similarly bent so as to pass through the remaining half of the pitch, and forming a complete cylindrical envelope on the outside of the armature

(Fig. 180). At the ends the two layers are soldered together, small copper clips being passed over the bare ends of a pair of bars,

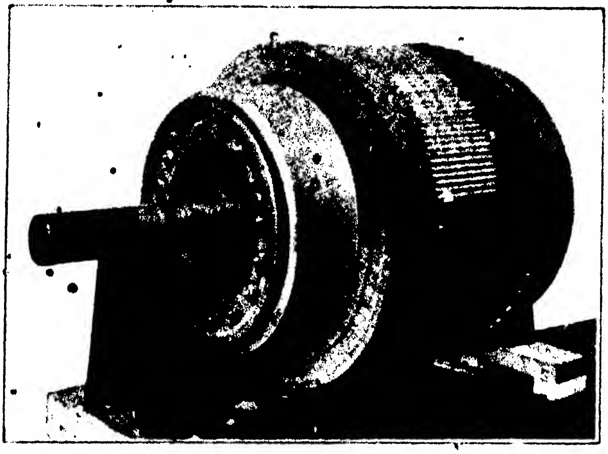


FIG. 180. Lap winding of bar armature with half-loops; upper layer.

one in the upper layer and the other in the bottom layer. Or at the commutator end the two layers of bars may be united by

FIG. 181.- Completed lap-wound bar armature.

soldering within the lug which leads to the commutator sector. Fig 181 shows the finished armature of Figs. 179 and 180 for an 8-pol 375-kilowatt machine ; as the slope of the end-connections at eithe

end is in opposite directions relatively to the armature core, it is lap-wound. The lower layer of end-connectors is frequently cranked down so as to allow of free circulation of air between the two layers, as shown in Fig. 200. With stout bars there is at the commutator end no necessity to support them on a special ring, the numerous commutator lugs helping to retain them in cylindrical shape unless the speed is very high.

The simple bars above described involve a greater number of soldered joints than the method of completely formed loops, but in either system the difference of potential between any two adjacent connectors in the same layer is never more than the E.M.F. due to two active conductors, so that there is little likelihood of the insulation breaking down. They are further easily repaired, especially

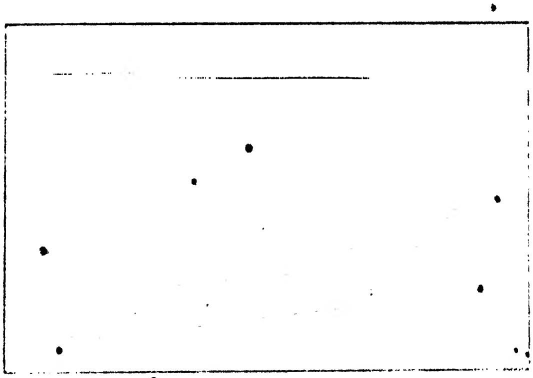


Fig. 182. Shaper for coil with involute end-connectors at one end.

the simply bar-wound form; if a bar of the lower layer be damaged it is only necessary to unsolder and take off a comparatively small number of bars in order to withdraw it. The sole disadvantage of the barrel-wound armature is, in fact, the axial length taken up by the end-connections, more especially when the number of poles is few. This axial length is for bars calculated on the same principles as in § 22 for coils.

In order to economize room in the axial length the involute type of end-connector is occasionally employed with bar armatures at the end farther from the commutator. The bare copper strip is first bent completely round on itself, leaving only a gap corresponding to the width between the two whorls of the end-connectors; if of massive section, this operation must be done when it is heated. It is then forced apart and hammered on a cast-iron shaper (Fig. 182) into a butterfly shape. Each end is then bent round through a right angle to form the straight inducing sides, and the whole is

lapped over with insulating tape (Fig. 201). A final class of drum armatures is that in which the axial length is a minimum owing to the employment of separate involute end-connectors at both ends of the armature, but this method of construction being practically confined to smooth-surface armatures is now seldom used.

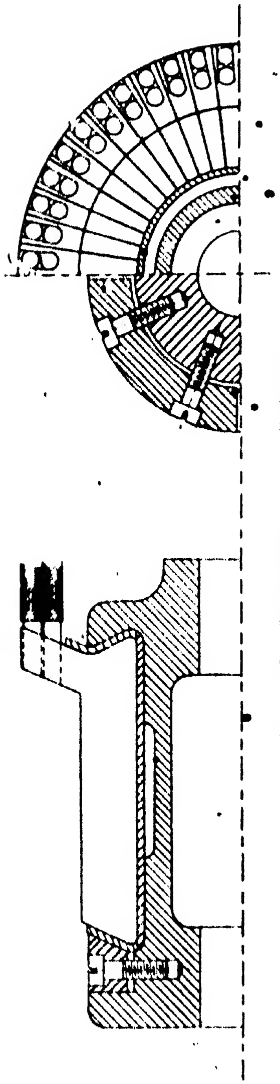


Fig. 183.—Commutator for small dynamo.

§ 25. Commutators.—In the construction of *the commutator* various modifications are possible, but, broadly speaking, two types may be distinguished. In the first (Fig. 183), which is only suitable for small machines, the wedge-shaped strips of copper and the intervening plates of mica are held in place by a sleeve or bush with a coned mushroom-shaped head and a coned ring divided into four or more segments and screwed down to the sleeve. The latter may be of gun-metal or cast iron, the ring being of wrought iron or gun-metal. The angle at which the coned surfaces are inclined to the horizontal axis should not exceed 50°. Complete insulation of the sectors from the supporting structure between which may exist the full stress of the voltage of the machine is obtained by conical rings on the sloping sides; these are of mica or micanite moulded exactly to the required taper. For 100 volts and upwards it becomes necessary to cover the entire sleeve with a moulding of micanite or paper to prevent sparking from copper to iron. Care must be exercised that the sectors do not bed down on to the sleeve before they are held thoroughly tight sideways. The sectors are forced inwards under the sloping faces at either end, and thus bind on one another

like the stones of an arch, but it is still possible, if their taper be but slight, that they may be driven inwards and put out of truth by an accidental blow.

In the second type (Fig. 184), which is more usual in larger

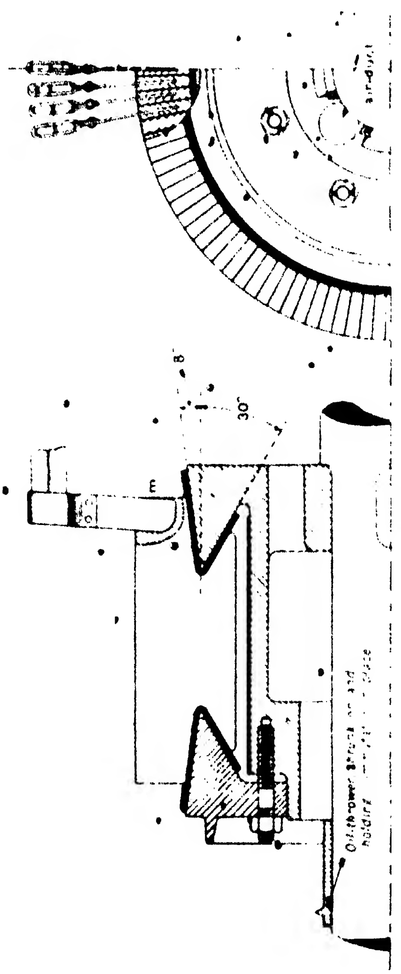


Fig. 184.—Commutator of second type, with double taper and set screws or studs.

machines, especially with carbon brushes, the coned ends of both sleeve and loose ring are let into recesses in the sectors, and for greater rigidity the sleeve is usually of cast iron with collar of cast iron or cast steel. The sectors and insulating strips of mica are

built up into a circle and tightly held together by a circular chuck or clamp having numerous radial screws compressing internal segmental pieces against the copper. At each end of the commutator

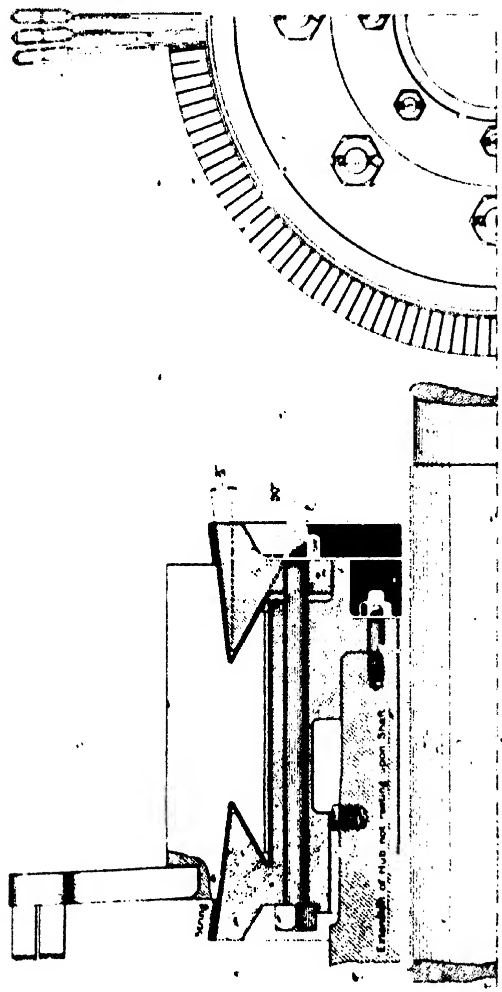


Fig. 185.—Commutator of second type with bolts.

a *V*-shaped groove is turned, into which a taper ring of built-up mica or micanite is fitted. The whole is passed over the sleeve until it engages with the coned fixed ring, the loose coned ring is put in place, and by hydraulic pressure the two rings are driven home and fixed by the screws after which the external chuck can be

removed. For greater strength in the present type the pressure is maintained by several set screws (Fig. 184), or in still larger machines by bolts, as in Fig. 185, and both coned rings may then be loose. Fig. 186 shows a component cast-iron sleeve and single loose cone, with one of the tapered micanite rings, while Figs. 187 and 188 show a small and a large finished commutator, the latter corresponding to the armature of Figs. 179-181. The bolts should be arranged as close as possible to the under side of the copper, and may for part of their length be let into pockets on the surface of the cast-iron body (Fig. 190); their material should have great tensile strength and a high elastic limit (nickel steel with an elastic limit of 65,000 lb. per square inch will be employed in special cases), and their length should be great, as giving more elasticity to take up the effect of heating and cooling.

FIG. 186.—Component parts of commutator.

When the dynamo is at work the temperature of the commutator is raised by the passage of the current over the contact-resistance of the brushes (especially if of carbon) and by their friction with its surface; the difference in the expansion of the copper and the cast-iron shell or sleeve then sets up very considerable stresses in the structure. It is consequently difficult to devise any mechanical construction that shall never fail to keep the surface perfectly cylindrical after repeated heating and cooling, since between metal and metal must intervene the layer of insulation, and upon this comparatively compressible medium falls all the stress. Any relative movement of neighbouring sectors by which one is raised above or lowered below the other even to a minute degree suffices with carbon brushes to cause them to jump as they pass over the displaced sector owing to the carbon having no elasticity such as is given by copper gauze brushes. Sparking is thereby set up, and the surface of the sector which is at fault is rapidly eaten away until finally the commutator is rendered unworkable. Smoothness of surface is in fact more essential than perfect concentricity with

the axis of the shaft; the brushes have time to follow any eccentricity of the commutator as a whole, but owing to the inertia of the brushes and brush-boxes their tension springs cannot take into account with sufficient rapidity any unevenness in adjacent sectors; the brush-box or brush is alternately jerked off and drawn on again to the surface—an operation accompanied by a chattering noise and excessive sparking. In order to prevent any shifting of the sectors relatively to one another, various constructions and various angles of taper have been tried, and experience seems to show that a double taper such as is shown in Figs. 184 and 185 is better than a single taper and a flat band. The two angles are, however, usually unequal, a general proportion being a total angle of 38°

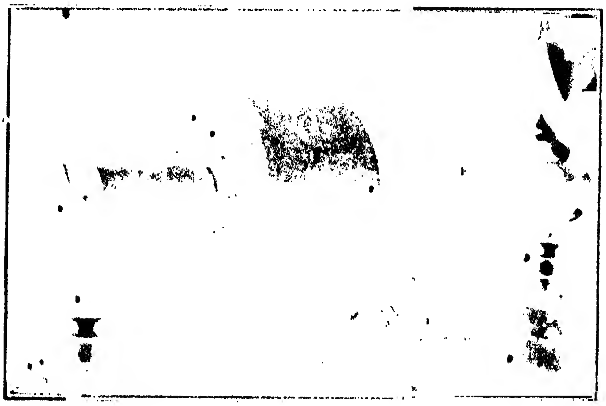


FIG. 187. Finished commutator.

divided into 30° below the horizontal and 8° above. By means of the double taper the sectors are held endways and centred round a circle corresponding to the diameter of the apex of the cone, and as the retaining rings expand and contract during heating and cooling all the sectors are maintained concentrically on this circle. Owing, however, to the greater angle below the horizontal, more force is exerted inwards than outwards as the metal cones are driven home in the initial process of construction, so that the sectors are at that time not only pinched together axially but jammed tight against each other sideways. The apex of the cones is rounded off so as not to press on the internal angle of the insulation. If the sectors are supported entirely by the V-rings with an internal air-space, the commutator is "arch-bound," while if the sectors are also compressed on to an insulated cylindrical seating, they may be styled "bed-bound." As to the relative advantages of the two methods of construction, it is not easy to decide. The

longer seating on the sleeve when the sectors are bedded thereon as compared with that on the upper cone perhaps affords greater security in the case of sectors having a very small taper; on the other hand, there is greater manufacturing difficulty in ensuring that the sectors do not lose some of their pressure sideways in virtue of their bedding on the sleeve before the arch is tightly closed, although with care this objection may to some extent be removed

FIG. 188. Finished commutator.

by inclining the sectors very slightly to a radial line by some angle not exceeding 5°, so that when further compressed inwards they tend to become more truly radial.[1]

It will be seen that for a given length of brush working surface commutators of the first type are slightly longer than those of the second type; on the other hand, the latter cannot be worn down below the level of the end rings, and therefore for a given diameter their radial depth of wear is less than in the first type. They present, however, a greater cooling surface, and if of considerable

[1] See Miles Walker, *The Diagnosing of Troubles in Electrical Machines*, pp. 296–302.

diameter air ways can be arranged so as to allow air to circulate through the inside of the central hub. The usual radial depth for turning down rises from $\frac{1}{2}''$ in small to $1''$ or $1\frac{1}{2}''$ in large machines.

No openings to the inner surface of the copper are permissible, for fear of dust or moisture finding an entrance and causing a breakdown of the insulation. In both types of commutator, in order to prevent the sleeve twisting round out of its proper position, it is secured either by a small sunk key under its head, as in Fig. 184, or by the fastening studs of Fig. 185. The small screw in Fig. 185 serves as a register when the commutator is being fitted in place.

The connections from the individual sectors to the sections of the armature winding are most commonly made by thin strips of copper. A saw-cut is made in each sector, as shown at *m* (Fig. 184), and into this one end of the strip (from $\frac{1}{2}''$ to $1\frac{1}{2}''$ wide according to the current) is soldered; the other end is carried up to the level of the ends of the sections and there embraces and is soldered to the armature wire. The strips must be bent over the tops of the armature bars, and may each be composed of a pair of thinner strips in parallel, as in Fig. 185. Since the current only flows through the commutator lugs for very short intervals of time, the current-density within them may be high, but should not exceed 5,000 amperes per square inch of section. When in small machines cast sectors of phosphor bronze alloyed with copper or gun-metal are used, a projecting lug is directly cast on each sector, as in Fig. 183; the wires are then soldered into a groove at the top of each lug. The same construction may be employed with hard-drawn copper sectors in small machines and in other cases where it is advisable, the wires being led down to upstanding lugs turned out of the solid copper. In such cases the mica is extended up to the full height of the lug, and an additional advantage claimed for the construction is that copper dust worn off the commutator by the brushes cannot be blown through the solid wall formed by the close-fitting lugs. On the other hand, a considerable amount of ventilating effect is lost, which on the first method is secured by the moving blades of copper. Drop-forged sectors have also been employed, and in these the lug can be forged in the mould in one piece with the sector. Nothing, however, is so suitable for commutators as bars of hard-drawn copper sawn up to the required lengths; homogeneity and perfect uniformity of the sectors in hardness is essential to good working, and in these qualities hard-drawn copper notably surpasses either drop-forgings or castings.

When mounted immediately upon the shaft, if there be any flexure of the shaft between the armature hub and the commutator sleeve, the connecting lugs are alternately extended and compressed every revolution. This stress in time hardens the copper, and causes it to break either at the junction with the armature bar or

at the root where it enters the commutator. This evil may be avoided by the employment of flexible stranded ribbon or cable for the connections. It is, however, in every way better in large machines subjected to heavy strains to fasten the commutator directly to the armature cast-iron hub either on a projecting sleeve (Figs. 185 and 199) or by bolting it up to its end, so that it is entirely free from the shaft. With this construction even though the shaft may bend in a minute degree, no relative displacement of the armature and commutator can take place.

§ 26. Mechanical design of V-type commutators. With large commutators running at high peripheral speeds, great care must be exercised in the design to ensure amply sufficient mechanical strength in the rings, bolts or screws, and in the copper sectors, so

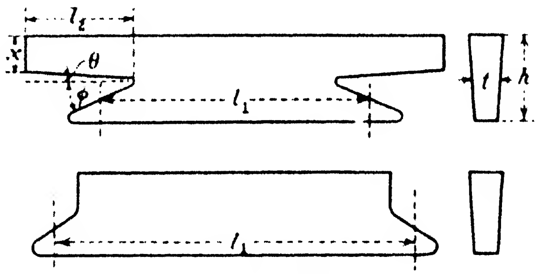


FIG. 189.

that there may be no fear of the commutator bursting under the stress of centrifugal force. If t_1 and t_2 = the thickness of a copper sector at the top and bottom respectively, and h = its depth, its mass-centre falls on the line bisecting its thickness at a distance

$$\bar{x} = \frac{h}{2}\left(1 + \frac{1}{3} \cdot \frac{t_1 - t_2}{t_1 + t_2}\right)$$

from the bottom. The radius to its centre of gravity is therefore

$$r_g = \frac{r_o + r_i}{2} + \frac{r_o - r_i}{6} \cdot \frac{t_1 - t_2}{t_1 + t_2}$$

and from the dimensions which the copper bars assume in practice this may in all except very small commutators be simply identified with the mean radius $\dfrac{r_o + r_i}{2}$.

If t = the mean thickness, and h be identified with $r_o - r_i$, as has been assumed above, the weight for a length of l inches is $0{\cdot}322\, l(r_o - r_i)t$ lb. The centrifugal force per sector of the given length, all dimensions being in inches, at N revs. per min., is therefore

$$\begin{aligned} f_{sc} &= 91 l(r_o - r_i)t N^2 . r_g \times 10^{-7}, \\ &= 45{\cdot}5 l(r_o^2 - r_i^2)t N^2 \times 10^{-7} \text{ lb.} \qquad (85) \end{aligned}$$

or per cubic inch $= 45{\cdot}5(r_o + r_i)N^2 \times 10^{-7}$.

In the first place, taking the bending moment due to this force distributed along the length l_1 between the supports of the commutator V's, the length l_1 being treated as a beam supported at both ends, the maximum stress on the copper is $\dfrac{f_{sc} \cdot l_1}{8Z}$.

The modulus Z of the tapering section in relation to tension along the outer edge is $\dfrac{h^2}{12}\left(\dfrac{l_1^2 + 4l_1 \cdot l_2 + l_2^2}{l_1 + 2l_2}\right)$, and the tensile stress upon the copper is

$$s_1 = 68 \cdot l_1^3 \cdot \frac{r_o + r_i}{r_o - r_i} \cdot N^2 \cdot \frac{t(l_1 + 2l_2)}{l_1^2 + 4l_1l_2 + l_2^2} \times 10^{-7} \text{ lb. per sq. in.} \quad \ldots \quad (86)$$

The support of the commutator bars is not such that they can be relied upon to reproduce the case of a beam built in at both ends, in which case the stress would be reduced to $\dfrac{f_{sc} \cdot l_1}{12Z}$.

In addition to the tensile stress of eq. (86) there is also the stress due to the axial pressure from the retaining end-rings. Let P = the total resultant axial load on the bolts from all causes; then the share taken by each sector is P/C. This force acting from each end of the bar causes an axial compression along l_1, and a bending moment $\dfrac{P}{C}(\bar{x} - c)$ where $\bar{x}$ is the distance of the mass-centre for the cross-section from the base of the copper, and c is the distance of the line l_1 from the base (Fig. 191). Above the neutral line, *i.e.* above the height $\bar{x}$, the stress changes to tension, which reaches its maximum value at the outer edge, namely, $\dfrac{P}{C}(\bar{x} - c)\left(\dfrac{h - \bar{x}}{I_c}\right)$, where I_c is the moment of inertia of the tapering section about the neutral axis, and is $= \dfrac{h^3}{36}\left(\dfrac{l_1^2 + 4l_1 \cdot l_2 + l_2^2}{l_1 + l_2}\right)$. The above-described stress from the axial pressure P is then additional to the tensile stress s_1 from centrifugal force.

The total tensile stress should not exceed 7,000 to 8,000 lb. per square inch for hard-drawn copper.

In very long commutators the deflection of the bar between the end-clamps may be appreciable and require calculation. Part of this deflection is due to the centrifugal force, and may be calculated as in § 9 for a beam of length l_1 under a distributed load, namely, $= \dfrac{f_{sc} \cdot l_1^3}{76 \cdot 8 \cdot E_c I_c}$, where E_c is the modulus of elasticity for copper = 16,000,000 lb. per square inch. But there is also a part due to the bending moment from the axial component of R opposing P,

which has a uniform value all along the line between the supports; the additional deflection due thereto is

$$\frac{1}{8} \cdot \frac{l_1^2}{E_c} \times \frac{(\bar{x} - c)}{C \times I_c} \times R \cdot \sin \phi$$

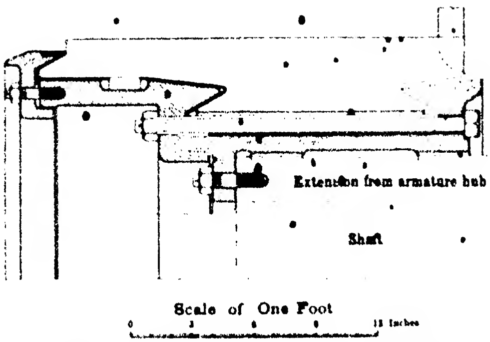


FIG. 190.—Construction for very long commutator

The above calculations must be worked out for the dimensions which the bar will assume when worn and turned down to the lowest permissible depth, since it is then that the stress reaches its

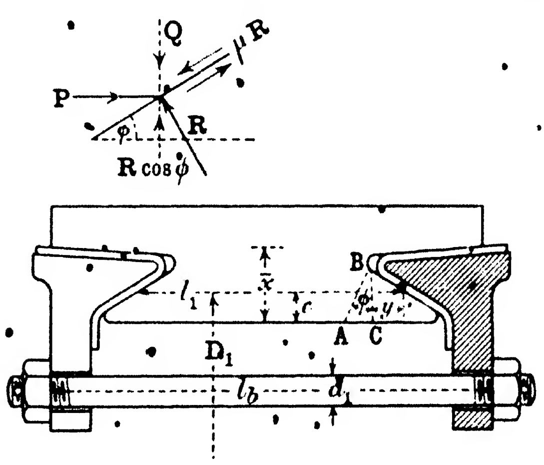


FIG. 191.

maximum value. If the deflection exceeds a few mils, as may happen with commutators for large currents at low voltages and high speeds, some central support becomes necessary, and this may be obtained either as in Fig. 190 or by a sleeve having a fixed cone

near the centre and two loose cones, one at each end, drawn together by tension bolts.

In a similar manner to the above may be calculated the bending stress upon the overhanging portion of the copper bar which projects beyond the nose of the V, the length l_2 (Fig. 189) being taken as a beam fixed at one end only. If the decrease of the centrifugal force due to the wearing down of the copper be neglected, *i.e.* assuming the centrifugal force per cubic inch of copper to remain constant, the following approximations are obtained for the stress and deflection of a plain overhanging portion[1] of depth x (Fig. 189), namely,

$$s_2 = \frac{45{\cdot}5\,(r_o + r_i)N^2 \times l_2^2\,(3x + l_2 \,.\, \tan\theta)}{(x + l_2 \,.\, \tan\theta)^2} \times 10^{-7} \text{ lb. per sq. in.},$$

and

$$\delta = \frac{s_2 \,.\, l_2^2}{2E_c(x + l_2 \tan\theta)}$$

These values, again, are usually a maximum when the commutator is worn down and x has its minimum working value. It is, in fact, important, especially with high speeds, that the amount by which the commutator may safely be turned down should be clearly marked upon its end face.

In the process of manufacture, while the whole commutator is heated up to a high temperature, the V-rings are squeezed tightly home by means of hydraulic power or external screws which enable a much higher pressure to be brought to bear than could be safely applied through the commutator retaining bolts only. Let the total axial pressure which is applied in this stage, and which is chiefly due to external means, reinforced it may be by some pressure from the retaining bolts, be $P_a + P_b = P_o$; this pressure calls forth a reaction R_o directed outwards and at right angles to the inclined surface of the copper sectors (Fig. 191). Taking the axis of the shaft as horizontal, then as the end-rings are forced up the inclined surfaces the horizontal components of R_o and of the frictional resistance μR_o by the principles of the inclined plane together balance P_o, *i.e.* $P_o = R_o \sin\phi + \mu R_o \cos\phi$. The vertical component of R_o is balanced by the combined effect of the vertical component of μR_o and the vertical pressure Q_o exerted inwards by the end ring, *i.e.* $R_o \cos\phi = Q_o + \mu R_o \sin\phi$. When expressed in terms of P_o,

$$Q_o = P_o \,.\, \frac{\cos\phi - \mu \sin\phi}{\sin\phi + \mu\cos\phi} = P_o \frac{1 - \mu\tan\phi}{\mu + \tan\phi}$$

and $R_o = \dfrac{P_o}{\sin\phi + \mu\cos\phi}$. The value of Q_o at this stage represents the total amount of the inward radial force acting uniformly all round the periphery of the ring of copper *at each end*. The case is analogous to that of § 5 for centrifugal force acting on a ring, but instead of a uniformly distributed outward radial pull we have a uniform inward radial push, and instead of a hoop tension we have a compressive stress produced on the flat sides of the copper and mica strips; the intensity of this compressive stress s_c is therefore given by the same expression as in § 5, namely, as equal to the total radial force divided by $2\pi ab$ where ab is the area on which it acts. Assuming the outer surface of the V to be nearly horizontal, so that there is little or no expanding action due to it on the overhanging wedges of copper,

[1] R. Livingstone, *The Mechanical Design and Construction of Commutators*, p. 8, where much practical and theoretical information on the subject is to be found.

the radial force *at each end* may be considered as producing a pressure distributed uniformly over the entire area of one half side of a copper bar or mica strip; *i.e.* the area *ab* now in question is half the entire surface A of one side of a sector or mica strip. The intensity of the peripheral stress between opposite faces of a longitudinal section through the commutator parallel to a mica strip is therefore, up to the present, when the commutator is at rest, $s_c = Q_o \cdot \dfrac{1}{\pi A}$.

During the stage of construction, $Q_o = P_o \left(\dfrac{1 - \mu \tan \phi}{\mu + \tan \phi}\right)$, and μ has its maximum positive value, so that $s_c = \dfrac{P_o(1 - \mu_{max} \tan \phi)}{\tan \phi + \mu_{max}} \cdot \dfrac{1}{\pi A}$.

If the same initial value is given to the intensity of this compression on the mica of different machines, it results that the axial pressure P_o that must be applied in building up becomes simply proportional to the area of a bar for a given angle of inclination and value of μ; *i.e.* $P_o = s_c \pi A \dfrac{\mu + \tan \phi}{1 - \mu \tan \phi}$. The coefficient of friction μ has great importance and can reach a high value, say, 0·4, whence with $\phi = 30°$, $P_o = 4 s_c A$.

Identifying P_o with the pressure P_r produced by external means only, since the nuts of the retaining bolts should not in the present stage be tightly screwed up so as to add any great amount to the total, a useful practical rule gives P_o as 4,800 A lb., so that s_c is virtually 1,200 lb. per square inch of the mica surface,[1] and this should be maintained until the commutator is cool. The nuts of the retaining bolts are now tightened up, and the pressure P_r removed, leaving only the pressure P_b from the bolts. In determining the new state which results from this removal of the external pressure, the guiding principle must be the consideration that should the copper expand or the end-rings contract even in the slightest degree, sliding takes place between the mica end-rings and the inclined metal surfaces, and this implies not only that the frictional resistance has changed its direction or sign, but also that μ has again reached its *maximum* value in the opposite or negative direction. The radial pressure on the end-rings, therefore, as soon as relative movement takes place, is $Q = P \cdot \dfrac{1 + \mu \tan \phi}{\tan \phi - \mu}$, and the axial pressure on the end-rings tending to force them apart becomes $P = R \sin \phi - \mu R \cos \phi$, or $R = \dfrac{P}{\sin \phi - \mu \cos \phi}$. For the same value of Q or R a value of P much smaller than P_o evidently suffices to retain the same total strain on copper and end-rings.

In order, then, to determine what happens when the external pressure is removed, the amount by which the copper has been contracted and the end-rings expanded under the pressure P_o must in the first place be calculated so as to judge whether it will alter, and this can be done as follows.

The strain of any component part of the structure is equal to its length multiplied by the ratio of its stress per unit area to its modulus of elasticity. The peripheral compression of the copper and mica is of chief importance, and the amount of the contraction of the circle through the middle of the sectors due to the peripheral compressive stress s_c is $\dfrac{lC \cdot s_c}{E_c} + \dfrac{mC \cdot s_c}{E_m}$, where E_c is the modulus of elasticity of copper $= 16 \times 10^6$ lb. per square inch as before, and E_m is the modulus of elasticity of the mica. The latter for built-up strips is very variable, and increases with increasing pressures; it can hardly, therefore, be regarded as constant, but from actual experiments analogous to the case of a commutator it may be taken as $0·5 \times 10^6$ to 0·75–

[1] If it be argued that the intensity of the pressure on the mica should increase as the number of sectors is increased, and their taper decreased, such an assumption would lead to s_c being made roughly proportional to the diameter of the commutator.

$\times 10^6$ lb. per square inch.[1] But this peripheral contraction virtually amounts to a shortening of the axial length along the line l_1, and it may also be expressed in terms of its axial equivalent by the following considerations. A radial contraction inwards of the inclined surface of the copper by an amount x has exactly the same effect as if the length of the copper was shortened by the amount $x/\tan\phi$. The radial contraction of both ends amounts, therefore, virtually to a contraction of the length by $2x/\tan\phi$, or since $2\pi x$ would be the contraction of the periphery the virtual axial shortening[2] from both ends is equal to $\dfrac{\text{peripheral contraction at either end}}{\pi\tan\phi}$

$$= \frac{1}{\pi\tan\phi}\cdot\frac{C}{\pi A}\left(\frac{l}{E_c}+\frac{m}{E_m}\right)\cdot\frac{P(1-\mu\tan\phi)}{\tan\phi+\mu}$$

The copper is also directly compressed axially, but for any value of P, the total axial compressive stress s_a is not simply equal to P divided by the area of the copper annulus; this direct stress is augmented by the fact that P does not act through the mass centre of the section, which gives the additional stress $P(x-c)\dfrac{x-c}{I_c \times C}$, and further by the fact that the compression is applied only at the supports while the radial resisting forces are distributed evenly between the supports. The average bending moment due to this latter fact is $\frac{1}{3}R\cos\phi\,.\,l_1$, yielding an additional compressive stress $\frac{1}{3}R\cos\phi\times l_1\times\dfrac{x-c}{I_c\times C}$. The total axial compressive stress is therefore

$$s_a = P\left(\frac{1}{\text{area of copper annulus}}+\frac{(x-c)^2}{C\times I_c}+\frac{l_1(x-c)}{3(C\times I_c)(\tan\phi+\mu)}\right)$$

and the compression $= l_1\,.\,\dfrac{s_a}{E_c}$.

The two additional terms within the bracket in general far outweigh the first, so that the axial compression of the copper is much more than might at first be expected. In order to shorten calculation as much as possible by expressing each quantity in the form in which it is finally required, and in terms of the data of construction, let

$$\frac{l_1}{E_c}\left(\frac{1}{\text{area of copper annulus}}+\frac{(x-c)^2}{C\times I_c}\right)=D$$

$$\frac{l_1^2}{3E_c}\times\frac{x-c}{C\times I_c}=G$$

$$\frac{1}{\pi\tan\phi}\cdot\frac{C}{\pi A}\cdot\left(\frac{l}{E_c}+\frac{m}{E_m}\right)=H$$

The total compression of the copper and mica, when reduced to the common basis of axial length, is thus in the construction stage when μ has its maximum positive value

$$\eta_{co} = P_o\left\{D+\frac{G+H(1-\mu_{max}\,.\,\tan\phi)}{\tan\phi+\mu_{max}}\right\}$$

Analogously to the case of the direct axial compressive stress on the copper, the peripheral or hoop tensile stress on the steel or iron end-ring is not simply equal to the radial pressure Q divided by $2\pi\,.\,ab$, where ab is the area of the section of the end-ring. This is augmented by the displacement of the line of action of Q from the neutral axis, by the direct action of the axial pressure

[1] The latter figure being that adopted by Mr. Livingstone, *The Mechanical Design and Construction of Commutators*, where the theory of commutator construction has been treated at length, but in the case of the V-ring type without allowance for the effect of the friction.

[2] Livingstone, *loc. cit.* p. 17 ff.

P, and also by the centrifugal force of the end-ring itself. The last-mentioned force may be neglected, and calculation from a specimen end-ring as typical of the class then shows that in relation to both Q and P the equivalent section A_e is about one-quarter of the total section, so that for A_e should be taken $\frac{1}{4}$th of the area shown shaded in Fig. 191 without deduction for the bolt holes. The tensile stress tending to expand the end-ring peripherally is thus

$$s_e = (Q + P) \cdot \frac{1}{2\pi A_e}$$

But $Q = P \cdot \dfrac{1 - \mu \tan\phi}{\tan\phi + \mu}$; therefore $s_e = P\left(\dfrac{1 - \mu\tan\phi}{\tan\phi + \mu} + 1\right)\dfrac{1}{2\pi A_e}$

and if D_1 is the mean diameter of the end-ring at the assumed line of contact between copper and ring, the peripheral elongation is $\pi D_1 \cdot s_e/E_e$, where E_e is its modulus of elasticity $= 30 \times 10^6$, if of steel or wrought iron. A radial expansion outwards of the inclined surface of an end ring by an amount x has exactly the same effect as if it were slid back along the bolts through a distance $x/\tan\phi$. The virtual increase in the axial length due to the peripheral expansion at both ends is therefore equal to $\dfrac{\text{peripheral expansion of one end}}{\pi \tan\phi}$

Let
$$\frac{1}{\pi \tan\phi} \cdot \frac{D_1}{2A_e E_e} = E$$

Then if P is given its value P_o, and μ has its maximum positive value, the equivalent increase in the axial length from the elongation of the end-rings is

$$\eta_{eo} = P_o\left\{E + \frac{E\,(1 - \mu_{max} \tan\phi)}{\mu_{max} + \tan\phi}\right\}$$

The sum of the contraction of the copper, axially and peripherally, and of the peripheral expansion of the end-rings, all reduced to their axial equivalents, is therefore

$$\eta_o = \eta_{co} + \eta_{eo} = P_o\left\{D + E + \frac{G + (F + H)\,(1 - \mu_{max} \tan\phi)}{\mu_{max} + \tan\phi}\right\}$$

Given this preliminary value for the compression of the copper and expansion of the end-rings, the question of what happens when the external pressure is removed entirely turns upon the minimum axial pressure P_r that will suffice to retain the same value η_o. When the copper is on the point of expanding and the end ring is on the point of contracting, μ has changed its sign and reached its maximum negative value, so that the required condition is

$$\eta_o = P_r \cdot \left\{D + E + \frac{G + (E + H)\,(1 + \mu_{max} \tan\phi)}{\tan\phi - \mu_{max}}\right\}$$

or the minimum retaining pressure is

$$P_r = P_o \cdot \frac{D + E + \dfrac{G + (E + H)\,(1 - \mu \tan\phi)}{\tan\phi + \mu}}{D + E + \dfrac{G + (E + H)\,(1 + \mu \tan\phi)}{\tan\phi - \mu}}$$

It is evident that unless μ_{max} is negligibly small, P_r may be much less than P_o, and if $\mu_{max} = 0{\cdot}4$, P_r usually works out only about $\frac{1}{5}$th of P_o, or say, $960\,A$ lb. In the supposed extreme case of $\mu_{max} = \tan\phi$, the commutator would be self-holding without any tension at all on the retaining bolts.

The effect of removing the external pressure can now be determined. If P_b exceeds P_r, although *ex hypothesi* less than P_o, no alteration of the length of the copper ensues. There is no further stretching of the bolts, and they remain subjected to a stress $= \dfrac{P_b}{n \cdot \dfrac{\pi d^2}{4}}$, where n is their number and d is their diameter at the bottom of the thread. The value of μ simply adjusts itself to take up the difference between P_b and P_r, and to the extent of this difference there is already a margin to meet the effect of heating or of

centrifugal force or of both combined before any further stress will fall upon the bolts. If P_b should happen to be exactly equal to P_r, any further stress from heating or centrifugal force will allow the copper to expand and slide on the V-rings, pushing them apart, and the bolts will be farther stretched. If P_b were less than P_r, the initial compression of the copper and elongation of the end-rings will not be maintained on the removal of P_s, and a smaller contraction of the copper and expansion of the end rings must be taken with a corresponding greater extension of the bolts, until the two values of P so obtained are one and the same, and the farther extension of the bolts is equal to the expansion of the copper and contraction of the end-rings when both are referred to the same dividing line. The pressure from the bolts has then failed to maintain the full value of the initial squeezing.

It is obvious that it is useless to give such a high value to P_o that the stress on the bolts from P_r approaches their elastic limit. But with good design there is no danger of this, and the stress from P_b, although exceeding P_r, always falls well below the elastic limit of the material of the bolts. The bolts can then at this stage be tightened up until the tensile stress on the bolts at the bottom of the thread has such a safe value as, say, 10,000 lb. per square inch, and from this may be reckoned the initial axial pressure $P_i = 10{,}000 \times n \cdot \frac{\pi d^2}{4}$ which the finished commutator has when at rest. If the preliminary P_b was less than P_i, the only effect of this will have been to again raise u to a lesser negative or a positive value without unduly stressing the bolts. The added effects of heating and centrifugal force may now be considered.

After prolonged running at full load, since the steel bolts and the copper reach different temperatures, and have different co-efficients of expansion, there would arise, if both were free to expand naturally, a certain *difference* of their lengths due to heating. There is, however, actually no resulting difference of length on the boundary surface between the copper and the iron, *i.e.* at the bearing lines along the inclined surfaces of the V-rings, the explanation being that the difference that would naturally arise is taken up by compression of the copper and mica and expansion of the end-rings, or by elongation of the bolts or partly by one and partly by the other. The linear co-efficient of expansion of copper being 0·0000173 per degree Cent., and of iron 0·000012, the natural expansion under heat of the copper with peripheral expansion reduced to its axial equivalent is $0{\cdot}0000173 \times T^\circ \left(l_1 + \frac{tC}{\pi \tan \phi}\right)$, and of the bolts and V-rings (assumed to have the same temperature) is $0{\cdot}000012\, t^\circ \left(l_1 + \frac{\pi D_1}{\pi \tan \phi}\right)$ where T° and t° are the respective rises of temperature of copper and iron. Although the bolts are of length l_b greater than l_1, yet the end-rings of iron expanding axially in the contrary direction counterbalance part of the length of the bolts, so that in each case the effective axial difference arises on the length l_1. If a rise of 34°·5 C. is assumed for the copper, and of 14°·5 C. for the iron retaining structure, the increases of length are respectively $0{\cdot}0006 \left(l_1 + \frac{tC}{\pi \tan \phi}\right)$ and $0{\cdot}000175 \left(l_1 + \frac{\pi D_1}{\pi \tan \phi}\right)$. The natural difference of length would therefore under the action of heat be

$$\eta_h = 0{\cdot}000425\, l_1 + \frac{0{\cdot}0006\, tC - 0{\cdot}000175\, \pi D_1}{\pi \tan \phi}$$

The whole of this could be taken up by compressing the copper and mica and expanding the end-rings. The axial force P_h required to do this when the copper is just about to expand, and μ has its maximum negative value, would, analogously to the case of P_r, be

$$P_h = \frac{\eta_h}{D + E + G + (F + H)\dfrac{(1 + \mu_{max} \tan \phi)}{\tan \phi - \mu_{max}}}$$

or

$$P_h = P_r \times \frac{\eta_h}{\eta_r}.$$

Lastly, the effect from the centrifugal force of the various parts when the machine is running at its full speed must be taken into account. Identifying the total radial centrifugal force of copper and mica with that of the copper sectors, this force[1] $F_{sc} = C \cdot f_{sc}$ acting radially against the inclined surfaces of the end-rings may be regarded as equally divided between the two ends, so that at each end there results an outward radial pressure $q = \frac{1}{2} F_{sc}$. The additional tensile stress thereby thrown upon the end-rings owing to the centrifugal force of the sectors tends to elongate them, but if they expand with increasing speed the peripheral compression on the copper and mica is automatically removed in a corresponding proportion. The actual proportions of the two changes will depend upon all the conditions as to the relative strengths of the copper and iron and the other forces present, but it will suffice here to observe that the total contraction of copper and elongation of rings for a given speed may or may not remain the same as when at rest. The hoop tension due to the centrifugal force from the end-rings themselves has the same effect, but may by comparison be neglected in the V-ring type of commutator.

As soon, therefore, as the commutator is rotated, from Q, the radial pressure of the retaining end-rings, there must be deducted $\frac{F_{sc}}{2}$, in order to find the resultant radial pressure which is really effective in compressing the copper and mica peripherally. Hence s_c in its complete form is $\frac{1}{\pi A}\left(\frac{P(1 + \mu \cdot \tan\phi)}{\tan\phi + \mu} - \frac{F_{sc}}{2}\right)$, and in the working state of the commutator, when it is heated and also rotating at full speed, the minimum axial pressure P_r' that will retain the original relative lengths of commutator and end-rings will be given by the equation

$$\eta_o + \eta_h = P_r'\left(D + E + \frac{G + E(1 + \mu_{max}\tan\phi)}{\tan\phi - \mu_{max}}\right) + H\left(\frac{P_r'(1 + \mu_{max}\tan\phi)}{\tan\phi - \mu_{max}} - \frac{F_{sc}}{2}\right)$$

whence

$$P_r' = \frac{\eta_o + \eta_h + H \cdot \frac{F_{sc}}{2}}{D + E + \frac{G + (E + H)(1 + \mu_{max}\tan\phi)}{\tan\phi - \mu_{max}}} \quad . \quad . \quad (87)$$

It is evident that P_r' is greater than P_r, not only owing to the action of heat but also owing to the action of centrifugal force which weakens the peripheral compression on the mica and copper.

If, then, P_t exceeds P_r', the copper and end-rings together still remain of the same length relatively to the bolts, although both have increased by the amount of expansion of the length of the bolts, and no further stress has been thrown upon the bolts, wherein at the threads we still have $s_t = \frac{P_t}{n \cdot \frac{\pi d^2}{4}}$.

Such must be the aim of the designer, and it will be seen that μ then acts as it were automatically to take up the additional stresses that would otherwise result to the bolts from heating and centrifugal force. The coefficient of friction assumes such a value μ_1, intermediate between its positive and negative maxima, that

$$\frac{\eta_o + \eta_h + H\frac{F_{sc}}{2}}{D + E + \frac{G + (E + H)(1 + \mu_1\tan\phi)}{\tan\phi - \mu_1}} = P_t$$

In making the calculations to determine what margin is actually assured in P_t, it is advisable to err on the side by taking μ at a fairly low value of, say, 0·3.

[1] As an approximation, allowing about 10 per cent. of the mean circumference for the mica strips, $F_{sc} \approx 175\ lN^2 (r_o^2 - r_i^2) \times 10^{-7}$ lb.

Although P_r' should not exceed P_t, it is of interest to determine what additional stress is thrown on the bolts when this occurs. Assuming that P_t is at least $> P_r + P_h$, let

$$P_t = \frac{\eta_o + \eta_h + H\frac{F_{sc}}{2}y}{D + E + \dfrac{G + (E + H)(1 + \mu_{max}.\tan\phi)}{\tan\phi - \mu_{max}}}$$

i.e. $\frac{F_{sc}}{2}.y$ is a certain proportion of the centrifugal force, or the centrifugal force for some speed lower than the full speed, for which with the heat effect the minimum retaining pressure is exactly P_t. As soon as the speed is increased above this value the bolts are more stressed, and the end-rings are expanded; the copper is more compressed axially, but on the other hand is less compressed peripherally. If $B = \dfrac{l_b}{n \cdot \dfrac{\pi d_1^2}{4} \cdot E_b}$ where E_b is the modulus of elasticity of the steel bolts $= 30 \times 10^6$ lb. per square inch, the additional elongation of the bolts is $(P - P_t)B$, and this together with the increased elongation of the end-rings and increased contraction of the copper end-wise must be equal to the amount by which the peripheral compression of the copper has been decreased. The latter at the commencement of the process was $H\left\{\dfrac{P_t(1 + \mu_{max}\tan\phi)}{\tan\phi - \mu_{max}} - \dfrac{F_{sc}}{2}y\right\}$, and ends at the value

$$H\left\{\frac{P(1 + \mu_{max}\tan\phi)}{\tan\phi - \mu_{max}} - \frac{F_{sc}}{2}\right\}.$$

Therefore

$$(P - P_t)\left\{B + D + E + \frac{G + E(1 + \mu_{max}.\tan\phi)}{\tan\phi - \mu_{max}}\right\}$$

$$= (P_t - P).\frac{H(1 + \mu_{max}\tan\phi)}{\tan\phi - \mu_{max}} + H\frac{F_{sc}}{2}(1 - y)$$

and finally,

$$P = P_t + \frac{H\frac{F_{sc}}{2}(1 - y)}{B + D + E + \dfrac{G + (E + H)(1 + \mu_{max}\tan\phi)}{\tan\phi - \mu_{max}}} \quad . \quad (87a)$$

where P is the final axial load resulting from all causes.

In order to calculate the tensile stress across the base of the copper V, if the total reaction R be considered which acts normally to the inclined surface, the section across which the stress reaches its maximum value may approximately be taken as coinciding with the line AB drawn at right angles to the inclined surface (Fig. 191). The bending moment on each sector is then $\frac{R}{C}\left(y + \frac{BC}{2}\tan\phi\right)$, and $R = \dfrac{P}{\sin\phi - \mu\cos\phi}$. When the appropriate values are given to t_1 and t_2 in the modulus Z of the tapering section, the quotient of the bending moment divided by the modulus gives the stress across AB. Although comparative values for different commutators may thus be obtained, the question is in reality complicated by the degree to which the peripheral compressive stress upon the copper lends support by its sideways pressure to the nose of the V.

§ 27. Mechanical design of commutators with shrink-rings.—In the case of high-speed dynamos driven by steam-turbines, where the peripheral velocity of the commutator may be as high as 7,000 to 8,000 feet per minute, it becomes necessary to employ

nickel-steel rings shrunk on to the commutator at both ends or at intervals along its length, a thin insulating band of mica intervening between the steel and the copper (Fig. 192)[1]. Three per cent. nickel steel has a breaking strength of 40 to 45 tons per square inch, and an elastic limit of about 25 tons per square inch, with an elongation of 20/25 per cent. in a length of 2 inches. The rings have a section of about 3 ins. × 2 ins., and the mica, built up in layers of segmental strips 20 or 30 mils thick, is held in place during the process of passing the rings over it by fine string which the ring heated to

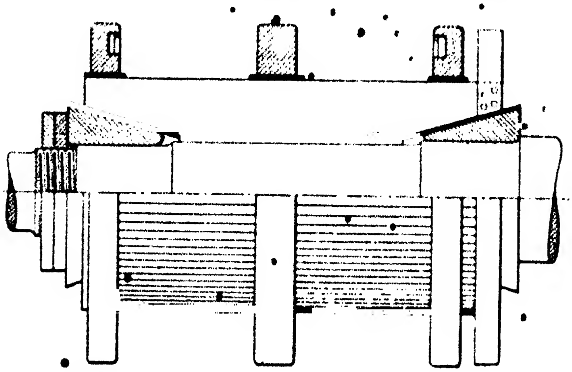


FIG. 192.—Commutator construction with shrink-rings for high speeds.

a cherry-red can burn off, or preferably by a band of fine steel wire upon which the ring beds when it cools.[2] To increase the leakage surface and still keep a firm face of mica which shall not spring away from the copper beyond the rings, it is best to make the shrink rings wider than will finally be required and to turn off some metal on their outer edges. No sticking varnish should be employed under the rings. Longitudinal grooves are sometimes scored along the commutator sectors to indicate the safe depth to which they may be worn down. The inner cone, shown separate in Fig. 192, may also be formed directly on the shaft. With high speeds above 3,000 feet per minute a perfect balance is essential, and the commutator is in some cases "seasoned" by running it at a temperature of 100° C. and at 20 per cent. excess speed to give it a permanent set.

[1] Cp. R. G. Ellis, "Steam Turbine Dynamos," *Journ. I.E.E.*, Vol. 37, pp. 322-24, where other constructions are also illustrated, and for radial commutators see Miles Walker, *Specification and Design of Dynamo-electric Machinery*, Chap. XVIII, pp. 516, 535-6.

[2] Cp. R. J. Roberts, "The Mechanical Design of Direct-current Turbo-generators," *Journ. I.E.E.*, Vol. 48, p. 138, and "Turbo-commutators," *Electr.*, Vol. 63, p. 121.

The theory of the shrink-ring commutator is simpler than that of the V-ring type, since the peripheral stresses and strains which alone come into question do not require to be brought to a common footing with any axial stresses and strains, and friction does not enter into the problem. Under the initial tension with which the rings are shrunk on, when the rings are cold and the commutator is at rest, since the respective circumferential lengths must be equal on any given boundary surface

(Natural length + forcible elongation) of the steel
= (natural length − compression) of the copper and mica.

Or if X and Y are the natural lengths of the copper and rings respectively, the sum of elongation of rings + compression of copper and mica $= X - Y = \eta_i$.

Since for the present purpose it is only the difference which is required, it is immaterial that X and Y should really be corrected to some actual dividing surface so long as the same respective diameters are retained for the copper and the steel throughout the following calculations. The consideration of the stresses in shrink-ring commutators must be based on the two laws which hold for the case of a thick cylinder subjected to a uniform external or internal pressure. If p_1 and p_2 are the intensities of the radial pressure per unit area on the inside and outside respectively of a thick cylinder of inner radius r_1 and outer radius r_2, then the intensity of the radial pressure p_r at any radius r within those limits is

$$p_r = \frac{(p_1 - p_2)\, r_1^2 r_2^2}{r^2 (r_2^2 - r_1^2)} + \frac{p_2 r_2^2 - p_1 r_1^2}{r_2^2 - r_1^2}$$

At the same time the circumferential or hoop stress s_r called out at the same radius r is

$$s_r = -\frac{(p_1 - p_2)\, r_1^2 r_2^2}{r^2 (r_2^2 - r_1^2)} + \frac{p_2 r_2^2 - p_1 r_1^2}{r_2^2 - r_1^2}$$

and is a tensile stress if it is negative.

If R_o and R_i are the outer and inner radii of the steel rings, r_o and r_i those of the commutator, let P_i and p_o be the intensities of the radial force per square inch of surface, producing tension on the inside of the steel ring and compression on the outside of the commutator, in both cases as due to the initial conditions after shrinking on. The external radial pressure on the shrink-ring being zero, the peripheral tensile stress on the inside of the ring is $s_{ei} = P_i \cdot \dfrac{R_o^2 + R_i^2}{R_o^2 - R_i^2}$ and if $1/\sigma$ = Poisson's ratio = 0·25, the elongation of the inside of the ring is $\dfrac{2\pi R_i}{E_s}\left(s_{ei} + \dfrac{1}{\sigma} \cdot P_i\right) = \dfrac{2\pi R_i}{E_s} \cdot s_{ei}\left(1 + \dfrac{1}{\sigma} \cdot \dfrac{R_o^2 - R_i^2}{R_o^2 + R_i^2}\right)$.
The internal radial pressure on the commutator being assumed to be zero, the peripheral compressive stress on the outside of the commutator in this initial stage is $s_{ci} = p_o \cdot \dfrac{r_o^2 + r_i^2}{r_o^2 - r_i^2}$, and the compression of the copper and mica is

$$\left(\frac{l_1 C}{E_c} + \frac{mC}{E_m}\right)\left(s_{ci} - \frac{1}{\sigma} \cdot p_i\right) = \left(\frac{l_1 C}{E_c} + \frac{mC}{E_m}\right) p_o \left(\frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} - \frac{1}{\sigma}\right)$$

If the insulating mica of thickness m' under the rings be assumed to be incompressible, $p_o = P_i \cdot \dfrac{R_i}{r_o} = s_{ei}\dfrac{R_o^2 - R_i^2}{R_o^2 + R_i^2} \times \dfrac{R_i}{r_o}$. The sum of the elongation of the rings and compression of the commutator expressed in terms of s_{ei} is then

$$s_{ei}\left[\frac{2\pi R_i}{E_s}\left(1 + \frac{1}{\sigma} \cdot \frac{R_o^2 - R_i^2}{R_o^2 + R_i^2}\right) + \left(\frac{l_1 C}{E_c} + \frac{mC}{E_m}\right)\frac{R_i}{r_o} \cdot \frac{R_o^2 - R_i^2}{R_o^2 + R_i^2}\left(\frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} - \frac{1}{\sigma}\right)\right]$$
$$= X - Y = \eta_i$$

The elongation and compression, although reckoned on different diameters, suffice to give the difference which we at present require. An estimate of the amount by which the ring when cold before shrinking on must be smaller than the mica backing upon which it is shrunk when the commutator is not truly compressed can, however, be obtained from the relation

$$\frac{t_1 C + mC}{\pi} + 2m'_s - D_i = \frac{\eta_i}{\pi}$$

or the difference of the two diameters. The calculated amount, although useful as a guide, must however be checked by the results of experience in view of the want of uniformity in the thickness of the mica, and other practical irregularities. Preferably, in order to pass the rings easily over it, the commutator is compressed to the full extent required by a clamp in the centre with 2 or 4 rings, and temporary steel wire bands over which the rings will pass. The difference of the diameters must then be reduced to the amount corresponding to the elongation of the steel ring only, and before the ring is cold the commutator should be released from the external pressure in order not to throw an undue stress upon the rings and cause an initial elongation above that which calculation shows to be safe. To avoid overheating of the rings they should not be raised above, say, 320° C.

By giving any required value to s_{ci}, say, 12,000 lb. per square inch, η_i is found, and assuming the same rises of temperature in the copper and steel as before,

$$\eta_h = 0{\cdot}0006 t_1 C - 0{\cdot}000175 \pi D_i$$

whence

$$s_{ci} + s_{ch} = s_{ci} \cdot \frac{\eta_i + \eta_h}{\eta_i}$$

If F_{sc} = the total radial force from the commutator sectors, the intensity of the pressure due thereto against the inner diameter of the rings is $P_{sc} = \frac{F_{sc}}{2\pi R_i l}$, where l is the joint axial length of the rings, and the tensile stress on the inner layer is

$$\frac{F_{sc}}{2\pi R_i l} \cdot \frac{R_o^2 + R_i^2}{R_o^2 - R_i^2}$$

The stress on the inner layer due to the centrifugal force of the rings themselves is as in § 15, $s_y = \frac{w_s}{g} \cdot \frac{\omega^2}{4}\left\{\left(3 + \frac{1}{\sigma}\right)R_o^2 + \left(1 - \frac{1}{\sigma}\right)R_i^2\right\}$. The total tensile stress due to centrifugal force regarded as acting independently would therefore be

$$s_{cc} = \frac{F_{sc}}{2\pi R_i l} \cdot \frac{R_o^2 + R_i^2}{R_o^2 - R_i^2} + s_y$$

But this cannot be combined with the stresses from initial tension and heating by simple addition; the tension and heating stresses are themselves affected by s_{cc}, and the latter can only be added to the former so far as they are simultaneously present. Let the speed be supposed to be raised until the *additional* elongation of the rings from centrifugal force is equal not only to the initial compressions of the copper and mica, but also to the amount by which they were compressed by the heating; the copper and mica can then, in the first place, resume their normal dimensions, and in the second place the copper becomes free to expand by the natural amount corresponding to its temperature. The compression of the copper and mica has thus at this particular speed entirely vanished. At the commencement of the supposed process of gradually increasing the speed when the commutator was at its full-load temperature, the rings were already elongated to the extent called for by the initial tension and heating; to this has now been added an amount equal to the compressions of the copper and mica, so that original elongation of rings from initial tension and heating + added elongation from centrifugal force = original elongation of rings + original compressions of copper and mica, or the final elongation of the rings is $\eta_{cf} = \eta_i + \eta_h$ as above calculated. The

elongation of the rings from centrifugal force alone is $\frac{2\pi R_i}{E_s}\left(s_{sc} + \frac{1}{\sigma} \cdot P_{sc}\right) = \eta_{ec}$. Hence, as η_{ec} rises from zero to η_{ef} with increasing speed, it is evident that $s_{ei} + s_{eh}$ is progressively decreased by the fraction η_{ec}/η_{ef}, or the actual proportion of the initial $s_{ei} + s_{eh}$ which is simultaneously present with s_{ec} is equal to $\left(1 - \frac{\eta_{ec}}{\eta_i + \eta_h}\right)$. When the commutator is at rest, $s_{ei} + s_{eh}$ is present to its full extent, and when the speed is so high that $\eta_{ec} = \eta_i + \eta_h$ it has vanished, and only s_{ec} remains. The resultant stress at any speed is therefore

$$s_e = (s_{ei} + s_{eh})\left(1 - \frac{\eta_{ec}}{\eta_i + \eta_h}\right) + s_{ec}$$

This may also be written in the form

$$s_e = s_{ei} + s_{eh} + s_{ec} \cdot \frac{s_{ef} - (s_{ei} + s_{eh})}{s_{ef}}$$

and this must fall below the elastic limit of the material, with allowance for a working factor of safety, or say below 18,000 lb. per square inch for nickel steel. Or inversely, from the above equation if 18,000 be inserted for s_e, the initial tension s_{ei} which the ring should have when it is shrunk on and is cold can be determined.

In order to give an equal stress in each of the rings and as far as possible to keep the face of the commutator parallel to the axis of the shaft under every condition, the rings must be given different sections according to their number and situation. Uniformly distributed forces from the copper have to be balanced by forces concentrated at certain points, and in order to do this, if there are three rings, one at the centre and one at either end, the former must have assigned to it $\frac{5}{8}$ths of the total area, and each of the latter $\frac{3}{16}$ths of the total area. If there are four rings, one at either end and two at equal distances apart, the latter must each have $\frac{11}{30}$ths, and the end rings each $\frac{4}{30}$ths of the total area.[1]

The same value for the resultant tensile stress s_e on the inner diameter of the ring can also be immediately reached by considering the actual resultant intensities of radial pressure, P_e and p_e, on the inside of the steel rings and outside of the commutator respectively. The resultant stress is $s_e = P_e \cdot \frac{R_o^2 + R_i^2}{R_o^2 - R_i^2} + s_y$, and the resultant elongation of the steel is

$$\frac{2\pi R_i}{E_s}\left(s_e + \frac{1}{\sigma} \cdot P_e\right) = \frac{2\pi R_i}{E_s}\left(s_e + \frac{1}{\sigma} \cdot s_e \frac{R_o^2 - R_i^2}{R_o^2 + R_i^2} - \frac{1}{\sigma} s_y \cdot \frac{R_o^2 - R_i^2}{R_o^2 + R_i^2}\right)$$

The resultant compression of the copper is $\left(\frac{t_1 C}{E_c} + \frac{mC}{E_m}\right) p_e \left(\frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} - \frac{1}{\sigma}\right)$ and $p_e = \left(P_e - \frac{F_{sc}}{2\pi R_i l}\right)\frac{R_i}{r_o}$, i.e. p_e, to which corresponds the resultant compressive stress upon the copper, is equal to P_e less the outward radial pressure from the centrifugal force of the sectors, both increased in the ratio of R_i to r_o, since p_e and P_i act at different diameters. Hence the resultant compression on the copper and mica by substitution becomes

$$\left(\frac{t_1 C}{E_c} + \frac{mC}{E_m}\right)\frac{R_i}{r_o} \cdot \frac{R_o^2 - R_i^2}{R_o^2 + R_i^2}\left(\frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} - \frac{1}{\sigma}\right)(s_e - s_{ec})$$

[1] R. Livingstone, *The Mechanical Design and Construction of Commutators*, p. 36.

In terms of the original data, calculation is economised by writing

$$\frac{2\pi R_i}{E_s} = G \qquad \frac{R_o^2 - R_i^2}{R_o^2 + R_i^2} = J \qquad \left(\frac{t_1 C}{E_c} + \frac{mC}{E_m}\right)\frac{R_i}{r_o}\cdot\left(\frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} - \frac{1}{\sigma}\right) = H$$

The sum of the above resultant elongation and compression must be equal to $\eta_t + \eta_h$.

Therefore $s_e\left(G + GJ\frac{1}{\sigma}\right) - s_y GJ\cdot\frac{1}{\sigma} + s_e\cdot JH - s_{ec}\, JH = \eta_t + \eta_h$

$$s_e\left(G + GJ\cdot\frac{1}{\sigma} + JH\right) = \eta_t + \eta_h + s_{ec}\, JH + s_y GJ\cdot\frac{1}{\sigma}$$

$$s_e = s_{et} + \frac{\eta_h + s_{ec}\, JH + s_y\, GJ\cdot\frac{1}{\sigma}}{G + GJ\frac{1}{\sigma} + JH}$$

Since $s_{ec} = \frac{F_{sc}}{2\pi R_i l}\cdot\frac{1}{J} + s_y$

$$s_e = s_{et} + \frac{\eta_h + \frac{F_{sc}}{2\pi R_i l}\cdot H + s_y\left(JH + GJ\frac{1}{\sigma}\right)}{G + GJ\frac{1}{\sigma} + JH} \qquad (87b)$$

The analogy to eq. (87a) is evident.

§ 28. Example of shrink-ring commutator calculation. As an example of the calculation of s_e, take the case of a commutator 18 in. diameter by 22 in. long, running at 1,500 revs. per min. with 150 sectors 3 in. deep, and mica strips 0·040 in. thick. Three nickel steel rings, each 2⅞ in. deep, the total width of the three being 6 in., are shrunk on wire and mica bands ⅛ in. thick with an initial tensile stress on their inner circumference $s_{et} = 10{,}000$ lb. per sq. inch. The moduli of elasticity of nickel steel, copper and mica, will be taken as $29{\cdot}7 \times 10^6$, 16×10^6, and $0{\cdot}75 \times 10^6$ respectively.

Assuming the copper to rise during working 34·5° C. and the steel rings to rise 14·5° C.,

$$\eta_h = 0{\cdot}0006 \times 0{\cdot}337 \times 150 \times 0{\cdot}000175 \times 2\pi \times 9\tfrac{1}{8}$$
$$= 0{\cdot}02 \text{ in.}$$

The total radial centrifugal force from the commutator sectors per inch length is when the commutator is new,

$$F_{sc} = 175 \times 1500^2\,(9^3 - 6^3)\,22 \times 10^{-7} \text{ lb.}$$
$$= 445{,}000 \text{ lb.}$$

and

$$\frac{F_{sc}}{2\pi R_i l} = \frac{445{,}000}{2\pi \times 9\tfrac{1}{8} \times 6} = 1{,}290 \text{ lb. per sq. inch.}$$

The stress on the inner layer of the steel ring due to its own centrifugal force is

$$s_y = 0{\cdot}00029 N^2\,(3{\cdot}25 + 0{\cdot}79 \times 0{\cdot}76^2)\,10^{-6} = 2{,}400 \text{ lb. per sq. inch.}$$

$$H = \left(\frac{0{\cdot}337 \times 150}{16 \times 10^6} + \frac{0{\cdot}04 \times 150}{0{\cdot}75 \times 10^6}\right)\frac{9\tfrac{1}{8}}{9}\left(\frac{9^2 + 6^2}{9^2 - 6^2} - \frac{1}{4}\right) = 26{\cdot}6 \times 10^{-6}$$

$$J = \frac{12^2 - (9\tfrac{1}{8})^2}{12^2 + (9\tfrac{1}{8})^2} = 0{\cdot}2685 \qquad JH = 7{\cdot}15 \times 10^{-6}$$

$$G = \frac{2\pi \times 9{\cdot}125}{29{\cdot}7 \times 10^6} = 1{\cdot}93 \times 10^{-6} \qquad GJ\frac{1}{\sigma} = 0{\cdot}13 \times 10^{-6}$$

$$JH + GJ\frac{1}{\sigma} = 7{\cdot}28 \times 10^{-6}$$

$$s_e = 10{,}000 + \frac{0{\cdot}02 + (1290 \times 26{\cdot}6 + 2400 \times 7{\cdot}28) \times 10^{-6}}{(1{\cdot}93 + 7{\cdot}28) \times 10^{-6}}$$
$$= 17{,}800 \text{ lb. per sq. inch.}$$

§ 29. Compression on mica strips in high-speed commutators.—It is a matter of importance to retain sufficient circumferential compressive stress on the mica-strips between the sectors to prevent any likelihood of flakes flying out when the machine is running at full speed, and this must be the case when the commutator is worn down to its lowest permissible radius. A method by which the calculation of the circumferential compressive stress could be approached was suggested by Mr. R. J. Roberts in his paper[1] on "The Mechanical Design of Direct-current Generators," and following on his suggestion the writer has put forward[2] an approximate practical formula which, if not a completely accurate solution of a very complex problem, enables comparative figures to be obtained for the purpose of checking new designs.

In the following let r be given the value of the radius of the commutator when worn down as low as is safe. Then, by the laws governing stress and pressure in thick cylinders, as given in § 27, if p_e = the resultant intensity of the radial pressure on the outside of the commutator underneath a steel

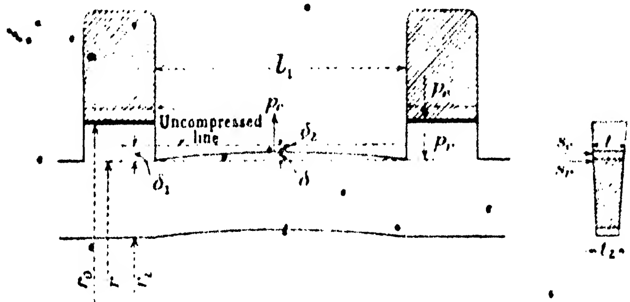


FIG. 193. Portion of commutator between two shrink-rings.

shrink-ring—*i.e.* at radius r_0, after all initial heating and centrifugal stresses have been taken into account, the intensity of the radial pressure under a shrink-ring and within the sectors at the radius r—*i.e.* level with the smallest working radius of the commutator—is

$$p_r = p_e \cdot \frac{r^2 r_0^2 - r_0^2 r_i^2}{r^2(r_0^2 - r_i^2)} \qquad (88)$$

and this calls out at the same depth a circumferential compressive stress

$$s_r = p_r \cdot \frac{r^2 + r_i^2}{r^2 - r_i^2}$$

Now, between a pair of shrink-rings, although the outward pull from centrifugal force per unit length along the sectors is always uniformly distributed throughout the span, yet so long as there is any circumferential compressive stress s_r under the rings, this stress and the radial pressure corresponding thereto must gradually decrease as the centre of the span is approached. For, the sectors being bowed outwards between the rings, the circumferential compressive stress becomes relieved to a greater or less degree depending on the span l_1, and correspondingly the portion of the radial force on each sector which is due to the compressive stress becomes reduced per unit length of the span as we approach the centre. Each sector between the two shrink-rings, therefore, resembles a beam subjected to a load which has its minimum value per unit of length at the centre where the deflection

[1] *Journ. I.E.E.*, Vol. 48, p. 140.

[2] "Note on High-speed Commutator Construction," by C. C. Hawkins, *Electr.*, 26 March, 1915, from which the following is quoted.

greatest, and its maximum value per unit of length at the ends, the decrease of load being proportional to the deflection.

If the load per unit length of the beam was reduced to zero at the centre, the line of deflection would be a parabolic curve. But when the machine is at work, although the circumferential compressive stress at the centre of the span may become zero, the outward radial force can never become zero, since there always remains the radial pull from centrifugal force per unit length of a sector. It may, however, as an approximation be assumed that the same parabolic law holds even when the radial force at the centre retains some value; the load per unit length is then given by the law,

$$w = w_{min} + Kx^2$$

where w_{min} is the minimum load per unit length, x is distance reckoned from the centre of the beam, and K is a constant. This reaches its maximum at the ends where $x = \frac{l_1}{2}$, *i.e.*

$$w_{max} = w_{min} + \frac{Kl_1^2}{4}$$

l_1 being throughout reckoned as the span between the edges of a pair of adjoining rings.

The maximum deflection[1] of the beam is then

$$\delta = \frac{1}{EI}\left(\frac{5}{384}w_{min} + \frac{56}{23{,}040} \cdot \frac{Kl_1^2}{4}\right)l_1^4$$

On unit length under a shrink-ring the total radial pressure at the radius r is $2\pi r$ times the intensity at that radius, and the proportion of this falling on a single sector is $t/2\pi r$, where t is the thickness of the sector at the radius r.

The maximum load per unit length at the ends of the sector regarded as a beam is, therefore,

$$w_{max} = tp_r$$

Similarly, if p_c is the intensity per square inch of the final radial force at the centre of the span

$$w_{min} = tp_c$$

Thence, $$\frac{Kl_1^2}{4} = w_{max} - w_{min} = t(p_r - p_c).$$

and the maximum deflection[2] is

$$\delta = \frac{t}{E_cI_c}\left\{\frac{5}{384}p_c + \frac{56}{23{,}040}(p_r - p_c)\right\}l_1^4.$$

$$= \frac{t(p_r + 4{\cdot}36p_c)}{411E_cI_c} \cdot l_1^4$$

The moment of inertia of the tapering section of a commutator bar is

$$I_c = \frac{(r - r_1)^3}{36} \times \frac{t^2 + 4tt_2 + t_2^2}{t + t_2}$$

t_2 being the minimum thickness on the inside. This moment decreases so rapidly with a reduction of radial depth that it is essential to consider the condition when the wear is greatest. The deflection is, therefore

$$\delta = \frac{36t(p_r + 4{\cdot}36p_c)}{411E_c(r - r_1)^3} \times \frac{t + t_2}{t^2 + 4tt_2 + t_2^2} \times l_1^4$$

[1] The solution is given by Mr. R. J. Roberts, *loc. cit.* p. 153 (where, however, a bracket has become misplaced in the final result), and it need not here be repeated.

[2] By giving different values to p_c/p_r from 0 to 1, the factor $411/(1 + 4{\cdot}36\ p_c/p_r)$ assumes values from 411 to 76·8, as given in curve I of Mr. Roberts' paper. The latter value is that which would hold in the imaginary case of the bars remaining perfectly straight between the rings and suffering no deformation whatever.

As already indicated, part of the outward radial force p_c when the machine is at work is due to the uniformly distributed centrifugal force; this portion of p_c is

$$p_f = 45{\cdot}5\,(r^2 - r_i^2)\,N^2 \times \frac{t + t_2}{2t} \times 10^{-7}\ \text{lb.}$$

where N = revs. per min. The remainder is the outward push of the sectors due to the circumferential compressive stress s_c at the centre of the span, and is

$$p_x = s_c \cdot \frac{r^2 - r_i^2}{r^2 + r_i^2}$$

In inch and lb. units, therefore,

$$p_c = p_f + p_x$$

$$= 45{\cdot}5\ N^2(r^2 - r_i^2)\,\frac{t + t_2}{2t} \times 10^{-7} + p_x$$

So long as there is any compressive stress s_c at the centre, the circle must have there contracted by the amount

$$\left(\frac{tC}{E_c} + \frac{mC}{E_m}\right)\left(s_c - \frac{1}{\sigma}\,p_x\right),$$

where E_c and E_m are the moduli of elasticity of the copper and mica respectively, and $1/\sigma$ is Poisson's ratio. The inaccuracy in the assumption that the latter ratio is the same for mica as for a metal may be here ignored. The contraction of the circumference implies a reduction in the radius to the amount $1/2\pi$ of the contraction, and this is equivalent to a deflection of the sector δ_2 as compared with its straight and unstressed condition when copper and mica are in close contact, but without any compression.

Now, the basis of the previous argument is that s_c still retains some value, and the reduction in the radius of the commutator at the centre of the span is thus

$$\delta_2 = \frac{1}{2\pi}\left(\frac{tC}{E_c} + \frac{mC}{E_m}\right)\left(s_c - \frac{1}{\sigma}\,p_x\right)$$

$$= \frac{1}{2\pi}\left(\frac{tC}{E_c} + \frac{mC}{E_m}\right)p_x\left(\frac{r^2 + r_i^2}{r^2 - r_i^2} - \frac{1}{\sigma}\right) \qquad (89)$$

If the commutator sectors were bowed outwards by this amount, at the centre they would reach the original state of close contact without compression. The sum $\delta + \delta_2$ must therefore be equal to the reduction in radius beneath the shrink-rings as effected by the actual conditions of stress under all the forces at that spot as compared with the initial unstressed condition. That is

$$\delta + \delta_2 = \delta_1,$$

or

$$\delta = \delta_1 - \delta_2.$$

The reduction in radius under the shrink-rings is

$$\delta_1 = \frac{1}{2\pi}\left(\frac{tC}{E_c} + \frac{mC}{E_m}\right)p_r\left(\frac{r^2 + r_i^2}{r^2 - r_i^2} - \frac{1}{\sigma}\right) \qquad (90)$$

Hence, finally,

$$\frac{36t(p_r + 4{\cdot}36p_c)}{411E_c(r - r_i)^3} \times \frac{t + t_2}{t^2 + 4tt_2 + t_2^2} \times l_1^4$$

$$= \frac{1}{2\pi}\left(\frac{tC}{E_c} + \frac{mC}{E_m}\right)\left(\frac{r^2 + r_i^2}{r^2 - r_i^2} - \frac{1}{\sigma}\right)(p_r - p_x)$$

$$\frac{p_r + 4{\cdot}36(p_x + p_f)}{(r - r_i)^3} \times \frac{t + t_2}{t^2 + 4tt_2 + t_2^2} \times l_1^4$$

$$= 1{\cdot}82C\left(1 + \frac{m}{t}\cdot\frac{E_c}{E_m}\right)\left(\frac{r^2 + r_i^2}{r^2 - r_i^2} - \frac{1}{\sigma}\right)(p_r - p_x) \qquad (91)$$

By inserting the values of p_r, p_f and of the known dimensions of the commutator, the value of p_z is found, and thence the value of $s_c = p_z \frac{r^2 + r_1^2}{r^2 - r_1^2}$. Or, by assuming s_c = say, 1,000 lb. per square inch, the permissible length of span l_1 can be found, and the number of rings can be determined for the given total length of commutator and joint width of the rings. The radial depth of the commutator must be known or assumed in either case, since upon this will also depend the value of the centrifugal force per unit length of the sector.

It will be seen that when the machine is at rest the radial force per square inch of surface at the centre can never with a finite length of span rise to equality with p_r, and that the sectors must be bowed outwards. If the speed rises to such an amount that the sectors at the centre of the span reach their initial radius when unstressed—*i.e.* if $\delta = \delta_1$—there is no compressive stress left at the centre, and δ_2 is zero. An increase of speed above this would lead to the sectors opening out sideways, and the compressive stress under the shrink rings would rapidly disappear towards the centre; the sector would then have a uniformly distributed load on it over a certain portion of its length at the centre of the span, and to this condition the parabolic equation no longer applies.

The bending moment on the sector as a beam of span l_1 between the rings is

$$w_{min}\frac{x^2}{2} + K\frac{x^4}{12} - w_{min}\frac{l_1^2}{8} - K\frac{l_1^4}{192}$$

This is a maximum at the centre, where $x = \frac{l_1}{2}$, and is there

$$\text{B.M.} = \frac{tl_1^2(p_r + 5p_c)}{48}$$

Taking the modulus Z as $(r - r_1)^2 t/6$, the bending stress on the outer edge of the copper is

$$\frac{l_1^2(p_r + 5p_c)}{8(r - r_1)^2}$$

When the commutator of § 28 has been worn down 1 in. in depth over a length of 16 in., the total centrifugal force is

$$F_{sc} = 175 \times 1500^2 [6 \times (9^2 - 6^2) + 16 \times (8^2 - 6^2)] \times 10^{-7}$$
$$= 307{,}700 \text{ lb.},$$

and

$$\frac{F_{sc}}{2\pi R_i l} = \frac{307{,}700}{2\pi \times 9\tfrac{1}{8} \times 6} = 890 \text{ lb. per sq. in.}$$

In these circumstances

$$s_c = 10{,}000 + \frac{0.02 + (890 \times 26.6 + 2{,}400 \times 7.28) \times 10^{-6}}{(1.93 + 7.28) \times 10^{-6}}$$
$$= 16{,}570 \text{ lb. per sq. inch.}$$

Thence

$$P_a = (s_a - s_y)\frac{R_o^2 - R_i^2}{R_o^2 + R_i^2} = (16{,}570 - 2{,}400) \times 0.2685$$
$$= 3{,}800 \text{ lb. per sq. in.,}$$

and

$$p_a = \left(P_a - \frac{F_{sc}}{2\pi R_i l}\right)\frac{R_i}{r_o} = (3800 - 890)\frac{9.125}{9} = 2{,}950 \text{ lb. per sq. in.}$$

At the radius of 8 in., by equation (88)

$$p_r = 2{,}950\frac{8^2 \times 9^2 - 9^2 \times 6^2}{8^2(9^2 - 6^2)} = 2{,}325 \text{ lb. per sq. in.}$$

$$p_f = 45.5 \times 1{,}500^2(8^2 - 6^2) \times \frac{0.295 + 0.211}{2 \times 0.295} \times 10^{-7} = 246 \text{ lb. per sq. in.}$$

$$\frac{t^2 + 4tt_2 + t_2^2}{t + t_2} = 0.75,$$

By equation (91)

$$\frac{2,325 + 4{\cdot}36p_x + 1,072}{(8-6)^2} \times \frac{1}{0{\cdot}75} \times 8^4$$

$$= 1{\cdot}82 \times 150\left(1 + \frac{0{\cdot}04}{0{\cdot}293} \times \frac{16}{0{\cdot}75}\right)\left(\frac{8^2+6^2}{8^2-6^2} - 0{\cdot}25\right)(2,325 - p_x)$$

whence

$$p_x = 908.$$

Finally, therefore,

$$s_0 = p_x \cdot \frac{r^2 + r_i^2}{r^2 - r_i^2} = 908 \times \frac{100}{28} = 3,240 \text{ lb per sq. in.}$$

which is on the safe side.

It is of interest to see what are then the actual deflections. By equation (90)

$$\delta_1 = 0{\cdot}0133 \text{ in.}$$

and by equation (89)

$$\delta_2 = \delta_1 \times \frac{p_x}{p} = 0{\cdot}0133 \times \frac{908}{2,325} = 0{\cdot}0052 \text{ in.}$$

so that $\delta = \delta_1 - \delta_2 = 0{\cdot}0081$ in. If the span is increased to 11 in. there is no compressive stress left at the centre, so that two steel-rings of the same total width would be entirely impracticable.

§ 30. Insulation of commutator sectors, etc. For the insulation between the separate sectors of the commutator up to 8 inches long *mica* is now almost universally used in plates about $\frac{1}{40}$th to $\frac{1}{25}$th of an inch thick (0·025″ to 0·040″; 1 cubic inch weighs $\frac{1}{10}$th of a pound), and these plates should project through at the inner end and between the lugs by some $\frac{3}{32}$″. The difference of potential between neighbouring sectors may be small, inasmuch as it is only the potential generated within the limits of one section of the winding. When, however, the sectors pass under the brushes, sparking is liable to occur, and a small arc is formed which bridges the insulation. Under this action almost every insulating substance, except mica, is apt to char and become conductive, and even with mica, if the thickness be less than $\frac{1}{40}$th of an inch, small particles of copper may occasionally bridge across adjoining sectors. In machines giving over 1,000 volts the thickness may be increased to $\frac{1}{20}$th of an inch (0·050″). Mica can be easily split into flat plates of very uniform thickness, and in commutators green or black-spotted mica is largely employed. It is extremely important, especially with carbon brushes, that the rate of wear of the mica should be the same as that of the copper sectors; it should therefore be specially soft, as in the amber and soft green kinds, even Indian mica being almost too hard. To avoid "high micas" due to greater wear of the copper and consequent sparking, the mica is often gouged out or milled down $\frac{1}{32}$″ or $\frac{1}{16}$″ below the level of the sectors—an operation that requires to be repeated at intervals.[1] Plates more than 8 to 10 inches long occur in nature with such comparative rarity that

[1] For hints on recessing commutators, see E. Murgatroyd, "Mica and Commutation Troubles," *Electr.*, Vol. 79, p. 545.

their price is prohibitive, and it becomes an economical necessity to employ either built-up plates or the artificial forms, micanite and megomit. When the latter are used for commutators, soft mica must be chosen for the component material, and there must be no tendency for the cement to ooze out when the material becomes heated; hence powerful pressure is employed in order to remove all excess of cement during the process of manufacture. The latter condition applies with equal force to end-rings and sleeves made of micanite or megomit, and when they are used it is advisable, after the commutator has been assembled, to heat it to a temperature of about 150° to 200° C. in a stove or by means of gas jets, and when at this high temperature to compress the whole structure by tightening up the fastening screws or bolts, at the same time carefully hammering the sectors so as to bed them well into the micanite until a compact and solid mass results. The taper end-rings are preferably not formed of two cones or of a coned ring and flat band cemented together, but are made in one piece by bending small strips of mica round at a sharp angle or by squeezing a sheet when hot in a mould. Thorough union between the insulating end-rings and the cylindrical sleeve must be ensured by allowing the latter to project slightly so that the one edge is crushed into the other. In machines for 250 volts and upwards the end-rings should project outwards beyond the edges of the sectors to prevent sparking to the metal case, and on the ledge of some ½" to ¾" width may be wound a layer of string to prevent flaking away of the mica (Figs. 184 and 185). The thickness of the insulation of the end-rings and on the body of the sleeve should rise from 0·06" for 100 volts and 0·1" for 250 volts up to 0·125" for 500 and 0·15" for 1,000 volts. In closed-circuit armatures the maximum volts per commutator sector, must not exceed, say, 40 to 45; otherwise an arc may be established between two adjacent sectors and short-circuit the winding between them. Except, however, in high-voltage machines or high-speed turbo-dynamos, the *average* volts per sector or $2p \cdot V_b/C$ seldom exceed 20 volts, since for other reasons the number of turns per sector must be limited. A large number of sectors implies a large diameter of commutator unless their width be very small, and mechanical considerations require that their width should not be less than $\frac{1}{16}$" at the top, tapering downwards say to $\frac{1}{32}$" at the bottom. Even then the top of the sector cannot be sawn to receive the lug, so that the latter must be riveted on to a flat recess milled against the side of each sector, or it becomes necessary to increase the depth of the sector until it measures 0·190" at the top, and perhaps to turn it down along the brush working surface. In any case the top of the commutator lugs should fall slightly below the level of the armature surface.

After it has been built up the insulation of the commutator is

tested with a high alternating difference of potential, all the sectors being temporarily thrown into contact by binding a wire round their exterior. It is then pressed on to its seating on the shaft either before the armature bars are inserted into the slots in a bar-wound armature or after the coils are in place, and in order to complete the armature winding it only remains to connect up each of the sectors by soldering to the loops or coils, junctions being made between the end of one section and the beginning of the next in succession. If the armature wires are brought down immediately into the commutator sectors, the soldering may be effected in one process by dipping the armature vertically into an annular vessel containing solder, and allowing the solder, melted by a gas jet, to soak into the junctions; but such a process is only possible with small armatures, and detracts from the neat appearance of the wires near to the commutator unless these are subsequently bound over with a string or other band.

The surface of the commutator is turned true after it is in place on the armature shaft. With carbon brushes, in order to secure the perfect smoothness so essential to their successful use, the commutator should be ground while slowly revolving by a small rapidly driven emery wheel.

It is evident that the commutator, speaking generally, introduces many difficulties, mechanical and electrical, and on this account it becomes practically impossible to build closed-circuit armatures to give more than 4,000 or at the outside 5,000 volts. A limitation is thereby set to their employment for the transmission of energy over very great distances (unless a number are placed in series, Chapter XXIII, § 2), and for this purpose especially the continuous-current dynamo must yield place to the alternator.

§ 31. Binding keys and binding wire.—When the winding of the armature is completed the next step is to secure the active conductors and their end-connectors in place; in order to counteract the tendency for the active wires to be thrown outwards from the core by centrifugal force, they are firmly held either by *wooden keys* or *wedges* (beechwood or hornbeam, boiled in paraffin wax or linseed oil) driven into sloping grooves at the tops of the slots (Figs. 150 and 151) or by bands of *binding wire* wound circumferentially round the armature.

The former are the more commonly used, especially on large armatures 4 ft. or more in diameter, where even with steel wire it becomes difficult to secure sufficient strength and also grip upon the armature. The keys have the disadvantage of increasing the inductance of the armature coils and of partially obstructing the radiation of heat from the conductors; but on the other hand, their use lessens the eddy-currents in solid massive bars, since the latter are buried deeper within the slots where there is less variation of

the flux-density. A hornbeam wedge $\frac{1}{4}''$ thick with sides making an angle of 70° with its base and 0·470" wide at the top has a breaking strength of about 550 lb. per inch of length ; above this the edges are split off, and the remainder forced out. But if the grain runs transversely across the slot, its limiting strength is raised to 1,200 lb. per inch length, above which it yields by bending of the fibres into curved bows across the slot. If $\frac{3}{16}''$ thick with slot 0·460 wide, and an angle of 60°, the figures become 500 and 1,000 respectively. A hard black fibre wedge of the same size has its edges sheared off at 3,600 lb. per inch length, and is therefore considerably stronger. The air-duct spaces should not be closed by the keys more than is necessary. With very high peripheral speeds, as in field-magnet rotors driven by steam-turbines, metal keys are also used of bronze, in order to obtain the necessary strength.

Considering, first, the centrifugal force within the limits of the armature core, let a = the cross-sectional area of an active conductor at right angles to its length in square inches, and let z = the number of conductors per slot ; the total area of copper in the conductors in a slot in a plane perpendicular to the axis of rotation is thus za, and taking any distance l inches along the armature core, the cubic inches of copper per slot in the considered length is zal. The corresponding weight in lb., allowing a little for the weight of the insulation surrounding the net cross-section of copper, is 0·33 zal, and by § 5 the centrifugal force per slot along the length l is $\frac{W}{g} \cdot \omega^2 R_m = \frac{0{\cdot}33 za \,.\, l}{32{\cdot}2} \left(\frac{2\pi N}{60}\right)^2 R_m$ lb., where R_m is the mean radius to the centre of the slot in feet ; or if d = the corresponding diameter in inches = $2 R_m \times 12$, it is

$$= 15\, za \,.\, l \,.\, \pi d N^2 \times 10^{-7} \text{ lb. per slot} \qquad (92)$$

The materials used for binding wire are phosphor bronze or silicium bronze, hard brass, Eureka wire, and steel, the requirements being great tensile strength with but little expansion under the heat of the soldering iron. On the armature core the wire should be non-magnetic ; and lastly, in the case of toothed armatures it is of great importance in order to avoid eddy-currents due to the variation of the flux-density opposite to teeth and slots respectively that it should possess high electrical resistivity. The ultimate breaking strengths of brass and phosphor bronze wires are about 70,000 and 90,000 lb. per square inch respectively, but they are good conductors, and are therefore not so suitable for toothed armatures as Eureka and steel wires. The breaking strength of Eureka wire is about 75,000 lb. per square inch, and its resistance per 1,000 feet and per square inch of area is 0·237 ohm, or 29 times that of copper. It is therefore suitable for small machines, but for larger armatures and in all cases of high stress preference must be given

to steel, of which the ultimate breaking strength is 200,000, and may even reach to 300,000 lb. per square inch. On the core, instead of ordinary pianoforte wire must be substituted special non-magnetic steel alloys, which can be obtained with a breaking strength of 200,000 lb. per square inch, and with a high resistance of 0·386 ohm per 1,000 feet and per square inch of area, or 40 times that of copper. In order to render it easy to solder and to prevent rusting, the steel wire should be nickel-plated. In all cases the stretching must be inappreciable if the wires are not to work loose, so that a large factor of safety of at least 8 is necessary in calculating the permissible tensile stress, especially seeing that the initial tension under which the wire is put on adds an indeterminate amount to the stress.

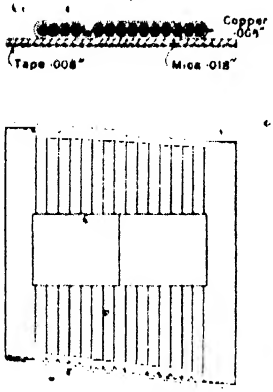


FIG. 194.—Clip for binding-wire band.

The diameter of the wire varies from 0·020" (No. 25 S.W.G.) for small armatures, to 0·056" (No. 17 S.W.G.) in large armatures. A band of thin tape covered with strips of thin mica slightly wider than the wires is fastened round the armature, and on this the wire is wound under considerable tension to form a belt about ¾" to 1¼" wide. If wider, eddy-currents are liable to cause heating of the bands. At intervals of about 8 to 10 inches along the circumferential length of the band are placed small strips of sheet copper about ⅜" wide; after the band has been wound on over these, the ends of the strips are turned over to form clips round the wire, and the whole is soldered together (Fig. 194). Such bands will be placed at intervals of about 3" along the entire length of the armature (cp. Fig. 198).

If we take No. 19 S.W.G. (= 0·042") and allow 0·026" for the mica on its tape backing, and $2 \times 0{\cdot}006'' = 0{\cdot}012''$ for the folded copper strip, a total addition of 0·08" is made to the radius of the armature by the binding wire, and the mechanical clearance must be reckoned from this over-all radius. In some cases, when the core of a toothed armature is being built up, batches of discs of slightly smaller diameter are interposed at intervals along the length; by this means shallow grooves are formed round the periphery just deep enough to receive the binding wire, so that the finished surface becomes practically flush.

The total centrifugal force of the conductors summed up all round the core for S slots is

$$F_c = 15\, Z \,.\, a_t\, l\pi d N^2 \times 10^{-7} \text{ lb.}$$

and per unit angle in circular measure is

$$f_c = \frac{F_c}{2\pi} = 7{\cdot}5\, Z \,.\, a \,.\, l \,.\, d \,.\, N^2 \times 10^{-7} \text{ lb.} \qquad (93)$$

which is by § 5 equal to the tension acting across the single section of the binding wires in the length l. If the cross-section of the binding-wires in the length l be $ab = \frac{\pi\delta^2}{4} \,.\, x$, where x is the number of wires each of diameter δ, the stress on the material is $\frac{f_c}{ab}$, and this must be $\leqq f_t$ where f_t is the safe permissible limit of tensile stress with the particular material employed. Thence x can be determined, and the requisite wires must be distributed among a suitable number of bands approximately equally spaced.

At the ends of the barrel-wound armature the slanting direction of the end-connectors implies that the mass of copper per unit of axial length is greater than along the core; a transverse section across the end-connectors in a plane at right angles to the axis cuts through $Z \,.\, a \,.\, \frac{\sqrt{m_1^2 + l^2}}{l}$ square inches of copper, where m_1 = the half-pitch at the mean circumference in the centre of a slot, and l is the axial length of the sloping portion of the end-connections, as in § 22. It is therefore only necessary in the above equation for the centrifugal force to substitute $a \,.\, \frac{\sqrt{m_1^2 + l^2}}{l}$ for a.

In toothed armatures of low peripheral speed and short length, the bars being tightly wedged in the slots and largely held by friction, bands and keys on the core may often be omitted, only the end-connectors requiring to be held by bands. At the ends of barrel armatures iron supporting rings can be cast on the hub upon which the end bands may firmly compress the bars (cp. Figs. 123 and 131), and in general, so far as safety permits, the bands should be wound outside the limits of the pole-pieces, so as to reduce the eddy-currents in them. Beyond the pole-faces thin bands of solid steel about 1¼ inches wide may alternatively be employed, these are fastened by a junction drawn together by a right- and left-handed screw (Fig. 181), or by a cotter, so that they can be easily removed and replaced in the case of repairs being necessary to the winding. The junction pieces must be arranged so as to balance one another on the armature. Or wire bands made up in segments fastened together by coned pins may be used, which can be taken off and replaced when occasion requires.

With very high peripheral speeds, as in machines driven by steam turbines, the retention of the mass of the end-windings in place is in practice a serious problem and calls for special means.

Involute end-connectors have been employed[1], and end bells of rolled phosphor-bronze, manganese bronze or nickel-steel[2], firmly clamping down the end-windings and reducing the risk of their shifting with consequent loss of balance. The mechanical strength of the end cylinders then fixes the maximum permissible speed of continuous-current dynamos.[3] If F_1 and F_2 are the total centrifugal forces of the retaining cylinder and of the copper end-winding, the stress in the cylinder is $\frac{F_1 + F_2}{\pi \times 2ab}$ where ab is the cross-section of one side. Taking 23,000 lb. per square inch as the ultimate tensile strength of good gun-metal or bronze, and 3,850 lb. as the safe working stress, the cylinder of gun-metal cannot itself be safely run at a higher peripheral speed than about 10,850 feet per minute, and when the additional load of the end-connections is also thrown on this must be largely reduced to 7,000 or 8,000 feet per minute, according to the depth of copper. With phosphor and manganese bronze of ultimate strength of 50,000 to 60,000 lb. per square inch, and elastic limit 30,000 lb., if 9,000 lb. per square inch be allowed as the safe working stress, the maximum safe speed of the cylinder alone is not more than about 16,500 feet per minute, and for higher speeds, nickel steel of 50,000 lb. per square inch elastic limit must be employed. The maximum peripheral speed of the continuous-current turbo-dynamo, with its projecting end-windings, cannot therefore be set much higher than about 16,750 feet per minute as given by, *e.g.*, an armature of 32″ diameter revolving at 2,000 revs. per minute.

§ 32. Insulation resistance of armature.—After the coils have been pressed into their final positions, but before they are soldered to the commutator[4], their insulation to "earth," *i.e.* to the iron core, is tested with a high pressure, and again after the completion and drying of the winding, the finished armature is subjected to a final test for *insulation resistance* between the winding and commutator as a whole and the iron core. A high insulation resistance of many megohms is not required, but a capability to withstand a high voltage without the insulation being punctured. In all cases when any fibrous material is employed in the insulation of any part of the electrical circuit of a dynamo, absorbed moisture will lower its insulation resistance, which may then be much improved by baking in a drying stove; the resistance will, however, once again fall when the dynamo is exposed to a damp atmosphere. The surface leakage is in especial entirely dependent upon the state of the

[1] See Miles Walker, *Specification and Design of Dynamo-electric Machinery*, pp. 632-7.
[2] A. G. Ellis, "Steam Turbine Dynamos," *Journ. I.E.E.*, Vol. 37, p. 321.
[3] Dr. R. Pohl, *Journ. I.E.E.*, Vol. 40, p. 240.
[4] For a method of testing the connections when the coils are coupled to the commutator, cp. O. Steels, *Rév. Gén. d'Electr.*, vol. 9, p. 875.

armature, brush gear, and of the exposed ends of the commutator. Prolonged baking at high temperatures only ages the insulating material prematurely, and a very high resistance of many megohms thus obtained may be easily broken down by a high voltage, while a comparatively low resistance may be perfectly sound to resist puncture. Surface moisture deposited by condensation from the air, as at night when the machinery is standing and a rapid fall of temperature occurs, is usually quite harmless, although when new machinery is started up for the first time after erection it may require a moderate amount of drying out if the voltage is high. It is evident, then, that a high-voltage puncturing test is far more valuable than any measurement of the actual insulation resistance. Under a difference of potential proportioned to its working pressure the insulation must not give way, even when warm. Thus if the armature is to give 100 volts, it should be tested when warm by the application of an alternating difference of potential of 1,200 R.M.S. volts between winding and shaft; or if its working pressure is 500 volts, by the application of 2,000 R.M.S. volts for one minute. It must be borne in mind that not only does the alternating difference of potential heat the dielectric more than an equivalent continuous potential, but from the shape of the alternating E.M.F. curve its maximum is about 1½ times its virtual value; hence such a test must not be pushed to such an extreme as to weaken permanently the insulation at some internal spot. The time of application should not exceed one minute, to avoid local heating of the dielectric and breakdown at the overheated spot. In all cases the pressure must be gradually raised to the required value to avoid abrupt discharge between contiguous parts of the electric circuit.[1] The factor of safety as compared with the test pressure and as compared with the working pressure should be particularly distinguished. Thus while a machine for 500 volts would be tested with 2,000 R.M.S. volts, or 4 times its working voltage, the pressure applied to a machine for very high voltages will not greatly exceed twice its working voltage, and its resultant factor of safety is therefore less.

When a machine is first run and heated up, its insulation resistance falls much below its value when cold, and this phenomenon is no doubt due to the initial action of the heat generated in all the interior parts of the winding, by which the moisture present in the cotton covering is driven out of the fine capillaries to form conducting passages for a leakage current. After a few runs the resistance steadily improves, and in a dynamo in regular work reaches a very high final value. When the high pressure test is applied, the British

[1] See especially *British Standardisation Rules for Electrical Machinery*, No. 72, Report of British Engineering Standards Committee (Crosby Lockwood & Son), where a standard test pressure of 1,000 volts *plus* twice the rated pressure of the machine is specified, and Miles Walker, *Specification and Design of Dynamo-electric Machinery*, pp. 187–190.

Standardisation Rules call for an insulation resistance in megohms not less than $\frac{\text{rated pressure in volts}}{1{,}000 + \text{rated output in KVA}}$.

Even in large dynamos and when new, the insulation resistance of the entire machine should never be less than one megohm.

§ 33. General.—Equalizing connections in multipolar machines are conveniently attached at the back end of the armature where they are easily accessible, as in Figs. 195 and 196. Thin strips of

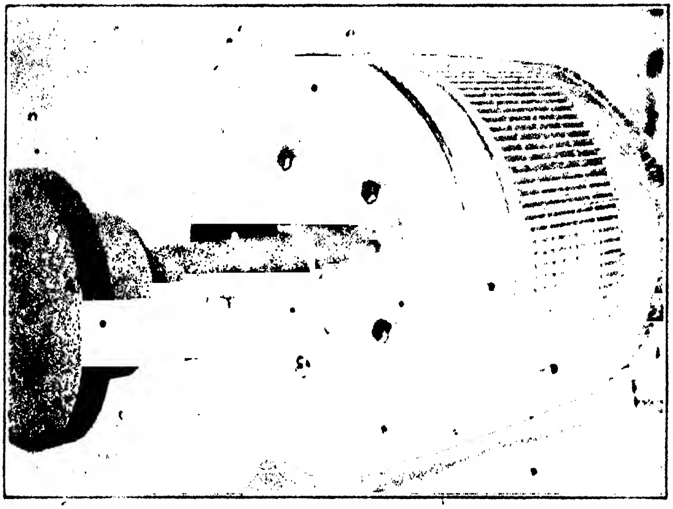


FIG. 195.—Equalizing connections at end of drum armature.

copper are soldered to the ends of the bars and to one of a number of insulated rings held in clamps of wood or ebonite. The section of each equalizing connection should be such that its resistance is less than the contact resistance of two sets of brushes, so that it may effectually abstract any equalizing current from the path through the brushes. Equalizing connections are also often arranged at the front end of the armature, being attached either to the bars close to the commutator or to the commutator sectors themselves, in which case they can be carried on an extension from the commutator at the back, and are built up thereon so as to be removable with the commutator.

In conclusion, there is added in Fig. 197 a view of an armature winding shop with its winding stands carrying a number of toothed armatures in various stages of construction, and finally in Figs. 198–201 are given sectional drawings illustrating different kinds

of armatures. The first of these is a coil-wound armature with core 18″ diameter × 9″ long for a 4-pole field, and the second a barrel-wound bar armature 21″ diameter × 11″ long, 4 layers per

FIG. 196.—Rear view of armature, showing equalizer rings (General Electric Company, U.S.A.).

slot with commutator in place; the latter has a cast-iron hub while in Fig. 200, which shows a complete 4-pole machine in section, the discs are again keyed directly to the shaft. Fig. 201 shows a toothed armature $25\frac{1}{4}''$ diameter × 10″ core-length for a 6-pole field with involute end-connectors formed of bent copper strip, and with wooden fixing wedges.

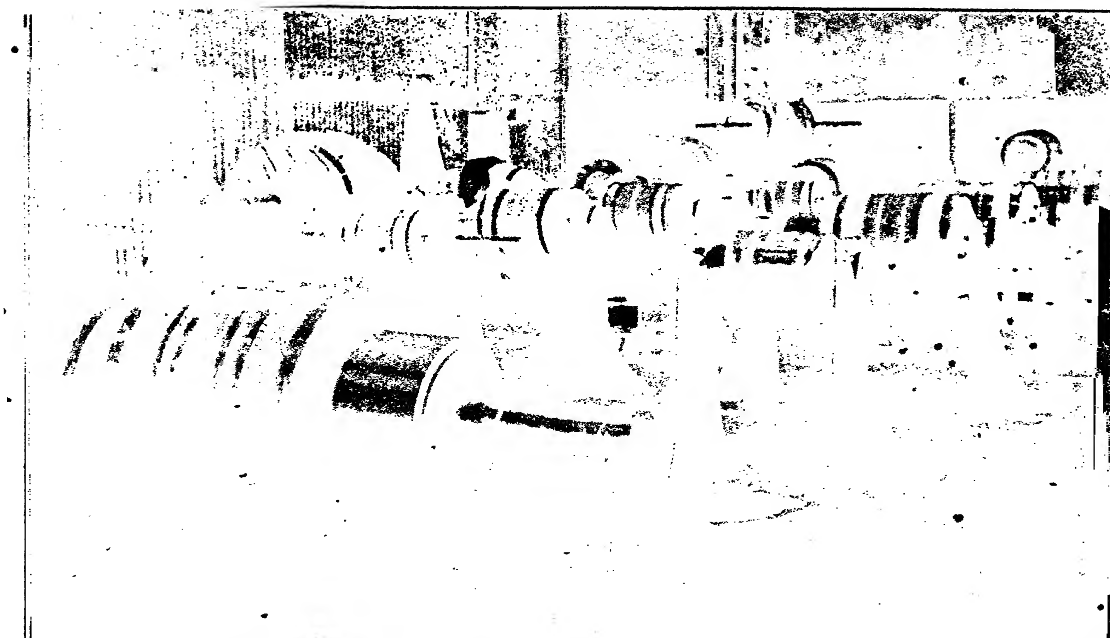


FIG. 197.—Armature winding shop.

§ 34. Magnetic humming of toothed armature.—If the width of opening of the slots of a toothed armature is great as compared

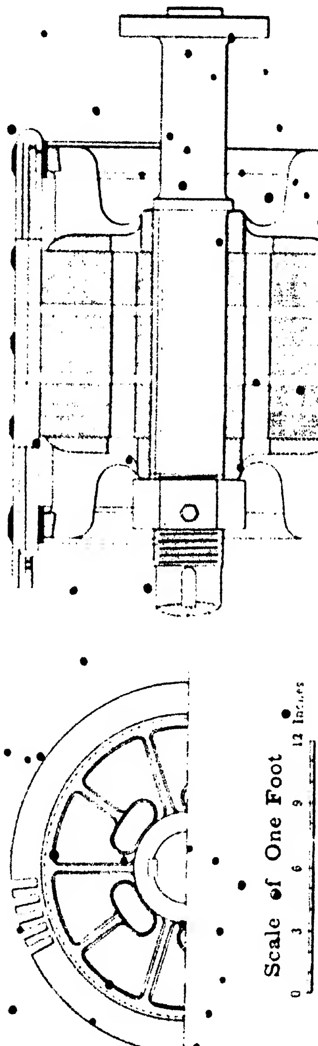


FIG. 199.—Barrel-wound armature 18 in. diameter, 9 in. long.

with the length of air-gap and the air-gap density is strong, the rapid change in the density of the lines between the trailing edge of one tooth and the leading edge of the next as each slot emerges from

under the pole face causes a rhythmic fluctuation in the powerful mechanical stress which draws together the tooth and the pole-edge. This sets up an alternate drawing together and springing apart of

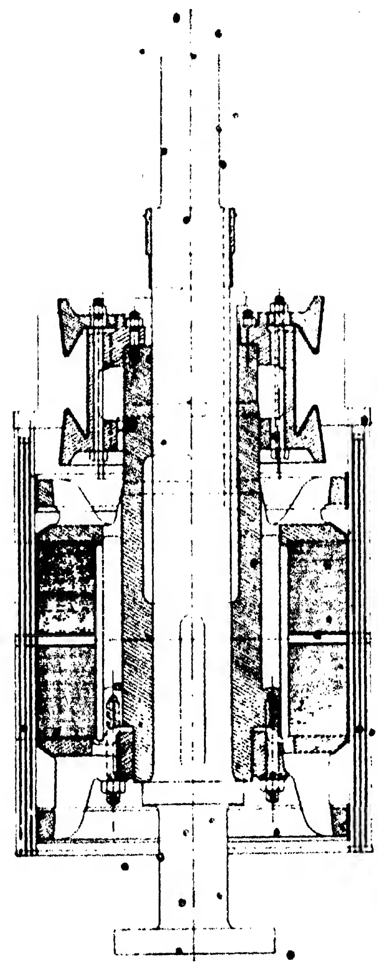


FIG. 199.—Bar-wound armature and commutator in section.

the tooth and pole-edge, which in turn imparts mechanical vibration to the air, whence a musical humming noise results of frequency proportional to the number of slots and the revolutions per minute. That the sound is to be attributed equally to the vibration of the

teeth and to the vibration of the pole-edge is probable from the observed fact that it is more prominent when the teeth are narrow at the root and comparatively weak, and also more marked with laminated than with solid pole-shoes.

In order to obtain the most economical design of machine it is

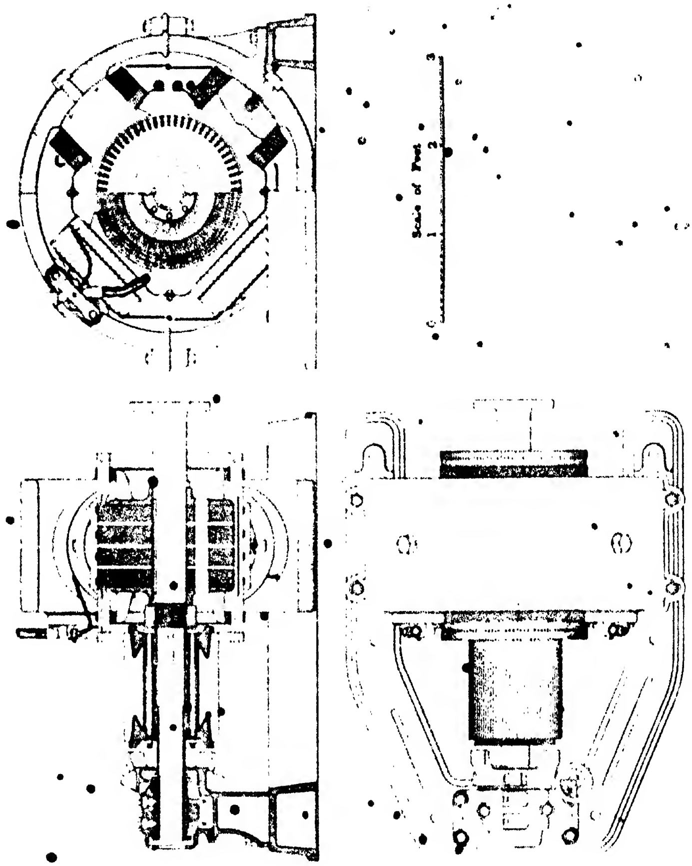


Fig. 200.—Sectional elevation and plan of 4-pole continuous-current dynamo with armature 21 in. diameter × 11 in. core-length.

desirable to employ as strong a field as possible, so that in cases where the musical hum of the toothed machine may be objectionable special care must be taken in its design to prevent or reduce it by eliminating the causes to which it is due. Particular relations of the polar arc to the toothed pitch have been suggested for this purpose,[1] but have not in the experience of the writer proved their

[1] As by Fischer-Hinnen; see G. W. Worrall, "Magnetic Oscillations in Alternators," *Journ. I.E.E.*, Vol. 40, p. 414.

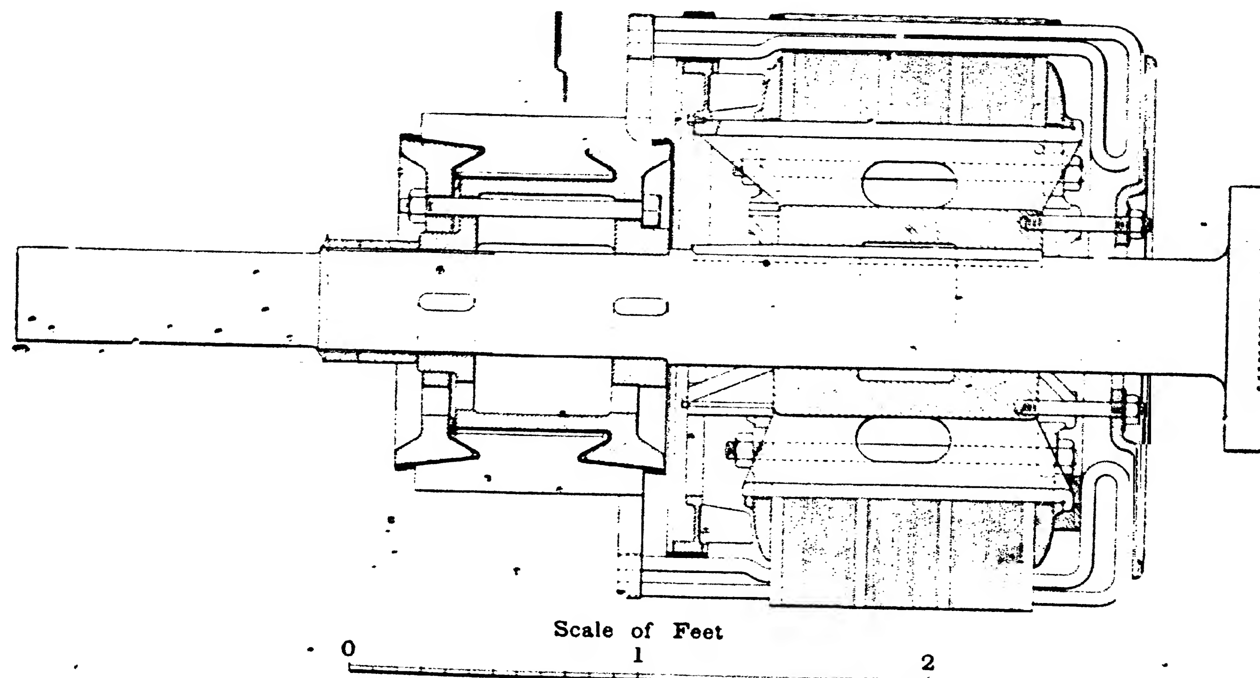


Fig. 201.—Drum armature $25\frac{1}{2}$ in. diameter × 10 in. long, with involute end-connectors at one end.

usefulness. The difficulty is that there is in reality no definite boundary or line of demarcation of the field, so that in practice no hard-and-fast relation of polar arc to tooth-pitch secures the desired end of noiselessness even when the pole-tips have rounded corners. All that can be done is to secure that there is no *abrupt* change of the conditions as the teeth leave the pole-edge by shading off the field to zero in as nearly a continuous manner as possible. The bore of the pole-faces should therefore be chamfered off until the air-gap is very small at the centre and gradually opens out to the edges, so that the fringe of lines diminishes as nearly as may be at a uniform rate. The best result is obtained by boring the poles to a larger radius and afterwards bringing them inwards, but a practical compromise is given by a careful shaping of the edges in the milling machine, so that the curved surface passes gently into a straight line, and the requisite clearance is obtained at the tips.

The width of the slot also plays an important part, so that in general the width of the opening should not exceed $\frac{1}{8}''$. When it is necessary to eliminate entirely this objectionable feature of the toothed machine it is best to employ half-closed slots, although care must be taken that the inductance of the short-circuited coils is not thereby increased too much. Another method of treating the same difficulty consists in making the pole-edges slant across the armature core, so that they are no longer parallel to the axis of the slots; but, in the case of laminated pole-shoes, this necessitates the building up of the laminations in small packets about an inch thick which are gradually stepped in relation to one another, and this again leaves the tips somewhat weakly supported, since the rivets cannot pass through the whole shoe near to the edges. R. Goldschmidt has obtained satisfactory results by laying a thin strip of sheet-iron 0·5 to 1 mm. thick, with a thin sheet of paper under it and bent over its sides, along the mouth of the slot.[1]

§ 35. Electrical resistance of armature. The calculation of the *electrical resistance of an armature* from brush to brush is made as follows. From the dimensions of the core, the length of one active conductor and of one connector by which it is joined to the next conductor in series can be estimated (see § 22). Let l be this length in some unit, and let ω = the resistance of unit length of copper wire of the given sectional area; then the entire length of conductor with which the armature is wound is $Z \times l$, where Z is the total number of active conductors, and its resistance if extended out in series is $Z \times l \times \omega$. Since the armature is divided into q parallel circuits, the resistance of each circuit is $Zl\omega/q$, and the resistance of q such circuits in parallel is

$$R_a = \frac{Z \times l \times \omega}{q^2} \quad . \quad . \quad . \quad . \quad (94)$$

[1] *E.T.Z.*, Vol. 28, p. 1166. *Electrician*, Vol. 60, p. 634. Cp. also Miles Walker, *The Diagnosing of Troubles in Electrical Machines*, pp. 151–4.

Thus in the simple bipolar or simplex wave-wound multipolar machine the resistance of the armature is one-fourth of the resistance of the total length of wire if in series; and in the multipolar lap-armature, unless multiplex wound, $q = P$, the number of poles, and $R_a = \frac{Z \times l \times \omega}{P^2}$.

§ 36. Peripheral speed of armature.—A high peripheral speed is to some extent desirable, as reducing the size and weight of a machine for a given output, but it must be limited by considerations of mechanical strength and durability, and therefore its permissible value depends largely on the method on which the armature is built up. But apart from the question of the mechanical strength of the rotating portion, its perfect balancing is of almost equal importance; with high speeds, even a comparatively light armature, unless accurately balanced, will set up such excessive vibration throughout the whole machine as will in the end considerably shorten its life. Moderately high-speed armatures are usually balanced on knife edges before leaving the workshop, and a little material is added as by solder or by a lead plug to whichever side is lighter. But even this only secures a statical balance when at rest, and does not necessarily imply balance under rotation unless the addition is made in the correct plane at right angles to the axis of rotation exactly opposite to the point of excess weight. To attain a running balance, each component of a high-speed machine, *i.e.* hub, discs, core with and without coils, commutator, should be balanced separately, and finally the complete rotor should be spun either vertically or horizontally in suspended bearings free to move laterally.[1]

With the ordinary slotted drum armature driven by belt a peripheral speed of about 3,000 feet per minute is common. In large continuous-current machines coupled directly to steam engines the peripheral speed reaches 5,250 feet per minute. When coupled to a large water turbine, a speed of 7,500 feet per minute has been reached in a continuous-current generator, and in turbo-dynamos coupled directly to steam turbines peripheral speeds as high as 10,000 feet per minute are found, but in all such cases only when the best materials and most careful construction are employed. A limit is set to the advantage of high speeds by the great increase in the eddy-current loss thence resulting, by the difficulty of securing sparkless commutation, and by the increased cost to manufacture.

§ 37. Ratio of length to diameter of armature.—The relative proportion of the length L to the diameter D of armature cores

[1] See *Electrical Engineer*, Vol. 38, pp. 866 A and 899 A, and also W. Hoult, "Direct-Current Turbo-Generators," *Journ. I.E.E.*, Vol. 40, p. 635; "Steam Turbines and Turbo-Generators" (W. J. A. London), *Journ. I.E.E.*, Vol. 35, p. 193; H. Holzwarth, *Power*, Vol. 28, p. 219; and § 10 of the present chapter.

depends on the type of magnet and on the number of poles, and in each type considerable variations are permissible without transgressing the limits fixed by mechanical or economical considerations.

In bipolar drums the length is usually considerably more than the diameter, but in multipolar machines the economical ratio of polar arc to length of core remains nearly a constant, with values from 1 to 1·4, so that the ratio of length to diameter rapidly decreases as the number of poles is increased. See Chapter XV, § 17.

§ 38. Radial depth of armature core.—Since the total flux which enters the armature from any one pole, or Φ_a, divides within the core, half of the lines passing in either direction onwards to an adjacent pole of opposite sign, the area of the iron through which the Φ_a lines flow is twice the cross-sectional area at any one part of the core if it be cut through from the outside to the shaft. Hence if h_c = the radial depth in centimetres of the discs below the level of the teeth, and L_i = the net length in centimetres of iron parallel to the shaft (after allowance has been made for the insulating varnish or paper between the discs and for any ventilating air-spaces that there may be in the core), the maximum flux-density in the core is $B_c = \frac{\Phi_a}{2h_cL_i}$. It will be understood that this is the maximum value which the induction reaches when averaged over a cross-section through the core in an interpolar gap; as Figs. 219 and 221 show, in this cross-section the induction is not actually uniform. With allowance for air-ducts, L_i varies from 0·75 to 0·85 of the gross length of the core L. The iron of which armature cores are composed being soft and permeable, a fairly high flux-density is permissible in the armature without impairing the efficiency or economy of excitation. In drum armatures the density B_c may be as high as 16,000 to 17,000, but is usually made to decrease somewhat as the frequency is increased; since the latter is partly dependent on the number of poles, average values of B_c are 16,000 to 15,000 in 4-pole, and 15,000 to 14,000 in 6-pole machines.

The nature and magnitude of the hysteresis loss in armatures will be treated in the next Chapter, but it will here be added that if the assumption be made that the hysteresis loss is proportional to $B_c^{1·6}$, then since $B_c = \frac{\Phi_a}{2h_cL_i}$ and V_c, the volume of iron in the core $= h_cL_i \,.\, 2\pi r$ where r is the mean radius of the core, it follows from equation (96) that with a given Φ_a the hysteresis loss is $\propto V_c\left(\frac{r}{V_c}\right)^{1·6}$, which is $\propto \frac{r^{1·6}}{V_c^{0·6}}$. Taking the external diameter and length as fixed by other considerations, it results that an increase in the radial

depth reduces the hysteresis loss more than in proportion to the increased volume and cost of the iron. On this score it might be thought that a low average density would be advisable; yet any such conclusion, if pressed too far, would in practice be erroneous for several reasons. In the first place, if the density is confined within the upper limits named above and is made to decrease slightly with an increase in the frequency, the absolute value of the hysteresis loss is so small that the slight gain in efficiency with a much lower density does not warrant the increased cost of the iron. In the second place, as will be shown in Chapter XIV, § 12, with a rotating field the loss does not with high densities increase in proportion to the 1·6th power of the induction, and even shows a decrease with values above 16,000. But more important than all, the actual flux-density over any cross-section of the core is not uniform, and if the curves of Figs. 219 and 221 are imagined to be prolonged it will be seen that the inner layers of iron carry less and less of their due proportion of lines; the maximum density in the outer layers is not, therefore, much reduced even if the radial depth is considerably increased, while the added inner layers of iron are not only inefficient from a magnetic point of view, but also are objectionable as shielding the layers which are most heated by hysteresis from the cooling effect of air-circulation through the core.[1] It is not, therefore, economical to reduce the density below 16,000 for frequencies from 5 to 15, below 14,000 for frequencies from 15 to 20, or below 12,000 for frequencies from 20 to 30.

On the other hand, considerations of the total loss limit the permissible values to which the flux-density in the core may be raised. Although with a true rotating field the hysteresis loss alone might be reduced per cycle with an increased average density above 16,000, it must be remembered that the loss by eddy-currents in the discs, which is proportional to the square of the frequency and of the density (Chapter XXI, § 17), and also the necessary excitation are increased. Further, when segmental core-discs have to be employed on large armatures, the breaks in the continuity of the discs at the joints may give rise to dissymmetry in the flux distribution and to eddy-currents in the shaft if the average flux-density is high and especially if the small air-gap at the breaks is not uniform over the whole width of the core.[2]

If the ratio of the polar arc to the pole-pitch be $\beta = 0{\cdot}7$, the constant k which reduces the maximum air-gap density $B_{g\,max}$ to an average value for the whole of the pole-pitch, may roughly be

[1] Dr. W. M. Thornton, "The Distribution of Magnetic Induction in Multipolar Armatures," *Electrician*, 26th August, 1904.

[2] See especially Miles Walker, *Specification and Design of Dynamo-electric Machinery*, pp. 83, 84, and *The Diagnosing of Troubles in Electrical Machines*, pp. 134 and 140.

identified with β, so that $\Phi_a = \beta \dfrac{\pi DL}{2p} B_{g\ max}$[1]. The net length of iron in the core, after making allowance for the insulation between the discs and for the ventilating ducts is $L_i = 0{\cdot}9 \times 0{\cdot}9L$, say, $0{\cdot}8L$, and the single section of the iron $= 0{\cdot}8 h_c L$. The average density over a section of the core below the teeth midway between a pair of poles is thus

$$B_c = \frac{0{\cdot}7 \dfrac{\pi DL}{2p}}{2 \times 0{\cdot}8 h_c L} B_{g\ max}$$

or

$$h_c = 0{\cdot}44 \times \text{pole-pitch} \times \frac{B_{g\ max}}{B_c}.$$

Taking a normal value for $B_{g\ max}$ as 7,500, and fixing the maximum limit of B_c as 16,800 for low frequencies and 13,000 for higher frequencies, h_c is approximately 20 to 25 per cent. of the pole-pitch, and $\dfrac{h_c}{D} =$ say, $0{\cdot}212 \dfrac{\pi}{2p} = \dfrac{0{\cdot}66}{\text{number of poles}}$. Thus the radial depth of core is to be regarded as related to the pole-pitch, and bears a ratio to the diameter which is entirely dependent upon $2p$. The larger the number of poles, the shallower become the discs.

§ 39. The effect of taper in the iron teeth.—The marked influence of the tapering tooth-width due to slots having parallel sides by way of increasing the saturation at the roots of the teeth, has already been mentioned in § 18. Its influence on the maximum density in the air-gap must also here be noted.

With slots having parallel sides the fraction of the width of the tooth at the top which is lost at the bottom owing to the tapering inwards of the sides is very closely $\dfrac{l_t(1 + w_s/w_n)}{R}$, where R is the radius of the armature and l_t is the depth of the slot. The fraction which the area at the root is of the area of the tooth at the top is therefore $1 - \dfrac{l_t(1 + w_s/w_n)}{R}$, and it is evident that for any constant values of w_s/w_n and of l_t the effect of the taper becomes more marked the smaller the diameter of the armature. But the depth of slot which is employed in small armatures itself decreases, which makes the variation of the ratio of the two areas less as between large and small armatures. The depth of the slot being assumed to have such

[1] The additional flux of the interpolar fringes is in the above approximation taken as making up for the reduction of the flux under the pole-face owing to the presence of the ventilating ducts in the armature core and also for the fact that the axial length of the pole-face is usually somewhat less than the over-all length L of the armature core between the end flanges. Cp. Chap. XVI. § 7.

normal values as those given at the end of § 18 of the present Chapter, and as shown in Fig. 202 for different diameters of armature, this Figure shows the fraction w_{t2}/w_{t1} for limiting ratios $w_s/w_{t1} =$ 0·5 and 1. The latter of these broadly speaking applies to small armatures and the former to large armatures, the passage being made gradually from one to the other curve with increasing diameter of the armature, as shown by the dotted line. Assuming no flux

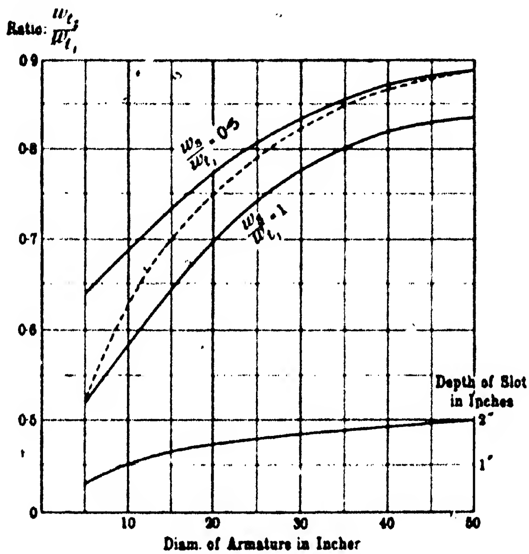


FIG. 202.—Ratios of areas at bottom and top of teeth.

to leak into the slots or ducts, if $B_{t2\ max}$ be the apparent flux-density at the root of the tooth before correction for such leakage

$$B_{g\ max} \times (w_{t1} + w_s) \times L = B_{t2\ max} \times w_{t2} \times L_i$$

whence

$$B_{g\ max} = 0{\cdot}81\ B_{t2\ max} \times \frac{w_{t2}}{w_{t1} + w_s} \quad . \quad . \quad . \quad . \quad (95)$$

From the point of view of sparkless commutation it is advantageous to saturate the teeth strongly up to about $B_{t2\ max} = 21{,}000$ (uncorrected) at the root, but if this value is much exceeded, the forcing of the flux outwards into the slots as shown in Chapters XVI § 8, and XXI § 19, causes greater eddy-currents in the solid conductors, and the increasing reluctivity of the iron demands a rapidly increasing number of ampere-turns to overcome it. Adopting then the normal value of 21,000 for $B_{t2\ max}$ and inserting it in the above equation, it follows from Fig. 202 that the value of $B_{g\ max}$ in small

machines cannot be nearly so high as in large machines, and that it must rise from, say, $B_{g\ max} = 5{,}000$ in an armature of 5 in. diameter to 10,000 in a 70 in. armature, approximately as shown in Fig. 203. Further reasons that forbid the use of such high air-gap densities in very small as in large armatures are the

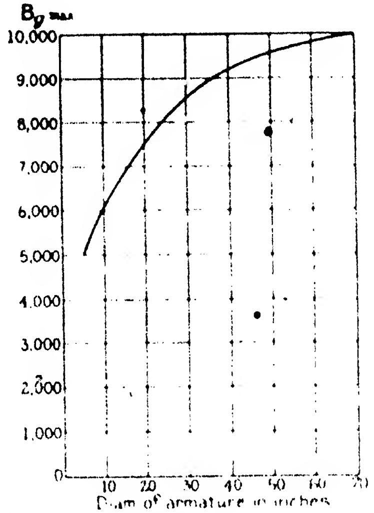


FIG. 203.—Relation of $B_{g\ max}$ to diameter of armature.

difficulty of bestowing the exciting ampere-turns on the poles and the liability of their shafts to bend under the unbalanced magnetic pull if the air-gaps are of slightly unequal lengths.

Since the large machines usually have the higher frequencies, it thus finally results when the considerations of the present section are applied to § 38 that the radial depth of core, although decreasing with an increased number of poles, bears a greater ratio to the pole-pitch in large machines.

CHAPTER XIV

THE MAGNETIC PROPERTIES OF IRON

§ 1. The conditions affecting the permeability of iron. The permeability, μ, of any material depends, not merely upon whether it is magnetic or non-magnetic, but, if it be the former, upon three conditions. These may be summed up as follows: (1) Its *physical and chemical state*, *e.g.* whether it be wrought iron or cast iron, annealed or hardened; or whether it be alloyed with other substances, and what is the percentage in which these are present, *e.g.* steel alloyed with 12 per cent. of manganese becomes almost non-magnetic. So far, the permeability is analogous to the electric conductivity of metals which is similarly affected by their physical state or by their purity. But the permeability even of a definite chemical substance in a definite physical state is not a constant quality; it also depends on (2) *the value of the flux-density*. Regarded from this point of view, the permeability is a function of B, the flux-density of the lines per square centimetre, or $\mu = f(B)$. Herein it differs decisively from the analogous property of electrical conductivity, for in the case of the electric circuit the resistivity of the material composing it is not a function of the current, but is independent of the amount of current flowing through it, save in so far as this indirectly affects the temperature of the conductor. Although, therefore, in the case of air and non-magnetic bodies, the analogy of magnetic permeability to electric conductivity is perfect, it is far from holding in the case of iron. While the permeability of iron may be 2,000, or even more, with an induction $B = 5{,}000$, yet when the flux-density has been raised by special methods to the extremely high value of 45,350, the value of μ is reduced to less than 2.[1] From this it will be seen that though air, gun-metal, zinc, etc., may be regarded as magnetic insulators relatively to iron when the latter is weakly magnetized, yet when the iron is "saturated" the difference in their permeabilities becomes greatly reduced; for values of B in wrought and cast iron, such as are ordinarily reached in practice, their permeabilities may be said to stand to that of air in the ratio of about 200 or 100 to 1, but for higher values it continually decreases, until for intense saturation they may be only about double that of air. Next, the permeability depends upon (3) the *previous magnetic history* of the metal, *i.e.* upon whether it has been previously subjected to a

[1] Actually 1·85 (Ewing, *Magnetic Induction in Iron and other Metals*, 3rd edit., Chap. VII, § 94).

larger or a smaller induction. The iron is affected, as it were, by its history, so that the number of lines passing through it differs in the two cases of an ascending or a descending induction. Lastly, the permeability of iron is affected by *temperature*, but as this effect is hardly noticeable within the range of temperature that is met with in the ordinary working of dynamos, it may for our present purpose be dismissed as negligible. Such being the complex nature of the permeability $\mu = B/H$, it is most convenient to represent the process of the magnetization of iron graphically by *curves of induction* or flux-density, connecting together corresponding values of B and H. It may here be recalled that the quantity H in C.G.S. units—in other words, the *magnetizing intensity* of the field in which the iron is placed, or, as it is sometimes called in the present connection the "magnetizing force," to which the iron is subjected—is also the fall of magnetic potential which takes place over a centimetre length of the substance forming, *e.g.* the core of a ring such as that of Fig. 4 when B lines flow through each square centimetre of its cross-section. A number of corresponding values of H and B are obtained, and these when plotted as abscissae and ordinates respectively are joined to form a curve of flux density.

§ 2. Ascending and descending curves of B and H. Starting with a soft annealed wrought-iron ring previously unmagnetized, and therefore in an entirely neutral state, or else carefully demagnetized by means of a gradually decreasing alternate current passed through the magnetizing helix, let its *ascending* curve be determined, the current being increased in strength step by step, and the flux-density being separately measured for each step. At the outset, for very small magnetizing intensities, the value of B rises at a certain slow rate almost proportionately to the increase of the intensity; when, however, the intensity has reached a value of about 1 or 2 C.G.S. units, the rate of rise of the induction changes very rapidly to a much increased value; at this new rate the induction again continues to rise almost proportionately to the increase of the magnetizing intensity, but when the latter is raised to a value of from 5 to 10, although the induction B continues to increase, its rate of rise falls off, and becomes gradually less and less rapid. The curve thus obtained for B and H is that marked OE in Fig. 204; since the iron was at the outset unmagnetized, it starts from the origin, and rises with the increasing magnetic intensity, as shown by the ascending arrow. It is divisible approximately into three different portions, marked a, b, c, corresponding to the three stages just mentioned.

After having reached the point E on the *ascending* curve of flux-density, let the magnetizing current be gradually reduced and the curve of *descending* induction be traced. This, as has been already stated, is by no means identical with the ascending curve.

It is marked II. with a downward arrow in Fig. 204, and is considerably higher than the ascending curve, so that when the magnetizing intensity is zero it cuts the vertical axis at some point R, considerably above the horizontal axis. The height OR, *i.e.* the number of lines still passing per square centimetre of the iron when

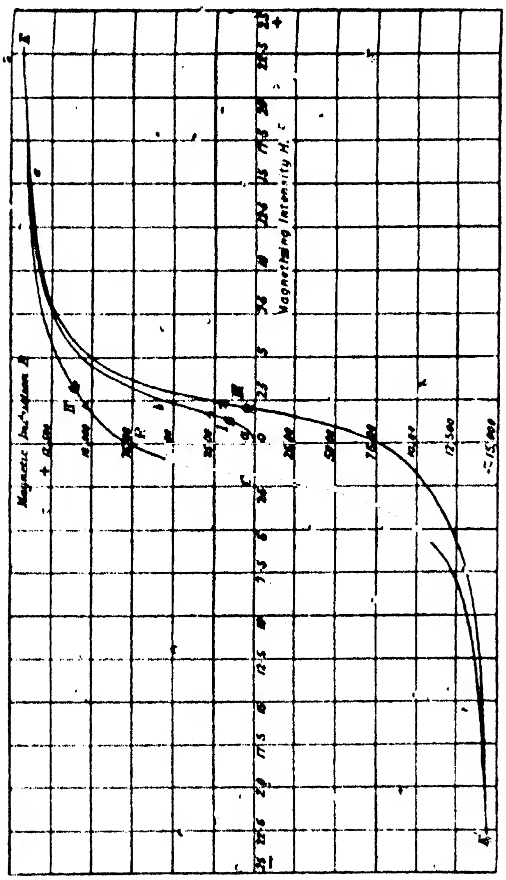


Fig. 204.

the magnetizing intensity has been gradually reduced to zero, is the *residual induction* or *remanence* of the iron; its exact amount depends upon the value to which the induction in the iron has been previously raised, and from which, as a starting-point, the reduction began, while the proportion which it bears to the maximum induction reached is the *retentivity* of the iron. After reaching point R on the downward curve, a negative magnetic intensity is required,

and the direction of the magnetizing current must be actually reversed by reversing the magnetizing current in order to reduce B to zero. The amount of the negative intensity, which has to be applied in order to reduce B exactly to zero after it has been

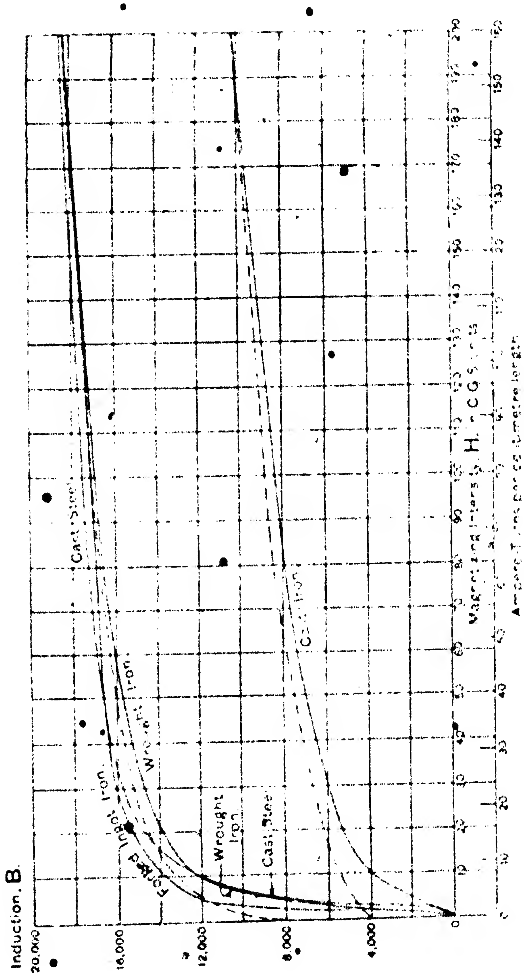


FIG 205 ·B-H curves for magnetic materials.

previously raised to a high value, is called the *coercive intensity* of the iron ; it is measured by the length of the line OC in Fig. 204.

§ 3. Saturation a relative term.—If, after reaching the point E on the ascending curve, the magnetizing intensity, instead of

being reduced, had been still further increased, the curve would become flatter and flatter, but it never becomes truly horizontal; although the iron may now be said to be "saturated," the induction B never ceases to increase when the magnetizing intensity is increased, so that there is no definite limiting value of B beyond which it cannot be raised; the permeability of the iron is enormously reduced, but even when the peculiar action of the iron in increasing the magnetic flux is almost imperceptible, more flux can always be passed through the helix, just as if it were a simple solenoid enclosing merely air and without any core of iron. The iron is commonly said to be "saturated" when the curve of flux-density has begun to bend over at the rounded knee between b and c, but the exact appearance of the curve and position of the bend largely depend on the respective scales to which B and H are plotted. Fig. 205 shows the curves of flux-density of three different kinds of iron, the ascending curve being shown full and the descending curve dotted; in these the induction and magnetizing intensity (the latter shown in the upper of the two scales along the horizontal axis) have been carried to much higher values than in Fig. 204. The scale of H has been altered to obtain a convenient size of diagram, and it will be seen that now the bend of the curve for annealed wrought iron appears to be at $B = 14{,}000$ instead of at $B = 11{,}000$, as in Fig. 204. Saturation is in fact, so far as lines of induction are concerned, a relative term, with no particular numerical significance.

But so far as "intensity of magnetization," J, as defined by the equation $B = H + 4\pi J$, is concerned, saturation has a more definite meaning. When H is raised above a value ranging from about 1,500 to 2,000 C.G.S. units, the value of B exceeds that of H by an amount which is practically constant for any given material. Thus with high grades of armature iron, for values of H above 2,000, experiment shows, say,

$$B = H + 21{,}400$$

whence the constant value of $J_s = 1{,}700$. This saturation intensity J_s or the "specific magnetism" may then be regarded as a true physical constant of the material.[1] The highest value of J_s so far recorded is 1,796 (in annealed Norwegian iron)[2], and an exceptionally good specimen of armature iron tested by Dr. Beattie and H. Gerrard gave a saturation intensity of 1,740 with a magnetizing

[1] See Sir R. A. Hadfield and Prof. B. Hopkinson, "The Magnetic Properties of Iron and its Alloys in intense fields," *Journ. I.E.E.*, Vol. 46, p. 235; Dr. R. Beattie and H. Gerrard, *Electr.*, Vol. 64, p. 750, Feb. 18th, 1910, F. Shaw, *Electr.*, Vol. 80, p. 790, and A. Campbell and D. W. Dye, *Journ. I.E.E.*, Vol. 54, p. 41.

[2] By B. O. Peirce, see *Electrician*, 20th Oct., 1909, p. 107.

intensity H, as low as 1,500. The results of the latter observers are given in the two upper and the lowest curves of Fig. 210; the saturation points are marked with a cross, after which the induction rises as an inclined straight line, the dotted parts being continuations beyond the experimentally observed portions.

§ 4. Physical and chemical conditions as bearing on magnetic properties of iron.—The permeability of iron is largely affected by the presence of impurities, intermixed or in combination with it, such as phosphorus, sulphur, tungsten, or manganese, and also by the closely connected physical quality of "hardness." The exact influence of any one ingredient it is not difficult to determine, since its absence or presence may affect not only the chemical composition but also the melting point, the process by which the metal must be treated during manufacture, and the interaction of the other impurities. It may, however, be said that the most important ingredient on all these scores is *carbon*; its presence in a combined state being detrimental to the permeability.[1] *Mild steel*, when rolled into thin sheets, may usually be relied upon as being soft, owing to the processes of manufacture through which it has to be carried, and, further, in this form it admits of a very thorough annealing. When, however, such sheets are stamped or punched out into the shapes suitable for dynamo purposes, it is usual to subject the stampings to a further *reannealing*, since by the process of stamping they become, to a certain extent, hardened. The annealing process should be carried out in air-tight closed boxes or chambers, since mere annealing in a fire open to the air reduces the permeability. In *cast iron* the effects of impurities and of hardness are again very marked. The iron used for castings should be specially soft and pure, and all hardening or chilling of it after casting should be avoided. The total amount of carbon present in a combined form or as graphite may vary from 3 to 4½ per cent., but should be low in order to reduce the percentage of combined carbon, if possible to less than 0·5 per cent. Approximately, it may be said that, for ordinary values of H between 40 and 80, the permeability of cast iron is less than half that of forged iron or cast steel; but different samples of cast iron show much more divergence among themselves in permeability than would be found in as many samples of wrought iron. In *steel castings* combined carbon is again objectionable, and should not be present to a greater extent than 0·2 per cent., the temperature at which it is cast being correspondingly high. An alloy of steel with 5 per cent. of nickel has been used with success to combine a

[1] See especially Barrett, Brown and Hadfield, "Researches on Different Alloys of Iron," *Journ. I.E.E.*, Vol. 31, p. 695 ff., and H. F. Parshall, "Magnetic Data of Iron and Steel," *Proc. C. E.*, 1896, Vol. 126. The effect of alloying sheet steel with silicon is dealt with later.

permeability as high as that of good cast steel with great mechanical strength after forging.[1]

The "retentivity" and "coercive intensity" likewise vary greatly in different qualities of iron, and are affected by their purity and hardness. It will be seen from Figs. 205 and 208 that when a high magnetizing intensity is gradually reduced to zero, the residual magnetic induction which persists in annealed wrought iron is about 7,000 or 8,000, while for cast iron it is about 3,000 to 4,000. The retentivity of soft annealed wrought iron is greater than that of any other material, since as much as 80 or 90 per cent. of the induction may be retained, but this amount is much reduced if the iron circuit be incomplete. Its coercive intensity is, however, very small, and the residual flux is quickly reduced by a feeble demagnetizing intensity, or by mechanical vibration, jarring or tapping. Hard iron and steel, although retaining less magnetic induction than soft iron, keep it much more strongly, and therefore permanent magnets, which are required to maintain their magnetic properties in spite of mechanical shocks or demagnetizing influences, are made of steel, the most suitable alloy being tungsten steel, of which the coercive intensity is as much as 70 C.G.S. units.[2] Were it not for the fact that a high flux-density cannot be obtained with hard steel owing to its low permeability, the field-magnets of dynamos would be made of steel permanently magnetized, and requiring no exciting current. In default of a material which is at once permeable and retentive, and possessing considerable coercive intensity, we are compelled to employ soft iron or mild steel magnets; these are permeable, but require the exciting current to be continuously maintained round them in order that they may not lose their magnetism entirely under the mechanical vibration to which the dynamo is subject when running, or through demagnetization by the current-turns of the armature.

§ 5. B-H curves of iron and steel.—While the curves of Fig. 205 serve for a general comparison of the permeabilities of various materials,[3] they need to be supplemented by further curves showing more in detail the relative merits of such materials as are in everyday use in the commercial manufacture of dynamos. These may be divided into the four main groups of iron or steel forgings, steel castings, cast-iron, and sheet steel stampings, the first three being

[1] A "B-H" curve for a nickel-steel forging, containing 3·5 per cent. nickel, with an elastic limit of 50,000 lb. per square inch, and an ultimate strength of 80,000 lb. per sq. in. is given by B. A. Behrend, *Trans. Amer. I.E.E.*, July, 1908, Vol. 27, p. 1061.

[2] Cp. S. P. Thompson, "The Magnetism of Permanent Magnets," *Journ. I.E.E.*, Vol. 50, p. 80, and S. Evershed, "Permanent Magnets in Theory and Practice," *Journ. I.E.E.*, Vol. 58, p. 818.

[3] B-H curves for various materials are shown in Fig. 12 of the paper by A. Campbell and D. W. Dye, *Journ. I.E.E.*, Vol. 54, p. 41.

used in field magnets, and the last in armature cores, pole-shoes, and in the rotors of some turbo-alternators.

For dynamo magnets, both wrought and forged-ingot iron, which were for long the favourite materials, have been almost entirely superseded by steel or iron castings. The yoke-rings of modern multipolar machines are better suited to castings, and cast steel for moderately high inductions above $B = 12{,}000$, such as are in practice required, is superior to even the best wrought iron, such as annealed Lowmoor or Swedish. The magnet-cores of poles, when circular in cross section, are, however, conveniently made from rolled bars

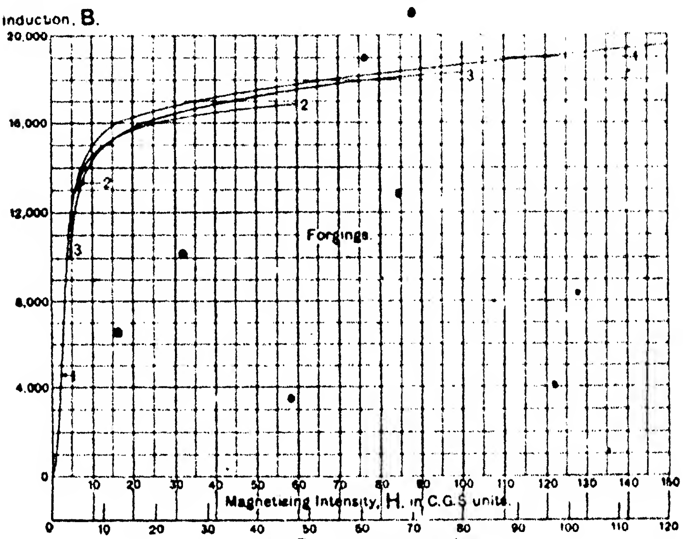


FIG. 206.—B-H curves for forgings.

of ingot iron. The three materials of Fig. 206 have this in common with wrought iron, that they are all either forged under the hammer or press, or are rolled into bars. But while wrought iron consists of puddled balls or scrap-iron pieces welded together by hammering, its fibrous structure still bearing witness to its method of manufacture, the *forged ingot irons* are homogeneous in their nature, having been at the outset thoroughly fused together in the furnace. While chemically they may be ranked as mild steels with a small percentage of carbon, the term "ingot iron," and its more expressive German equivalent "Flusseisen," indicate that they have been melted before they reach the hammer or the rolls. Fig. 207 gives the *B-H* curves of four favourable specimens of the second group, namely, *steel castings*; it also illustrates the fact that a low initial

permeability does not necessarily imply any inferiority at high inductions. The curves cross one another, and they are further remarkable from the facts that the crossing-point in each case is practically coincident with $B = 15{,}000$, and that the relative position of the curves becomes exactly reversed. On the whole there is little to choose between the permeabilities of the two groups, the forgings being slightly more permeable between $B = 13{,}000$ and $B = 16{,}000$, and the castings more permeable at still higher inductions. There is indeed but little difference in the chemical

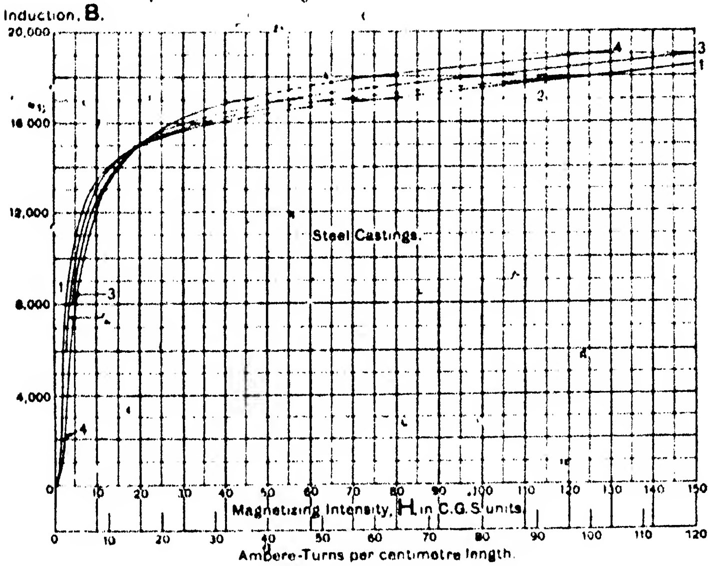


FIG. 207.—B-H curves for steel castings.

composition of the two groups; both are as nearly as possible pure iron with only such admixture of carbon or other substances (silicon, phosphorus, and manganese) as may enable them to be conveniently worked during the process of manufacture. An average analysis for magnetic cast steel would give, *e.g.*, carbon 0·08 to 0·15 per cent., silicon 0·2 to 2 per cent., manganese a trace, and phosphorus and sulphur as low as possible. In fact, the total impurities present, including carbon, may not exceed 0·3 per cent., the remaining 99·7 per cent. being pure iron. The material of curve 1 in Fig. 206 may be credited with the highest permeability over an extended range; at low inductions it is slightly superior to wrought iron, and it is only surpassed by an exceptionally good cast steel, when the induction is pressed beyond $B = 18{,}000$. Its curve is nearly

identical with that of an almost perfectly pure specimen of iron prepared specially for laboratory purposes.

The curve for a cast iron of specially good magnetic quality is given in Fig. 208; its composition was approximately 70 per cent. of machinery scrap and 30 per cent. of a soft hematite pig-iron.

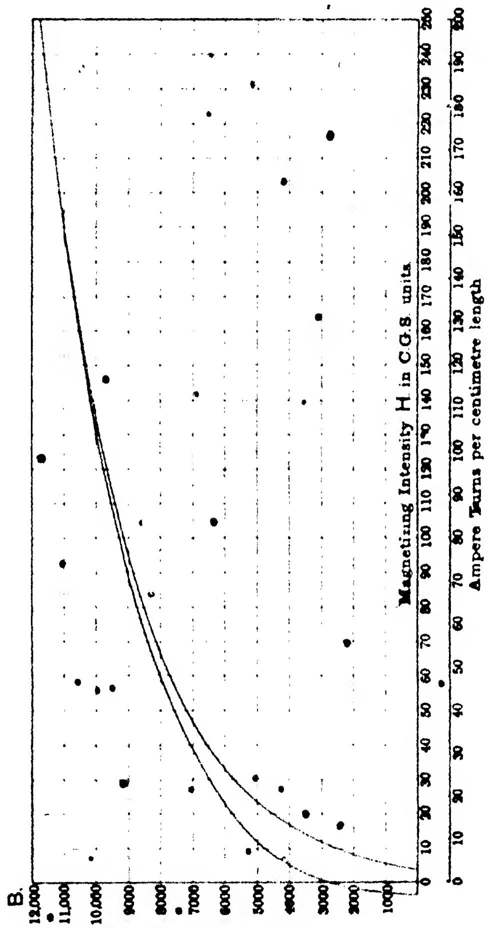


Fig. 208.—B-H curve for cast iron of good quality.

For the core plates of armatures the material in general use is a mild steel of very nearly the same composition as that of the larger masses of ingot iron or steel forgings. From the similarity of their chemical analysis *armature stampings* (Fig. 209) may be expected to have a permeability similar to that of the same iron or steel when

tested in bulk, and such is in fact the case. The only differences are that owing to its repeated mechanical treatment in the rolling-mill, and its subsequent annealing, the sheet-metal has a permeability at low inductions rather higher than that of group II, but falling off at higher inductions, and that, as mentioned later, the sheet-metal is now often definitely alloyed with silicon. Four qualities of electrical sheet steel are supplied by Messrs. Joseph Sankey & Sons, Ltd., of Bilston and Messrs. John Lysaght, Ltd., of Newport, Mon. Of these "Lohys" is the quality most commonly used for armature

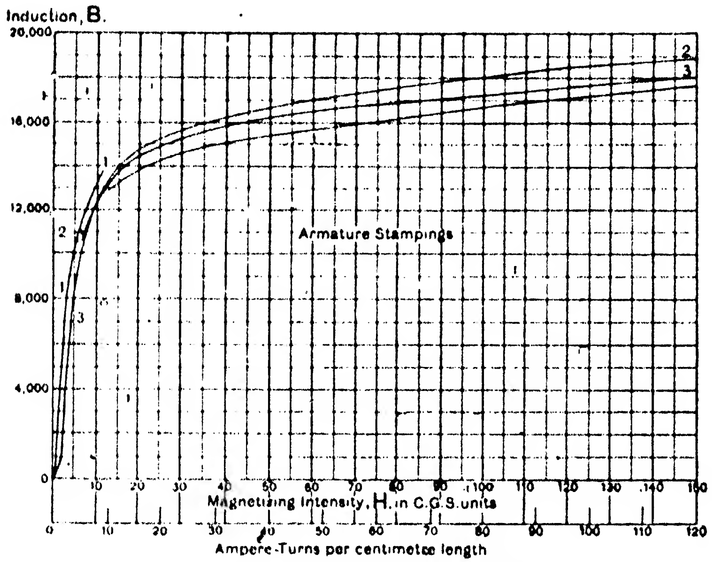


FIG. 209.—B-H curves for stampings.

stampings; thence rising in silicon content, "Special Lohys" of which the permeability is slightly inferior but the total loss from hysteresis and eddy-currents less, "Medium Resistance," again with a lower permeability but less loss, and lastly "Stalloy" with about 3 per cent. of silicon which is largely used in transformer work but is not suitable for armature stampings owing to its brittleness and the difficulty of notching it. Curves 1 and 2 of Fig. 209 may be taken as roughly representative of "Lohys," and 3 as representing "Medium Resistance." For $B = 15{,}000$, H should in no case exceed 30, the percentage of silicon in the alloyed sheet-steel of the lowest curve of Fig. 209 being higher than is necessary or advisable for armature stampings. In the teeth of slotted armature cores the induction may often reach the high figure of $B = 20{,}000$,

and in order to embrace even higher values, Fig. 210 is added,[1] in which the lowest curve may be taken as slightly inferior to "Medium Resistance" quality at high inductions above 21,000. For use over the whole range of practice is given Fig. 211.[2] Lastly in the

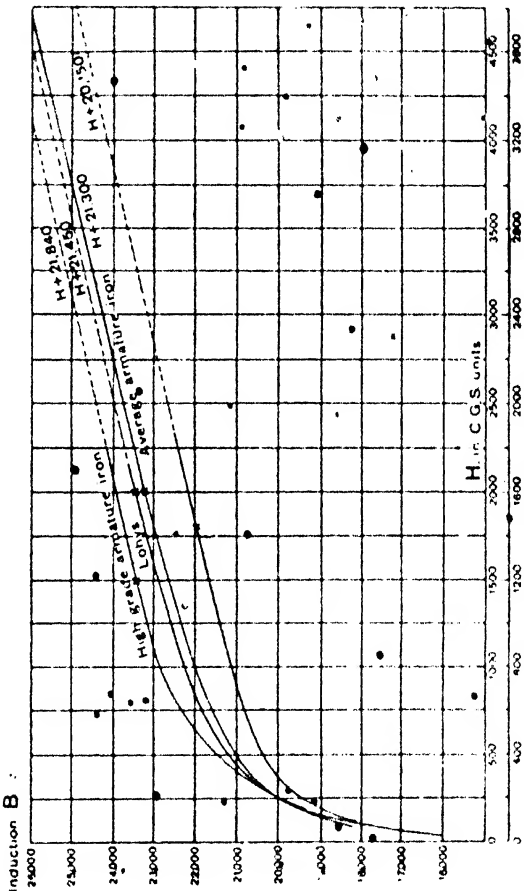


FIG. 210.—B-H curve for high flux-densities in stampings.

two lower curves of Fig. 212 are given permeability curves, which may be taken as representative of average steel, such as is commonly

[1] For the influence of scale on steel sheets, see F. Shaw, "The Measurement of the Permeability of Iron Stampings by Ewing's Double Bar and Yoke Method," *Electr.*, Vol. 80, p. 787.

[2] Based on the curves of Fig. 23 in Miles Walker, *Specification and Design of Dynamo-electric Machinery*, so far as the two lower curves are concerned.

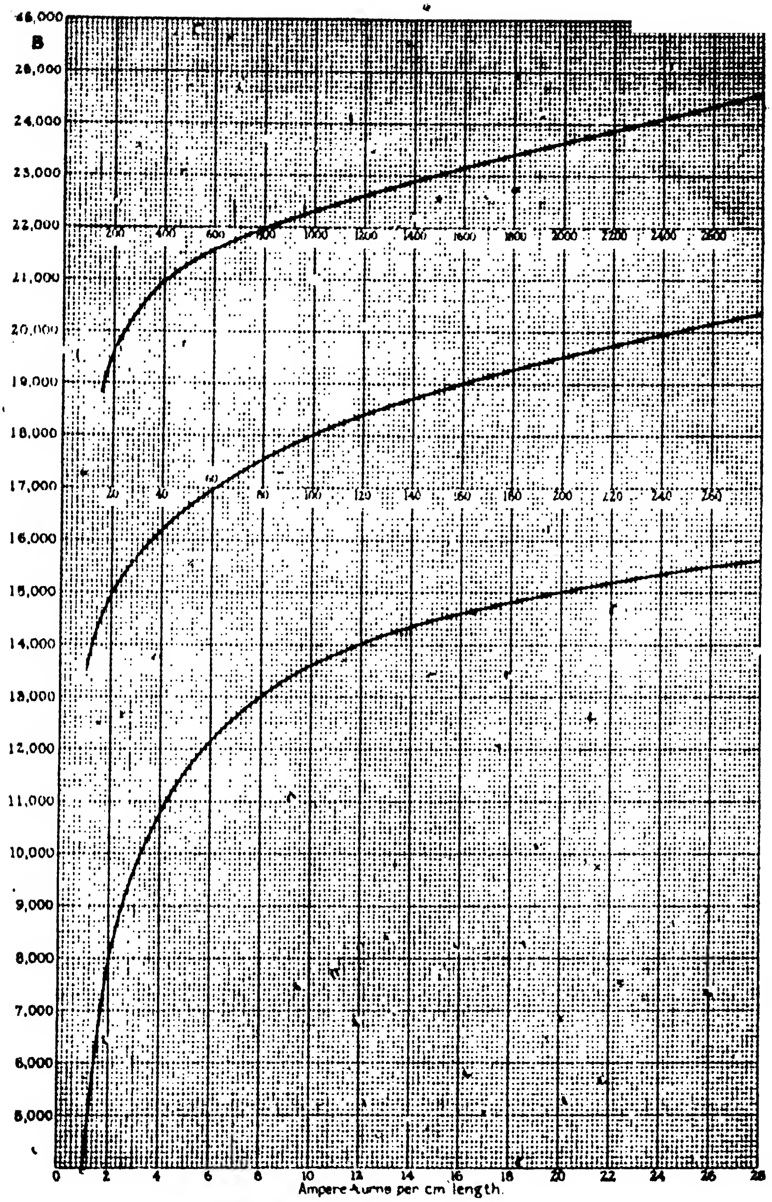


FIG. 211.—Magnetization curve of dynamo sheet steel.

used in dynamo work; they show how great is the variation of μ for different values of B.[1]

It will now be fully evident that the relation between B and H, or the permeability of a given sample of iron, even with a definite magnetizing intensity, cannot be absolutely specified, and may take any one of a certain range of values according to the previous treatment to which it has been subjected. Since, however, the

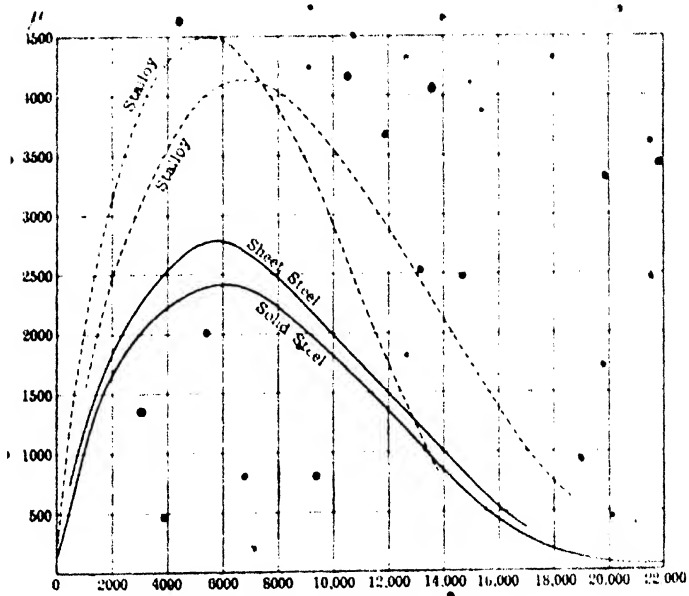


FIG. 212. Permeability curves.

effect of hysteresis is very slight when the iron is strongly magnetized or "saturated," and since in dynamo field-magnets the iron is usually magnetized to a fairly high induction of over 14,000 in wrought iron and steel or 6,500 in cast iron, the value of B for a given magnetizing intensity will differ little, however the latter has been arrived at—that is, whether the magnetizing intensity

[1] For various methods of comparing or measuring permeabilities, some of them suitable for practical use in the workshop, see the paper quoted in the last note but one, and Du Bois, *The Magnetic Circuit*, Chap. XI; Prof. Ewing, "The Magnetic Testing of Iron and Steel," *Magnetic Induction in Iron and other Metals*, 3rd ed., Chap. XII; Lamb and Walker, *Journ. I.E.E.*, Vol. 30, p. 930; Drysdale, *Journ. I.E.E.*, Vol. 31, p. 283; W. H. F. Murdoch, *Journ. I.E.E.*, Vol. 40, p. 137; A. Campbell and D. W. Dye, "The Magnetic Testing of Bars of Straight or Curved Form," *Journ. I.E.E.*, Vol. 54, p. 35, and for a comparison of results obtained by different methods, Prof. J. T. Morris and T. H. Langford, *Phys. Soc. Proc.*, Vol. 23, Pt. IV, p. 277, 1911.

has been decreased down to or increased up to that value (see Chapter XVI, § 3).

§ 6. Magnetic hysteresis.—Reverting to the downward curve of Fig. 204, it is seen that the changes in the magnetic state of the iron are not coincident with the changes in the strength of the magnetizing intensity, but lag behind them; this is most forcibly exemplified by the residual magnetic induction which persists when the positive magnetizing intensity has been reduced to zero, and by the fact that the induction only becomes zero after the magnetizing intensity has reached a definite negative value. The physical fact here described is known shortly as the *magnetic hysteresis*[1] of the iron. It should be noted that this is not a lagging behind in point of *time*, since the actual time taken for the changes in the value of the magnetizing intensity (provided they be not extremely rapid) is immaterial. The magnetic hysteresis depends merely on the order of succession of the different current strengths, and even long intervals of waiting, during which the magnetizing intensity is kept constant, do not obliterate the distinction between ascending and descending curves. What is really implied by the term "magnetic hysteresis" is that if the magnetizing intensity is reduced from a stronger to some weaker value, the rate at which the magnetic induction becomes reduced with reference to the magnetizing intensity, or the slope of the downward curve, is less than the rate at which it increased when the magnetizing intensity was raised from the weaker up to the stronger value; thus, if at point R the magnetizing intensity be again reapplied in a positive direction, the rate at which the induction is recovered is again less than the rate at which it was lost for a change of magnetizing intensity within the same limits.

Let us now continue the descending curve from the point C onwards by increasing the negative magnetizing intensity, and so reversing the direction of the induction. After reaching some point E', let us gradually reduce the negative magnetizing intensity, reverse it, and increase its strength in the former positive direction. A new ascending curve marked III is thus traced, which again shows hysteresis; it differs in shape from the previous ascending curve, which started with the iron in a neutral or "virgin" state, but is analogous to the descending curve II. By carrying the new ascending curve up far enough it will eventually cut curve II, say at E, and we thus arrive at the same point whence we started to trace the descending curve. A complete *loop* has therefore been described, and the two curves Nos. II and III enclose a certain area depending upon the extent to which they diverge from one another between the points E and E'. The magnetizing intensity has been taken through a cycle of changes in direction and value,

[1] From Greek ὑστερέω, to lag behind.

eventually returning to the same point as that from which the cycle began, and the iron has similarly been taken from a positive to a negative induction and back again. It is not, however, necessary that the magnetizing intensity should be actually reversed in order that the curve of induction may describe a complete loop; it is sufficient to withdraw partially the magnetizing intensity and then reapply it. Thus, at any point on the ascending curve the gradual increase of the magnetizing current might be suspended, and it might be first reduced and then again increased; a small loop would then be traced on the induction curve inside the last one.[1]

§ 7. Dissipation of energy in heat by magnetic hysteresis.—Now it can be shown that the area of any complete loop formed by

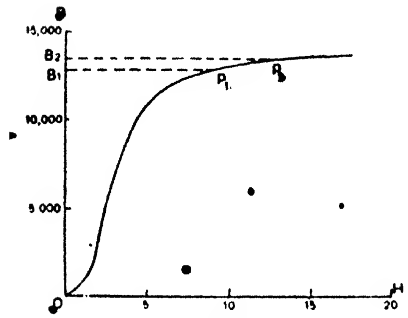


FIG. 213.

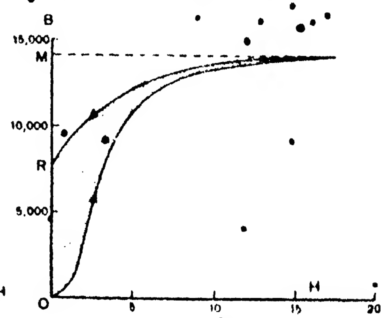


FIG. 214.

taking iron through a complete cycle of magnetic induction represents a certain amount of energy which must be spent per unit of volume in performing this cycle; *i.e.* it represents on the C.G.S. system a definite number of ergs of work done in taking each cubic centimetre of iron through the cycle of induction traced by the loop.

The principle from which this follows is established as follows. Taking the ascending *B-H* curve for a closed ring of iron magnetized by an exciting coil of T turns carrying A amperes, let the induction be raised from B_1 to B_2 (Fig. 213) in time t. All the lines are linked with all the turns T, and the increase of the former from B_1a to B_2a (where a is the cross-sectional area of the iron in square centimetres) must have produced in the exciting coil a back E.M.F. of average value $a \cdot \dfrac{B_1 - B_2}{t} \cdot T$. Or, if the points p_1 and p_2 are taken so close together that the infinitely small increase dB in the infinitely small time dt is considered, the time-rate of increase of the linkages of lines is $\dfrac{dB}{dt}$, and the back E.M.F. in the exciting coil is $-a \cdot \dfrac{dB}{dt} \cdot T$.

[1] *Vide* Ewing, *Magnetic Induction in Iron and other Metals*, 3rd ed., Chap. V, pp. 94–96.

The source of the magnetizing current must accordingly do work in overcoming this back E.M.F. at a rate equal to the product of the current and the back E.M.F., or $a \cdot \frac{dB}{dt} T \frac{A}{10}$ ergs per second, where $\frac{A}{10}$ is the value of the magnetizing current in C.G.S. units which may be regarded as constant during the infinitely short time dt. Hence the total work done during the time dt by the current in producing the change of induction dB is $a \cdot \frac{dB}{dt} T \frac{A}{10} \cdot dt = a \frac{AT}{10} \cdot dB$ ergs. But $\frac{AT}{10} = \frac{Hl}{4\pi}$, so that the work done or energy expended on the magnetic field is $a\,l \frac{H}{4\pi} dB$. Further, al is equal to the total volume of iron in cubic centimetres, and as each cubic centimetre may be considered as having a proportionate amount of work done on it, the work expended per cubic centimetre is $\frac{1}{4\pi} HdB$, or $\frac{1}{4\pi}$ of the small area $p_1B_1B_2p_2$, if one unit of length along the horizontal represents one C.G.S. unit of magnetic intensity, and along the vertical represents one C.G.S. unit of induction. If we extend the same process and raise the induction from any value B_1 to any other value B_2, the work expended per cubic centimetre is equal to

$$\frac{1}{4\pi}\int_{B_1}^{B_2} H \cdot dB;$$

thus, for example, if in Fig. 214, which repeats a portion of Fig. 204, the induction is raised from zero up to E, a total amount of work is expended in each cubic centimetre of

$$\frac{1}{4\pi}\int_0^E HdB,$$

which is equal to $1/4\pi$ of the area $OEMO$. The maintenance of the magnetic field at E or at any other constant value involves no expenditure of energy; the passage of the exciting current I in the magnet bobbins of resistance R does indeed involve a continuous expenditure of energy at the rate of I^2R watts, but as this is dissipated entirely in heating the copper wires of the coils, it is not to be debited directly to the magnetic field, and as we have said is only necessary owing to the lack of retentiveness in soft iron or steel. Next, let the magnetizing current be lowered from the value corresponding to E down to zero; the decrease of the lines now causes a forward E.M.F. assisting the current in the exciting coil, and work is thereby done in virtue of the energy stored in the field. The amount so recovered is, however, only equal to $\frac{1}{4\pi}$

of the area EMR, since the descending curve does not follow the same course as the ascending curve; hence it is less than the amount expended by $1/4\pi$ of the shaded area ORE. It is not, however, proved how much is irrecoverable until a complete cycle of induction has been traced, and the iron has been brought back to the same state as at starting. Such a complete cycle is given in Fig. 204, from which it is clear that the total amount lost per cubic centimetre of iron is equal to $1/4\pi$ of the area of the closed loop terminated by the maxima values E and E' of the induction. The energy that is irrecoverably lost in any such cyclic process is dissipated throughout the iron in heat. Thus the creation of a magnetic field in iron as opposed to air is not a perfectly reversible process, since some portion of the energy expended is not stored but entirely lost in heating the iron. This loss is to be sharply distinguished from the loss by eddy-currents which must to a small extent be present even in finely laminated iron. In a given mass the latter loss is proportional to the square of the frequency or number of cycles per second, while the former is simply proportional thereto, *i.e.* the one varies as the square of the speed, the other as the speed, and by this difference the two losses may be separated out (Chapter XXI, § 16). Eddy-currents may be practically eliminated by very carefully subdividing the iron (Chapter XIII, § 1), or by causing the induction to change very slowly. But no subdivision or lamination of the iron will eliminate the loss by hysteresis; nor is it reduced, however slowly the cycle be performed, as is shown by the expression $a \cdot \dfrac{dB}{dt} T \dfrac{A}{10} dt$, from which dt disappears. Further, the loss by hysteresis does not react magnetically upon the field, while eddy-currents, however minute, screen the iron against induction, and by reason of their M.M.F. affect the field-strength. Indeed the only deviation from the law, that the hysteresis loss is proportional to the frequency or speed, is due to the screening action of eddy-currents which may still be present if the thickness of the iron laminations be appreciable.

§ 8. Hysteresis loss in alternating field.—The effect of hysteresis is most marked on the second or steep part of the curve (b, Fig. 204), *i.e.* if the limits within which the magnetizing intensity is cyclically varied fall within or embrace that portion of the curve; cyclic changes of a strong magnetizing intensity taking place entirely on the upper flat portion of the curves of Fig. 205, *i.e.* when the iron is "saturated," show so little hysteresis that its effect is almost negligible, and no difference is discernible in the ascending or descending curves (*vide* Fig. 205). On the steep portion of the curve a cycle of changes in the strength of the magnetizing intensity once performed will cause the induction curve to describe a loop, but the crossing of the descending and ascending curves may not coincide

with the starting-point whence the descent was begun; hence a repetition of the same cycle of changes will not cause the loop to be exactly retraced. If, however, the magnetizing-intensity cycle be repeated a few times, the iron will eventually reach such a state that the same loop will be continually retraced, and the change of magnetic induction will itself become strictly cyclic and coincident with the magnetizing-intensity cycle. The need for several repetitions of the magnetizing cycle in order that the same loop may be exactly retraced each time is much less marked when the two limiting values of the loop are high up on the positive and negative curves, and in the drawing of Fig. 204 it has been assumed that the maximum negative induction at E' is equal to the positive induction at E, and is reached for the same value of magnetizing intensity as corresponds with E on both the two ascending curves. Thus, for simplicity's sake, only three curves are shown, and the cycle of induction of curves II and III is assumed to have been reached at once, and to be immediately capable of exact repetition. But, even if it cannot be so quickly established, yet when the magnetizing intensity is continuously varied between two fixed values in either direction, the magnetic effect soon also becomes cyclic, and the induction likewise varies between two values in each cycle. If, therefore, the loops obtained when the iron is carried from a strong positive induction to an *equally* strong negative induction, and back again to the original starting-point, are determined for several different values of the maximum induction, it will be found that the area of the loops, and therefore the energy dissipated in heat in each complete cycle, depends upon the nature of the material and also upon the maximum induction up to which the iron is carried. The area of a loop being approximately equal to a rectangle having a base equal to twice the coercive intensity and a height equal to twice the maximum induction, the amount of the "loss by hysteresis" in ergs for any cycle, or $1/4\pi$ of this area, may approximately be said to be equal to $\dfrac{\text{coercive intensity} \times \text{maximum induction}}{\pi}$; thus in soft annealed iron it is very small, even for a cycle of high induction, but in certain steels it becomes very considerable, amounting to as much as 200,000 ergs per cubic centimetre per cycle in tungsten steel. Taking, however, any one substance, it has been found that the loss of energy by hysteresis in an *alternating field* is not proportional to the maximum induction, but increases more rapidly at least up to values of the induction below 16,000.

In Fig. 215[1] is given a curve showing for different values of the maximum induction of the cycle the hysteresis loss of a sample of

[1] From *Phil. Trans.*, 1896, Vol. 187, A., pp. 715–746; "The Hysteresis of Iron and Steel in a Rotating Magnetic Field."

soft iron tested by Prof. F. G. Baily in an alternating field. Steinmetz has shown that over a considerable range, say, from $B = 1000$ to $B = 14{,}000$, the loss from the alternating cycle is approximately proportional to the 1·6th power of the maximum induction. Hence the specific loss in ergs per cubic centimetre of iron and per cycle could be expressed by the empirical formula $\eta B^{1\cdot6}$, where η is the *hysteretic constant* of the iron in question. Since one erg $= 10^{-7}$ joule, the specific loss in joules is

$$h = \eta B^{1\cdot6} \times 10^{-7}$$

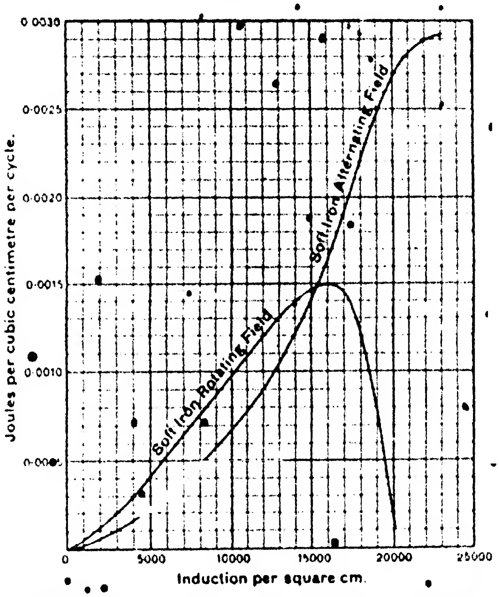


FIG. 215. Dissipation of energy in iron through hysteresis in alternating and rotating fields.

The value of η varies with different kinds of iron and steel between such wide limits as from 0·0015 to 0·08. It is reduced by careful annealing, but the exact effects of various processes or of chemical impurities cannot be said to be settled. For average samples of annealed sheet iron or sheet steel such as are used in the construction of armature cores, η may be taken as $= 0{\cdot}003$, while for exceptionally good transformer iron it is as low as 0·0015. In Fig. 216 is shown the curve connecting the joules expended in a cubic centimetre in each complete cycle with the maximum induction for a hysteretic constant of $\eta = 0{\cdot}003$, and a comparison with Fig. 215 will show the closeness of the correspondence of the calculated and the experimental curves up to $B = 14{,}000$. The right-hand scale

gives the corresponding joules per lb. of metal (1 lb. = 58·5 to 59 cm.^{3} of iron, according to the density of the material, so that 1 joule per c. cm. = approximately 60 joules per lb.). If V_e = the volume of a given mass in cubic centimetres, and $f = \frac{pN}{60}$ is the number of complete cycles per second, the total hysteresis loss per second, or the rate at which heat is generated in the iron is

$$H_w = hfV_e = \eta B^{1\cdot 6} \, . \, 10^{-7} \, \frac{pN}{60} \, . \, V_e \text{ joules per second or watts} \quad . \quad (96)$$

and the value of $\eta B^{1\cdot 6} 10^{-7}$ or h may be read off the left-hand scale.

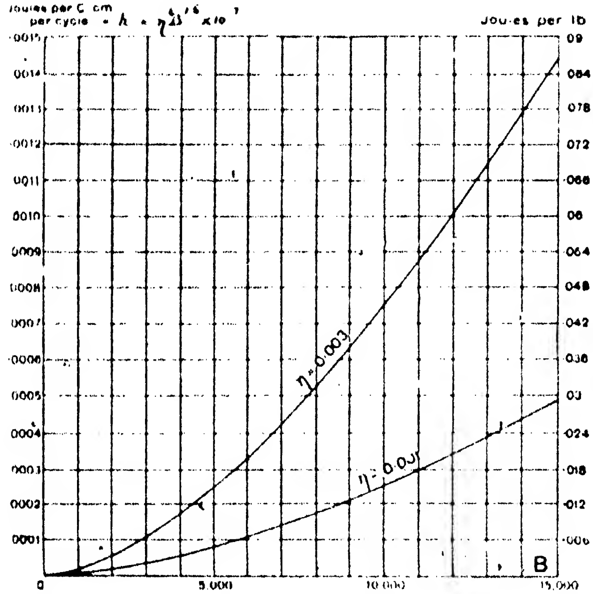


FIG. 216.—Value of h in an alternating field for η = 0·003 and 0·001.

The exponent of B is, however, not really constant over any long range, and when the induction is raised to a very high value, the empirical formula of Steinmetz no longer holds even approximately true ; the rate at which the curve rises falls off as the iron approaches saturation,[1] and eventually the loss reaches a nearly constant value (Fig. 215). The point of flexure occurs in good soft iron at about B = 15,000, and the curve becomes fairly flat for values of B above 23,000. The tendency towards a constant value does

[1] Professor Baily, *Electr.*, Vol. 36, p. 116: "The Hysteresis of Iron in an Alternating Magnetic Field." See *also especially*, Prof. W. M. Thornton, "The Hysteresis Loop and Index," *Electr.*, Vol. 71, p. 214.

not, however, appear to be so clearly marked in all kinds of iron as in the case of steel.[1]

The loss by hysteresis when iron is taken through an unsymmetrical cycle is greater than when it is taken through a symmetrical cycle for the same total difference of flux-density,[2] and increases with the maximum flux-density reached during the cycle.

§ 9. Comparison of different grades of iron and steel. For the practical comparison in the workshop of the hysteresis loss in different samples of sheet steel, the hysteresis tester of Professor Ewing is the most convenient instrument. The specimen is rotated between the poles of a magnet free to rotate about the same centre, and the deflection of the latter is compared with that produced by standards with a known hysteresis loss.[3] A number of specimens should be tested in order to obtain an average result, since even samples taken from the same sheet show considerable variations among themselves.

According to the experiments of Prof. E. Wilson, V. H. Winson, and G. F. O'Dell[4] a sample of "Lohys" as then supplied by Messrs. Joseph Sankey & Sons, Ltd., gave between $B = 3{,}780$ and $7{,}970$,

$$h = 0{\cdot}00148\ B^{1{\cdot}59} \times 10^{-7} \text{ joules per cycle and per c. cm.}$$

or, say, approximately

$$h = 0{\cdot}0015\ B^{1{\cdot}6} \times 10^{-7}.$$

Both hysteresis and eddy-current losses are reduced by alloying the steel with silicon,[5] the latter effect being due to the greatly increased electrical resistivity of the material, of which more will be said later. A sample of "Stalloy" supplied at the same time to the experimenters above quoted gave

$$h = 0{\cdot}000363\ B^{1{\cdot}71} \times 10^{-7} \text{ between } B = 629 - 6{,}050$$
$$\text{and } \quad = 0{\cdot}000321\ B^{1{\cdot}72} \times 10^{-7} \text{ between } B = 6{,}050 - 11{,}500,$$

which may also be approximately represented by

$$h = 0{\cdot}00095\ B^{1{\cdot}6} \times 10^{-7} \text{ joules per cycle and per c. cm.,}$$

or more accurately with η rising from 0·000735 to 0·0011 between $B = 629$ and $B = 13{,}480$. For values of B from 17,500 to 19,500, Mr. J. S. Nicholson[6] found $h = 0{\cdot}00125\ B^{1{\cdot}58} \times 10^{-7}$.

[1] Cp. Beattie and Clinker, *Electr.*, Vol. 37, p. 727.

[2] M. Rosenbaum, *Journ. I.E.E.*, Vol. 48, p. 534; F. Holm, *E.T.Z.*, Vol. 33, p. 928.

[3] *Journ. I.E.E.*, Vol. 24, p. 398, and *Proc. C.E.*, Vol. 126, May, 1896. Cp. Prof. Fleming, *Electrical Engineer*, Vol. 20, p. 366. Although a rotary motion is employed, the process of reversal resembles that of the alternating rather than of the rotating field.

[4] *Electr.*, Vol. 61, p. 721 (1908, 21st Aug.).

[5] *See* Barrett, Brown and Hadfield, *Journ. I.E.E.*, Vol. 31, p. 715 ff.

[6] *Journ. I.E.E.*, Vol. 53, p. 253.

Other tests of high-resistivity materials[1] have given

$$h = 0{\cdot}00155 \text{ to } 0{\cdot}0009\ B^{1\cdot6} \times 10^{-7} \text{ joules,}$$

and Dr. Kollben[2] has given for low-carbon iron sheets alloyed with 1·07, 2·28, 3·25 and 3·5 per cent. of silicon respectively the values of η as 0·0014, 0·001, 0·0009 and 0·0009. The above figures are cited to show the empirical nature of the exponent of B, and the great variations in the values of η which follow from it.[3]

If in an alternating field $\eta = 0{\cdot}001$ and the 1·6th power of B be assumed as correct (Fig. 216), a convenient formula is

$$hf = 63 f\left(\frac{B}{1{,}000}\right)^{1\cdot6} \times 10^{-7} \text{ watts per c. cm.}$$

$$= 0{\cdot}00037 f\left(\frac{B}{1{,}000}\right)^{1\cdot6} \text{ watts per lb.}$$

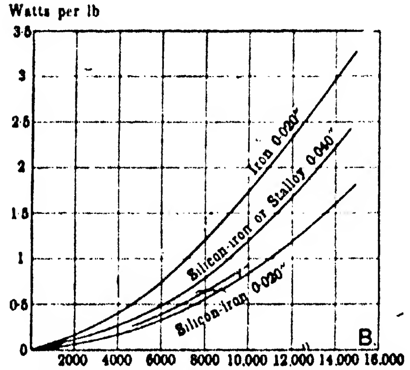


FIG. 217.—Total loss by hysteresis and eddy currents in an alternating field at 50 cycles per second.

whence for any other value η' it is only necessary to multiply the constant or the lower curve of Fig. 216 by $\eta' \times 10^3$.

But more usually the *total* loss from both hysteresis and eddy-currents in an alternating field is measured by a wattmeter with an *alternating* current,[4] and the results are therefore partly dependent

[1] A. Campbell, *Journ. I.E.E.*, Vol. 43, p. 560.

[2] Abstracted in *Engineering*, Vol. 87, p. 732.

[3] Cp. F. Stroude, *Phys. Soc. Proc.*, Vol. 24, p. 238, abstracted in *Electr.*, Vol. 69, p. 606, 19th July, 1912.

[4] See especially Prof. J. Epstein, "The Testing of Electrical Machinery, and of the Materials for its Construction," *Journ. I.E.E.*, Vol. 38, p. 33 ff.; A. Campbell, "On Magnetic Testing of Iron with Alternating Current," *Journ. I.E.E.*, Vol. 43, p. 553; and L. Wild, "The Testing of Transformer Iron," *Journ. I.E.E.*, Vol. 46, p. 217; Dr. S. Guggenheim, *E. u. M.*, Vol. 28, pp. 336, 337, abstracted in *Electr.*, Vol. 64, p. 539, and other papers, cited by Prof. Miles Walker in *Specification and Design of Dynamo-electric Machinery*, p. 86.

upon the thickness of the stampings. For purposes of comparison it is usual to take as the standard conditions a frequency of 50 cycles per second and maximum $B = 10{,}000$ (Fig. 217). The total loss in watts per lb. by both hysteresis and eddy-currents in an alternating field should then not exceed the following amounts—

Thickness in inches.	*Lohys.*	*Special Lohys.*	*Medium Resistance.*	*Stalloy.*
0·025	2·00	1·63	1·43	0·88
0·020	1·59	1·33	1·15	0·8
0·014	1·32	1·09	1·00	0·69

The total loss is naturally reduced to about two-thirds by the high electrical resistivity of silicon steel to which allusion has already been made. Dr. Kolben's tests gave for sheets, 0·020" thick, alloyed with 1·07, 2·28, 3·25 and 3·52 per cent. of silicon 1·36, 1·07, 0·93 and 0·84 watts per lb. respectively. The specific resistance of these sheets was 28·7, 35·5, 50 and 59·5 microhms per cm. cube, and their increase of resistance per 100° C. rise 19, 14, and 5 per cent. respectively, so that the high-silicon alloys have a low temperature coefficient. The resistivity of the high-resistance alloyed materials[1] ranges from 43 to 60 microhms as compared with 11 to 15 microhms for ordinary sheet steel with 3 or more per cent. of carbon.

The specific resistivity of Stalloy[2] is about 49·7 microhms or four times that of ordinary stampings. Its permeability when carefully annealed can reach very high figures[3] such as 4,140 at $B = 6{,}500$, and Prof. Wilson found 4,470 at $B = 6{,}050$. Thus at low inductions the permeability exceeds that of the best iron, but this is not maintained at high inductions; *e.g.* the permeability at $B = 13{,}480$ may fall to 832 (Fig. 212). At $H = 140$, Dr. Kolben gives values for B of 17,300, 17,000, 16,700 and 16,400, falling with the percentage of silicon.

Owing to the greater cost of alloyed steel its use in dynamo work at present comes into question only in the case of extra high-speed turbo-generators and turbo-alternators, where the frequency is high, and it is very necessary to limit to the utmost the heat generated per unit volume of the armature. Dr. Kolben's tests showed that with 3 or more per cent. of silicon the sheets become very brittle with little elongation, although the ultimate tensile strength increases; such sheets are therefore difficult to stamp and are unsuitable for armature cores. But with a lesser silicon content as in "Medium Resistance" quality, mechanical properties

[1] Cp. also A. Campbell, *Journ. I.E.E.*, Vol. 43, p. 560; Barrett, Brown and Hadfield, Vol. 31, p. 681.

[2] Prof. E. Wilson, V. H. Winson, and G. F. O'Dell, *Electr.*, Vol. 61, p. 721.

[3] H. R. Hamley and L. Rossiter, *Electn.*, 3rd Nov., 1911.

may be obtained in comparison with those of "Stalloy" as shown by such approximate figures as the following—

	Elongation on 4 ins.	*Yield point in tons per sq. in.*
Stalloy	6 per cent.	25
Medium Resistance . .	20 ,,	22
Lohys	18 ,,	12

§ 10 Ewing's molecular theory of magnetism. At this point a brief outline may be given of the modern theory of magnetism which satisfactorily accounts for the very large range of facts that have been brought to light by experimental research.[1] The *molecular theory*, as it is termed, starts with the fundamental assumption that the molecules of a magnetic body are already and always permanent magnets endowed with polarity, and that the process of magnetization consists in turning these small magnets so that the direction of their magnetic axes coincides more or less with the direction of the magnetizing intensity due to some external cause. The actual distances between the centres of the molecular magnets are supposed not to be changed (save by the effects of mechanical stress or heating), so that each magnet is only capable of rotation about its centre. Since, however (as the ascending curve of Fig. 204 shows), a strong magnetizing intensity is required to produce a high magnetic induction in iron, it is evident that the molecules cannot be perfectly free to turn and set themselves along the direction of the magnetizing intensity; for in that case, even when a weak magnetizing intensity was applied, they would all swing round into line with its direction, and the iron would at once become entirely "saturated." They must therefore experience some restraint, and the exact nature of this restraint was for long a stumbling-block. But in 1890 the experiments of Professor Ewing led him to return to the simplest hypothesis, namely, that it is the mutual attraction and repulsion of the molecular magnets which supplies the constraining force, and hinders their immediate alignment into perfect parallelism with the magnetizing intensity. By means of a model visibly representing the supposed molecular structure of a magnetic body he was able to imitate almost all of the phenomena of magnetism, and thence to deduce the following theory.[2]

The molecules of a piece of iron when in a neutral state are assumed to be arranged in groups; these are not necessarily identical in configuration, but each of them is stable, and has no external magnetic effect, the attractions and repulsions of each magnet being satisfied within its own group. For small displacements of its members the whole group remains stable; but if the members are turned through a sufficiently great angle, and the group is distorted, one or more members become unstable, and their equilibrium is liable to be upset. The result is that for a slight increase of the displacement the whole group becomes broken up, and has to be partially or wholly rearranged and reconstructed after a new plan. From the internal structure above described it follows that when a group is subjected to a magnetizing intensity which is gradually increased in strength from zero, the first effect is to produce a *stable* deflection of all the component members, except those which lie exactly along or opposite to the direction of H, and the general lines of the group are still retained. This corresponds with the initial stage of magnetization (*a*, Fig. 204) when the induction increases at a slight rate almost proportional to the increase of the magnetizing intensity. But now let the value of H be increased still further; the members of the group are still further deflected, until at last one or more become *unstable*. The ties which

[1] For a further theoretical development, which combines Ampere's hypothesis of molecular currents with Ewing's theory, see the valuable paper on "Permanent Magnets in Theory and Practice," by S. Evershed, *Journ. I.E.E.*, Vol. 58, p. 780.

[2] "Contributions to the Molecular Theory of Induced Magnetism," by Professor Ewing, now embodied with additions in the same author's *Magnetic Induction in Iron and other Metals*, Chap. XI.

bound them are then ruptured, and by the intermolecular attractions and repulsions they and their neighbours are constrained to change entirely their grouping, and take up some new configuration, the main lines of which agree more closely with the direction of H. In general, with a considerable number of different groups, this stage of instability will not be reached by them all simultaneously at one given value of H, but group after group will gradually become unstable and break up. This second stage, when the groups are one after another passing through an unstable condition, corresponds to the steep part of the ascending curve (*b*, Fig. 204) when the induction increases at a rapid rate. If H be still further increased in strength, we have the third stage (*c*, Fig. 204), in which the alignment of the molecules becomes gradually more and more perfect; as each molecule is pulled more and more into line with H, the iron becomes more and more "saturated," and its permeability decreases. The deflections of the molecules in their third stage are, however, again stable, as in the first stage. When the saturation intensity has been reached, all the magnetic molecules will set themselves along the line of direction of the magnetizing intensity without further interaction among themselves.

But now, if after stage 2 the intensity H is gradually removed, the majority of the groups, having swung over to a new stable condition, retain their new configuration; hence, if the intensity be reduced to zero, there is still a considerable amount of residual magnetic induction, and it will require the magnetizing intensity to be actually reversed to produce a condition of instability in the new groupings, and so cause them to be in their turn upset and replaced by fresh configurations. Thus the gradual removal of the magnetizing intensity does not lead to the exact and complete repetition backwards of what happened when the magnetization was being increased; in other words, the movements of the molecules are not reversible without qualification. Given, however, a sufficient number of groups composing the body, if the magnetizing intensity be removed and reapplied, then, unless it be very weak, there are, in general, some groups which pass through a condition of instability. Especially will this be the case if the piece of iron be not perfectly homogeneous, and if, therefore, the lines of the different groups or chains are differently inclined at different places. This exactly corresponds with the observed facts, that hysteresis is always present in all cyclic changes of H, but that its effect increases rapidly for changes extending over the second stage of the curve of induction. The approach to a steady value of the hysteresis loss in the third stage may also be expected; thus in Fig. 204 if the iron had reached its saturation intensity at point E, the area of the closed curve should give the final maximum of the hysteresis, any increase of the induction beyond E simply carrying the molecules with it in complete alignment without the passage through fresh configurations.

The phenomenon of hysteresis is thus amply accounted for. It occurs whenever on the ascending curve a molecule is deflected from one stable position to another through a position of instability; since then, on the descending curve, the new position will persist until the change in the magnetizing intensity becomes so marked as to cause it to pass again through a position of instability. Further, whenever a molecule passes through a position of instability, energy is dissipated in the form of heat; its equilibrium being upset, it acquires kinetic energy in falling over towards a new position of equilibrium, about which it then oscillates and causes its neighbours to oscillate. How the oscillations are damped out and converted into heat is not yet precisely known; the damping cannot be due to mechanical friction, and is more probably due to some form of molecular eddy currents. That mechanical vibration should lessen the residual magnetic induction is easily explicable on the molecular hypothesis, since it will cause changes in the distances between the molecular centres. During the swinging thus set up, as the magnets recede from each other their stability is reduced, so that they respond more easily to change of magnetizing intensity, and hysteresis is lessened.

§ 11. Hysteresis in dynamo armatures.—All that is required in any material for dynamo magnets is high permeability under strong magnetizing intensities; its hysteresis and loss of energy in consequence thereof are of no

interest to the designer. It is different when we come to the materials for armature cores, where both high permeability and low hysteresis are desirable. The question of the hysteresis in armatures requires, therefore, some further consideration.

The results described in § 8 are obtained when iron is subjected to a magnetizing intensity, of which the direction is always along the same line, although it alternates in "sense" from a + to a − maximum. But such linear alternation of magnetization, as it may be called, does not strictly correspond to the case of the armature core of a dynamo.

In the heteropolar ring or drum armature, whether of an alternator or continuous current machine, the magnetizing intensity varies also in its spacial direction relatively to the armature core, and the characteristic feature is that the molecules of iron must adjust themselves *spacially* to this varying direction. As they are carried round with the armature, they would tend to adjust themselves continuously to the changing direction of the flux. The maximum strength of field occurs approximately midway between the poles, and the minimum at the centre of a pole, where the magnetizing intensity only dies away to zero as we penetrate to the inner layers of a core of considerable radial depth. Hence, as the armature revolves, each molecule of iron in the body of the armature core retaining its alignment with the direction of the field from pole to pole, even while rotating, is in effect twisted through an entire circle of 360° in each period corresponding to the passage past a pair of poles.

§ 12. Hysteresis in a rotating field.—It is therefore clear that to imitate the case of a heteropolar ring or drum armature a piece of divided iron must be rotated in a constant magnetic field, or conversely a constant magnetic field must be rotated about a stationary iron armature. The logical deduction from Professor Ewing's theory as applied to such a case was first pointed out by Mr. Swinburne. During the first or quasi-elastic stage corresponding to a weak field, the hysteresis loss should increase but slowly; during the second stage, when the groups of little magnets are passing through irregular combinations, the loss should increase rapidly and approximately in proportion to the induction, but finally as saturation is approached the hysteresis curve should reach a maximum and then not merely remain constant as in the case of an alternating field, but bend over and fall rapidly towards zero. When the magnetizing intensity is never removed, and at each point in the path of the molecules is of such strength as to keep them in perfect alignment, even though rotated, there would be no opportunity for them to pass through a position of instability and so to break up into new combinations. This view, which was first advanced as an objection to the molecular theory, has since been amply verified by direct experiment, and furnishes additional evidence of the correctness of the theory. Indeed, by means of an ingenious model in which a number of little magnets are rotated in a constant magnetic field we are enabled to watch the actual progress of the phenomena; the spasmodic breaking up of the initial configurations under the strain of rotation in a field of moderate strength is rendered visible, and we can trace the gradual increase of alignment, until finally the field becomes of sufficient strength to maintain the magnets pointing always in a definite direction, however quickly the frame which carries them is spun round.[1] In the early stages the *hysteresis loss of iron in a rotating field*, or of iron rotating in a constant field, although of the same order as in an alternating field, is somewhat greater.[2] This may be due to the more gradual change of the rotating field, and also to the smaller choice in the direction of movement of the molecules,

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[1] The first experimental verification of this fact was communicated to the British Association by Professor F. G. Baily in 1894 (*Electrician*, Vol. 33, p. 516), and a model illustrating it was shown by Professor Ewing in 1895 (*Electr.*, Vol. 34, p. 670), and *Magnetic Induction in Iron and other Metals*, Chap. XI).

[2] Cp. Holden, *Electr.*, Vol. 35, p. 329; Hiecke, *E.T.Z.*, Vol. 23, p. 142; P. Weiss and V. Planer, *Journ. de Physique*, Vol. 6, p. 5, and G. Vallauri, *Atti dell' Assoc. Elettr. Ital.*, Vol. 13, p. 437, p. 1909.

so that some combinations offer more resistance to dissociation, and on their rupture the oscillation is greater.

In Fig. 215 the second curve shows the results obtained by Professor Baily for the hysteresis loss in a rotating field[1] for a small cylinder made up of soft-iron discs cut from the same sheet as that for which the first curve shows the loss in an alternating field; the latter corresponds to a hysteretic constant η about 0·003, the former to a value somewhat greater than 0·004 so long as the curve is rising rapidly. Indeed, for $B = 8{,}000$ or 10,000 the hysteresis loss in the rotating field is some 50 per cent. higher than in the alternating field. But when B the average induction across a section of the core at the spot where it has its maximum value $\left(\dfrac{\text{half flux of pole}}{\text{cross section of core}}\right)$, reaches 15,000, the loss becomes less when

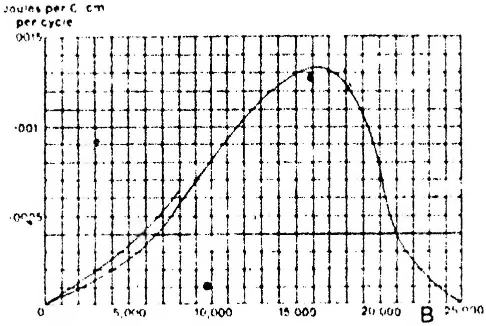


Fig. 218.—Hysteresis loss in rotating field.

reversal takes place by rotation, and at about 16,000 or 17,000 it attains a maximum; it thence decreases rapidly, and almost disappears at $B = 21{,}000$, but with a slight tendency for the curve to turn off as it approaches the zero axis.[2] In the complete curve of Fig. 218 are given the results of tests made by Messrs. Beattie and Clinker[3] on sheet iron; the initial portion up to $B = 14{,}000$ agrees roughly with a hysteretic constant of 0·003. The upper short curve of the same figure is obtained from tests made by Professor Baily[4] on an actual dynamo armature. Of two specimens tested by Mr. Holden[5] up to $B = 8{,}000$, one gave results agreeing fairly closely with the upper and the other with the lower of the curves of Fig. 218, so that they may be taken as representing average

[1] *Phil. Trans.*, 1896, Vol. 187, pp. 715–746.
[2] R. Czepek's experiments (*E. u. M.*, Vol. 28, p. 325, April, 1910) do not, however, confirm this.
[3] *Electr.*, Vol. 37, p. 723.
[4] *Ibid.*, Vol. 44, p. 323.
[5] *Ibid.*, Vol. 35, p. 327.

cases. The exponent was found by Mr. Holden to vary in a number of samples from 1·4 to 1·7, the average being 1·5.

The flux in a cylindrical armature core follows such lines that the total reluctance which their paths present is a minimum. Between any two points symmetrically situated within a pair of neighbouring poles the shortest path is the chord joining them together; if this were rigidly followed, it would cause such a concentration of the flux along a narrow band situated some little distance below the outer periphery on an interpolar section of the core that the permeability would be appreciably reduced owing to the great density, and a longer path with higher permeability would then offer less reluctance. The lines therefore spread out, but their density

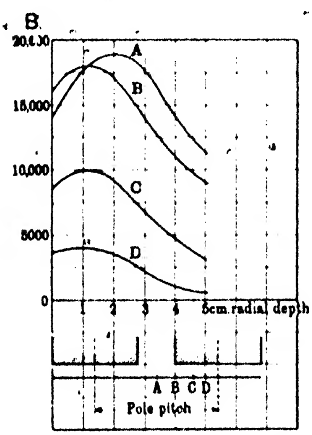


FIG. 219.—Inductions in smooth-core internal armature at different radial depths and different cross sections.

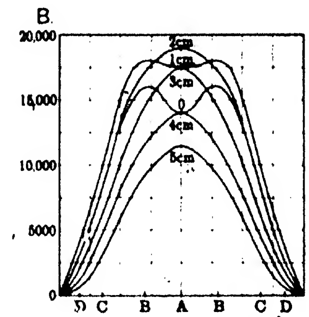


FIG. 220.—Half-cycles of magnetization at different depths in smooth-core internal armature.

across any section never becomes completely uniform. The distribution of the induction within a small armature core, both smooth and toothed, has been described by Prof. W. M. Thornton in a paper on "The Distribution of Magnetic Induction in Multipolar Armatures," read at the meeting of the British Association in 1904,[1] and in a second paper on the same subject published in *Journal I.E.E.* (Vol. 37, p. 125), on which the following is based (cp. Fig. 249). It is there shown that the depth within the core at which the maximum density is found is almost independent of the total flux and of the air-gap length, but depends upon the pole-pitch. In a smooth-core multipolar machine it occurs practically at the centre of a chord joining the points at which the centre lines of

[1] Reprinted in *Electrician*, 26th August, 1904. Cp. also W. E. Goldsborough, *Trans. Amer. I.E.E.*, Vol. 16, pp. 481–500.

the poles intersect the periphery of the armature. Further, if the induction is measured at different depths in a radial section of the core and with such radial section occupying the different planes marked A, B, C, D, curves similar to Fig. 219 are obtained in a smooth-core armature with particular values of the air-gap length and density, ratio of polar arc to pole-pitch, and shape of pole-tip; with other values of these latter quantities the curves are of different shape, but Fig. 219 may be taken as a typical construction to show the approximate values of the inductions when the polar arc is 70 per cent. of the pole-pitch and a mean induction of $B_c = 16{,}000$ at the centre of the interpolar gap is

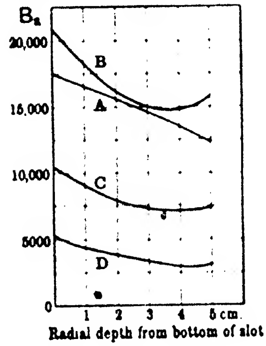


FIG. 221. Inductions in multipolar toothed armature core

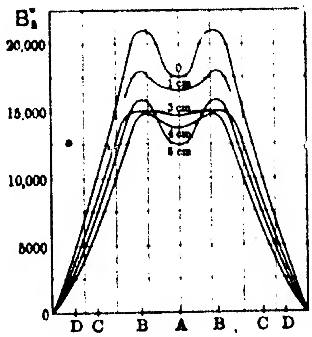


FIG. 222. Half cycles of magnetization at different depths in multipolar toothed armature core.

assumed in a smooth-core armature. From these curves, by replotting the ordinates for a particular depth in relation to the pole-pitch are obtained the curves of Fig. 220, which show the nature of the magnetic change which the molecules of iron at each depth pass through in a half-cycle. In a toothed armature core (Figs. 221 and 222) the density over the section midway between the poles with normal ratios of armature dimensions falls more nearly in a sloping straight line, and under the pole the density near the inner edge of the core may even rise again.[1] The true calculation of the hysteresis of a rotating armature would thus only be obtained if the core were divided into a number of small coaxial cylinders, and the hysteresis loss of each elementary cylinder passing through

[1] This rise is, however, only apparent; as pointed out by Prof. W. M. Thornton to the writer, the reason for it is that search coils in a radial plane under a pole are perpendicular to the flux at the outer face immediately under the slot of a toothed armature and also near the inner face of the core, but are not so strictly perpendicular near the centre of the core where the lines pass through the coils in a more slanting direction; hence the central search coils give a lower than the true value for the induction at right angles to the flux.

its particular cycle of magnetization could be estimated, their separate losses being finally added together. Such a process would be unnecessarily tedious in practice, but the above analysis serves to show that in the experimentally observed curve for a rotating field of, *e.g.* Fig. 215, for each value of the total flux-density averaged over the cross-section of the core between the poles where it reaches its maximum value, the separate losses in the indefinitely small cylinders are summed up for us in the particular case of the armature which was actually under test. The results are therefore to some extent dependent upon the length of air-gap, ratio of the polar arc to the pole-pitch, and shape of pole-tip which were actually employed. The air-gap in the case of Fig. 215 was 0·275 cm., and the ratio of

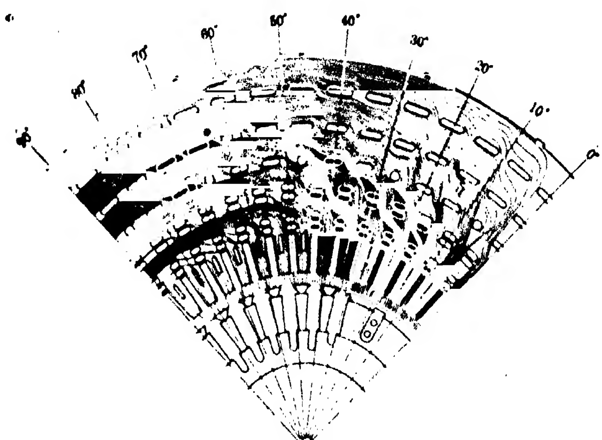


FIG. 223. Flux distribution in external armature core.

the polar arc to the pole-pitch was 0·666, which may be regarded as fairly approximating to practical conditions.

In the external stator core of a 15,000 kVA two-pole turbo-alternator, 110 in. in external diameter with a core depth of 25 in., one quadrant of which is shown in section in Fig. 223, with the approximate paths of the flux, experiment[1] showed that at each level behind the stator slots, the curve of flux density from pole to pole was practically sinusoidal at no load (Fig. 224). When the average density over the section of core where the flux was at its maximum (*i.e.* = one half of the total flux of the machine divided by the net area of the single core-section) was about 18,000 and 12,700, corresponding respectively to 12,800 and 8,500 volts,

[1] Carl J. Fechheimer, "Flux-distribution in the Core of a Turbo-alternator," *Journ. Amer. I.E.E.*, Vol. 39, p. 669, July, 1920. Through the courtesy of Mr. Fechheimer the writer has been enabled to reproduce here Fig. 223.

the density at the outside was about 0·9 times, and on the inside 1·2 times the average (Fig. 225), showing that the flux reached well back to the outside of the core, in spite of its greater length of path thereat. The range of density was not therefore great.

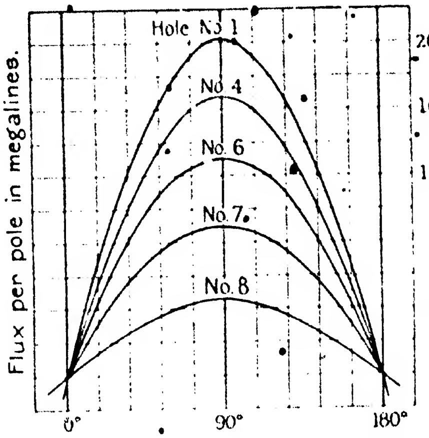


FIG. 224.—Flux-density over pole pitch of armature core.

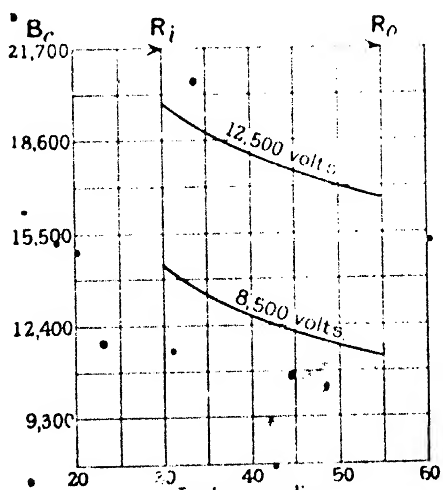


FIG. 225.—Flux-density in armature core.

§ 13. Influence of vibration and temperature on hysteresis in dynamos.—The mechanical vibration[1] to which an armature

[1] Cp. *Phil. Trans.*, 1896, Vol. 187, pp. 715-746.

is subjected when running reduces the loss from hysteresis only slightly, but possibly the much more rapid vibration due to the alternations of the current in an alternator armature, even when stationary, may affect the amount of the loss. Experiment shows that with increasing temperatures the hysteresis loss decreases by about 1 per cent. for each 10° C., so that when a dynamo is at work and thoroughly heated there may be a reduction of some 4 per cent. from the loss when cold. On the other hand, the body of evidence points to the fact that after the iron has been subjected to continuous and prolonged heating at a fairly high temperature, such as 65° C., there is a permanent increase in the loss[1] ("ageing"). The effect of temperature in the case of dynamos is, however, on the whole of small moment.

§ 14. **Practical calculation of hysteresis loss.**—In all calculations, therefore, as to the heating of armatures, or as to the efficiency of dynamos, some allowance must be made for a loss of energy by hysteresis, as estimated from the formula

$$H_w = h \cdot \frac{pN}{60} \cdot V_c \text{ watts} \quad . \quad . \quad . \quad (97)$$

In the teeth of a slotted armature, Prof. W. M. Thornton has shown by the analogy of stream-line photographs that the direction of the flux through a tooth while remaining vertically down its length under a pole gradually becomes oblique as the pole-tip is approached, and finally becomes horizontal or directly across tooth and slot mid-way between the poles, though never falling to zero (Fig. 249 *b*). The magnetization is therefore rotating, and at the same time fluctuates between such limits as 20,000 and 5,000 when the former limit is the maximum flux-density. In default of complete knowledge of the laws governing the hysteresis in such a case, it must suffice to fall back upon the rotating-field curves of a solid core as summing up for us the average effect for a given maximum value of B. Thus for both the body of the core and the teeth the value of h is to be taken from the rotating-field curves of either Fig. 215 or 219, and on this assumption it follows that the distortion of the field under load (as will be described in Chapter XIX), although probably increasing the loss, will not produce any great effect, since at very high densities the hysteresis loss does not with a rotating field continue to rise. In order to be on the safe side, the density at the top of the teeth may be taken as the value for B, since with a rotating field in the teeth this will usually give the largest hysteresis loss The process of calculating H_w is again referred to in Chapter XXI, § 21.

In continuous-current machines the frequency $pN/60$ on an average ranges from 10 to 40 for small outputs of a few kilowatts, and

[1] Cp. Professor J. Epstein, *Journ. I.E.E.*, Vol. 38, p. 36.

from 5 to 20 for large outputs up to 1,000 kilowatts. In dynamos driven by steam turbines the frequency even in large machines may be as much as 50, as *e.g.* would be given by 1500 revolutions per minute with a 4-pole field, and such a frequency is not uncommon in generators driven by steam turbines through gearing.

It will be found that in the above and even in the case of alternators with the normal frequency of 50 the total heat generated by hysteresis is comparatively small, and only becomes of consequence as increasing the heat due to other and more serious losses. Hence the permeability of the armature stampings is of more importance than their hysteresis; and since the one bears no direct relation, as far as is known, to the other, it is to the permeability that attention must be chiefly directed.

CHAPTER XV

FIELD-MAGNETS

§ 1. Economical limits to flux-density in magnets.—That the number of ampere-turns required per unit length of iron in order to produce a certain flux-density therein, increases very rapidly as the density is itself increased, is sufficiently shown by the *B-H* curves of Fig. 205. Thus in the case of iron forgings or steel castings, while a density $B = 16,000$ may be obtained with 20 to 30 ampere-turns for each centimetre length of the circuit in the iron, it requires nearly treble that number of ampere-turns per centimetre length to obtain the increased density of 18,000. But the excitation of the field by magnetizing coils implies, not only a certain first cost for the copper wire used therein, but also a continuous outlay during the working of the machine; for electrical energy is absorbed by the passage of the amperes through the turns, so long as the machine is at work. Evidently, therefore, there must be some approximate limits within which it is economical to keep the value of the flux-density in the field magnet. If the density falls below a certain minimum limiting value, the iron magnet becomes too heavy and expensive; in a self-exciting dynamo there is the further disadvantage that the magnetism may be unstable (as will be more particularly described in Chapter XVII, § 10), and the machine may become difficult to excite at all. On the other hand, if the density in the magnet be pushed up to a very high figure the ampere-turns required thereby will bear an excessive proportion to the whole number, and will involve too large an expenditure either in the first cost of the copper or in watt-hours during the working of the machine. Since, when designing a dynamo, the number of lines Φ_m to be carried by the iron of the magnet is approximately known, we are enabled from the above considerations to determine approximately the area which the field-magnet must present for the flow of lines. For magnets of forged ingot iron or cast steel the limits within which it is advisable to keep the density may be set at $B_m = 15,500$, and $B_m = 17,900$, a good intermediate value being $B_m = 17,000$; while for cast iron the limits are 5,500 and 9,000, a usual figure being $B_m = 7,500$. If pressed beyond these values, the horizontal divergence of the curves, even with materials of the same class, becomes so marked that any slight inferiority of the metal may lead to difficulties owing to the impossibility of increasing the ampere-turns sufficiently to obtain the desired voltage and speed. A small error in the estimate of the densities will, in fact, produce a disproportionately great error in the result.

§ 2. Comparison of cast iron with forged iron and cast steel.—From the above average values for the density it follows that the weight of a cast-iron magnet as compared with that of a forged-iron or cast-steel magnet to carry the same flux must be roughly as 2¼ to 1. Where considerations of weight and bulk are paramount, it is therefore essential to build up the magnet out of ingot-iron forgings or rolled bars or to cast it in steel. The forgings in their rough state as they come from the steam-hammer may be cheaper than castings of good soft iron but when machined to the required dimensions their cost is so far increased that they become more expensive per hundredweight than cast iron; yet this increased price is more than counterbalanced by the lesser total weight that is required. In the same way, steel castings are considerably more expensive than iron, but not more than twice as costly, so that for the same magnetic work, when their lesser weight in handling, lesser freight, etc., are taken into account, the balance of advantage is again in favour of their use rather than of cast iron.

On the other hand, castings of complex shape can be produced easily and cheaply in iron, and of such accuracy of dimension that they require but little further machining. In large machines an additional set-off which sometimes counterbalances the higher cost of cast iron is found in the shape of its flux-density curve as compared with that of forged iron or cast-steel. A reference to Fig. 205 shows that the curve for cast iron over the working range rises in a gentle sweep, while that for forgings or cast steel has a marked point of flexure and then rises very slowly. Hence in the case of a dynamo which has to work over a very long range of voltage, if forged iron or steel are employed for the material of the field-magnet, and these are worked on the steep part of the curve for the lower limit of voltage, the machine may prove magnetically unstable (Chapter XVII, § 11); or if the working point be carried further up, an undue and unexpected increase in the exciting turns may be required to give the higher voltage, should the material prove slightly inferior to the curve. But with cast iron both the lower and upper limits of voltage can be reached with more certainty, and the designer is less at the mercy of the quality of the material.

§ 3. Composite magnets.—It is evident that a composite magnet may be made up out of two or more materials, by which their several advantages may be more or less completely combined, and such is, in fact, usually the case. The poles or magnet-cores carrying the exciting coils are almost always either made from rolled bars of ingot iron or castings of steel, or built up of thin laminated sheet-steel stampings. The yoke-ring of the ordinary multipolar type of Fig. 6 is then either of cast steel or cast iron. In the latter case the reason for the employment of cast iron may be either the necessity for partially combining the advantages of stability of the

voltage and accurate compounding over a long range (as above mentioned) with economy of excitation by means of the joint use of two materials, or the fact that it is desired to cast the supporting framework or bedplate of the dynamo in one with the magnetic circuit. In small belt-driven dynamos (up to an output of about 5 kilowatts), in which the cost of machining bears the largest ratio to the total cost of manufacture, it may frequently be advantageous for the bedplate and plummer-block pedestals, which form a large item in the total cost, to be a single casting integral with the lower half of the yoke-ring; as so much of the casting is not required to be magnetic, cast iron then becomes the most suitable material. But still more often in the case of the large stators of alternators the greater depth of the cast-iron frame is an advantage as adding to its mechanical strength to resist deflection.

Since the permeability of any casting is much reduced if it be hard, it is important, in designing the shape of a casting that will form part of the magnetic circuit, to avoid any thin flange or narrow, outstanding edge; such a piece of small area is likely to be chilled during casting, and so to become hard and magnetically inferior. For both mechanical and magnetic reasons, all corners and projections should be well rounded or, if need be, cast massive, and subsequently machined to the required dimensions.

The design, in short, should be such that there is no disproportionate difference in area between different portions of the same casting, whether of iron or steel; otherwise as it cools unequal contraction takes place, and hollow cavities or sponginess in the casting are difficult to avoid. For this reason, in large multipolar machines, even if entirely of cast steel, it is on the whole better for the massive poles to be cast separately from the yoke-ring.

§ 4. Comparison of sectional shapes for magnet cores.—The same economical considerations which determine approximately the sectional area of iron required to carry a given flux, namely, the first cost of the copper coils and the allowable expenditure of energy in magnetizing them, also bear upon the geometrical shape or figure by which the required area is obtained—only, however, in that portion of the magnetic circuit whereon the magnetizing coils are actually wound.

Since for a given area the circle has the least perimeter, the theoretically best section which can be given to the magnet cores whereon the field-winding is placed is the circle; the length and resistance of the wire for a given number of turns embracing a given sectional area have then their minima values. For this reason, in the multipolar continuous-current dynamo as Fig. 6, or in multipolar alternators similar to Fig. 71, the magnet cores, *m*, *m* are often of circular section. Such a circular core will require to have a rectangular pole-shoe at the one end in order to present a proper

area of polar surface where the lines pass through the air-gap, and so to minimize the reluctance of the interferric space between the iron of the pole and the armature core. Instead, therefore, of a circular section an oval or trapezoidal section of magnet core may be arranged so as to approximate to the required polar area at the air-gap, or again it may be more economical to give the magnet cores a rectangular shape, to avoid an undue amount of overhang of the pole-shoe beyond the magnet core.

Next to the circle, the oval composed of a square and two semicircular ends and the square must be ranked as the most economical sections, their perimeters being respectively 1·08 and 1·13 times that of the circle containing the same area. Finally, the larger the ratio between the length and breadth of any rectangular figure, the more uneconomical becomes the shape as regards length of wire in each turn encircling it. If the two sides of the rectangular be as 2 to 1, the perimeter becomes 1·2 times that of the equivalent circle, while for a ratio of 3 to 1 it increases to 1·3. When wound, therefore, with the same number of turns and the same size of wire, the weight of copper employed will be increased respectively 20 and 30 per cent. above that of the circular magnet, and the resistance being similarly increased the rate of expenditure of electrical energy to produce the same excitation will likewise be increased by 20 and 30 per cent. If the same efficiency is to be retained in all cases, the area of the increased length of copper wire must also be increased, making in all, for a ratio of 3 to 1, an increase in the amount of copper of nearly 70 per cent. A larger ratio, therefore, is very uneconomical, and is seldom required in ordinary designs. Rectangular magnet-cores with sharp edges have a higher leakage permeance than equivalent round or oval poles.

§ 5. Length of the magnet-core.—Since the maintenance of the excited field during the working of the dynamo requires the continuous expenditure of energy at a certain rate in watts, and this expenditure, being simply due to the passage of the magnetizing current through the electrical resistance of the wire, appears as heat in the coils, radiation, convection, and conduction must be relied on to avoid such a rise of temperature in the coils as will endanger the insulation of the field-winding; and this is secured by so disposing the coils that they present a considerable cooling surface to the air. As soon as the dynamo is set to work the temperature of the coils begins to rise above that of the surrounding air; this gradual rise continues until, finally, the rate at which the heat is generated in the coils is balanced by the rate at which it is dissipated under the combined action of radiation, convection, and conduction. After reaching this limit the temperature of the coils remains stationary so long as the conditions are unchanged. The effect of radiation, etc., being dependent on the entire cooling

surface of the coils, it is evident that the rise of the temperature of the coils above that of the surrounding air may be kept within any required limit by allowing a due proportion of cooling surface for each watt expended in the coils. The effective cooling surface of a given coil wound round an iron magnet will depend on a large number of conditions, difficult to calculate, since the mass of iron itself to some extent helps to dissipate the heat by conduction, and draughts of air set up by the revolving armature may increase the convection more in one machine than in another. Again, the depth of a winding of many layers will largely affect the temperature of the turns forming the middle layers. Roughly speaking, however, in machines of similar type the comparative cooling power of a coil may be taken as proportional to its external surface, reckoned as the product of its external perimeter, multiplied by its axial length, *plus* the area of the end-surfaces at the top and bottom of the coil, the cooling influence of the iron inside the coil being regarded as bearing a fixed ratio to that of the exposed wire surface in normal cases. Thus if the over-all dimensions of one circular coil, as shown in Fig. 266, are a diameter of 15″ and an axial length of $6\frac{1}{2}$″, with $2\frac{1}{2}$″ depth of winding (mean length of a turn 39·2″), the external surface, namely,

$$\begin{aligned} 15 \times \pi \times 6{\cdot}5 &= 306 \\ 2 \times 2\tfrac{1}{2} \times 39{\cdot}2 &= 196 \\ \hline &\ 502 \text{ sq. inches} \end{aligned}$$

will be a measure of its cooling power, and is to be reckoned in our present connection as its cooling surface. The basis for the estimation of the cooling surface having been thus decided, experience guides the designer to a certain ratio which the cooling surface in analogous cases must bear to the watts in order that the rise of temperature of the coil may not exceed a certain assigned limit. Of the values for this ratio more will be said in Chapter XXI. If the rate in watts at which energy may be lost in the field winding is approximately determined by the efficiency required in the machine, the length of the magnet coils need only be such as to provide a suitable amount of cooling surface proportioned to the rate in watts at which heat is dissipated in them. More generally, however, the settlement of the number of watts which may be regarded as an allowable rate of dissipation in the field coils will be a matter of compromise between the efficiency of the machine and the first cost of the copper wire; since the smaller the number of watts expended in the field-coils, the greater the weight of copper wire required. Further, the total number of ampere-turns required for the field excitation is not exactly known until the length of the magnetic circuit has been determined; experience, therefore, alone can furnish guidance for a first

approximation to the required length of coil, and a method of trial and error must be resorted to in order to determine exactly the ampere-turns on the field, the loss of energy in magnetizing the field, the dimensions of the coils, and their weight of copper. When, however, the designer has assigned an adequate length to those portions of the magnetic circuit whereon the coils are to be placed, he will complete the magnetic circuit in as direct a line as possible; since any length beyond the minimum thus required unnecessarily adds to the reluctance, and is therefore uneconomical in both iron and copper.

§ 6. Types of field-magnets.—Out of the great variety of forms which have been tried for the field-magnet system of dynamos, practically only three have survived, namely, the multipolar form of Figs. 6, 227, etc., for continuous-current machines with external yoke-ring and internal radial poles; and for alternators, the reverse of the last-mentioned, either with radial poles projecting from an internal yoke-ring (salient-pole type), or with a smooth-surface cylindrical rotor, as in turbo-alternators (non-salient pole type).

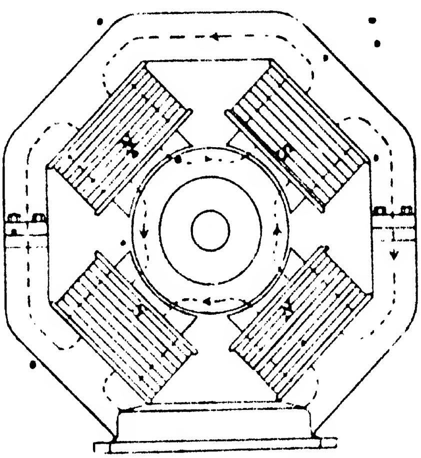


FIG. 226. Four-pole dynamo.

Considering first the magnet system of multipolar continuous-current machines, the yoke-ring is usually divided on the horizontal diameter to enable the upper half to be removed vertically for examination or removal of the armature. The lugs by which the two halves are bolted together may either be on the outer periphery of the yoke-ring or, if the increased width be an objection, may be arranged on the side flanks of the ring, as in Fig. 226. On either side of the lower half are usually cast projecting brackets or feet of sufficient width to give stability to the machine (Fig. 200 or 227). They are planed on the under side, and by them the machine is bolted to the base-plate; a steady pin on either side registers the exact position, and thin distance-pieces of sheet metal may be inserted underneath by means of which the height of the magnet may be adjusted in relation to the armature. Very large machines may be divided into four quarters, and occasionally, when the

machine stands between the cranks of a steam-engine, the yoke-ring is divided on a vertical diameter, so that the two halves may be drawn apart horizontally. For small machines the yoke may be of cast iron, since the section of metal required in cast steel makes the depth of the ring somewhat thin in appearance, even if it be strong enough mechanically. Or if steel is preferred the yoke may be given a channel or T-section, as in Figs. 227 and 228,

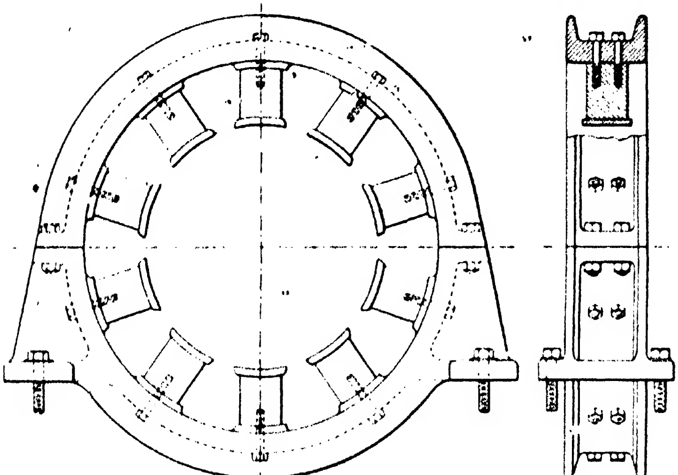


FIG. 227. Multipolar magnet frame for continuous-current dynamo.

in order to obtain sufficient strength against bending, although in general a slightly rounded section, as in Fig. 230, gives the most pleasing external appearance. The separate magnet cores, usually solid of cast steel or cut off from rolled bars of ingot iron, but occasionally laminated, are either fastened to the ring by screws or less often are cast into the yoke. The facings on which the poles are seated in the first case are usually for cheapness of manufacture bored out, the pole tops being themselves rounded to an equal radius (cp. Fig. 200), but may also be planed flat. The proportion of magnetic leakage in this type is but small, and it is economical in both iron and copper, especially if the section of the poles on which the bobbins are placed is circular. The shape of the yoke-ring may be a polygon with a number of sides depending on the number of poles (cp. Fig. 226), but is more usually made circular even with four poles (Fig. 200).

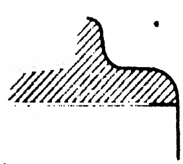


FIG. 228. Section of yoke.

§ 7. Commutating poles.—In order to obtain fixity of brush position in continuous-current machines under varying loads and sparkless running in difficult cases, such as high-speed dynamos driven by steam turbines, or machines to give a very wide range of voltage, auxiliary *commutating* or *reversing poles*, or as they are also called "interpoles," are an indispensable addition. Even in ordinary cases, when the speed is not very high, or the machine large, they enable windings which would otherwise be of doubtful

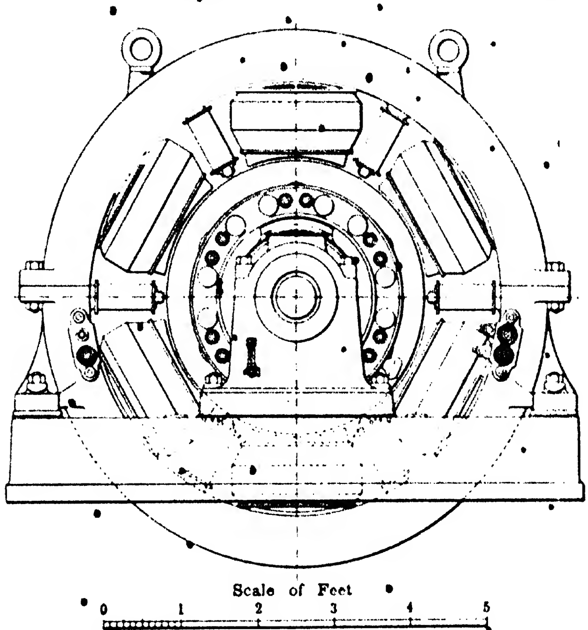


FIG. 229.—2[illegible]0-kilowatt dynamo with commutating poles.

success to be more readily employed, and a net saving in cost can be effected; thus the use of wave instead of lap windings can be extended to larger outputs, and the advantages are thereby gained of an armature cheaper to manufacture with a more open and mechanical winding as well as of a commutator with open lugs, small diameter and peripheral speed, while the freedom of wave-windings from local currents due to unequal pole-strengths or incorrect centring renders them better adapted to short air-gaps. The use of commutating poles is therefore now common throughout almost the whole range of continuous-current machines, both large and small, owing to the improvement in commutation which they give.

Such poles project from the yoke-ring of the field-magnet in the middle of each interpolar gap, so that their pole-faces are presented to the armature core on the interpolar line of symmetry (Figs. 229 and 230), and the brushes are so set that the coils when short-circuited are brought under the influence of the field between commutating pole and armature. They are introduced here as forming part of the magnet system, and—to anticipate their fuller

FIG. 230. Six-pole magnet and plummer-blocks of continuous-current dynamo having commutating poles.

discussion in Chaps. XIX and XX—their function is to supply the right strength of reversing field so as to cause the current in a short-circuited coil to pass smoothly from its value in the one direction to the same value in the opposite direction during the commutation period under all conditions of load from zero up to the required maximum overload. They are excited by magnetizing coils which are in series with the armature winding and carry the full armature current so that their effect may be proportional thereto; or in some cases the current out of each brush arm of the multipolar machine is taken directly round an adjacent commutating pole before it joins the combined stream.

The nature of the flux distribution is indicated diagrammatically in Fig. 231. This shows the application of commutating poles to the two-pole machine, but the same holds in principle for the multipolar machine. It will be seen that the commutating pole is essentially a strip from the leading edge of the adjacent pole transferred backwards against the direction of rotation into the centre of the interpolar gap, this detached portion being furnished with a separate exciting coil.[1]

The design and excitation of commutating poles will be further considered in Chaps. XIX, § 13, and XX, §§ 41-45, but it may here be added that it is not necessary that the axial length of the commutating pole-faces should be equal to that of the armature core.

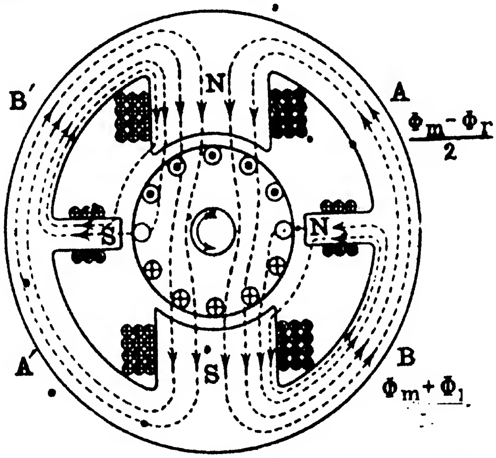


Fig. 231.—Flux distribution with commutating poles.

It may often with advantage be shorter, and an extreme case of this which is occasionally advisable is the use of only half as many commutating poles as there are main poles, every other interpolar gap being alone furnished with a commutating pole (cp. Fig. 273).

§ 8. Types of internal field-magnets with salient or non-salient poles.—The change of the armature core from its position as the internal rotating member to its position as external stationary member in alternators does not involve any radically new feature: its chief effect is that with parallel-sided slots the maximum flux-density now occurs at the tips of the teeth instead of at the roots, and excessive tooth-saturation is not to be feared. Neither does the change of the field-magnet system from external stator to internal rotor involve any great change of principle, provided that the number of pole-pairs is fairly large. But when the number of poles is only

[1] Cp. Dr. R. Pohl, *Electr. Eng.*, Vol. 37, p. 346.

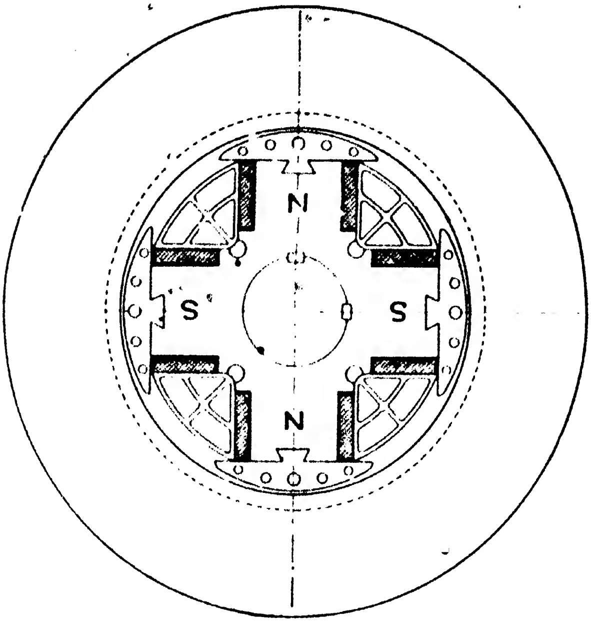


FIG. 232.—Four-pole internal magnet with salient-poles.

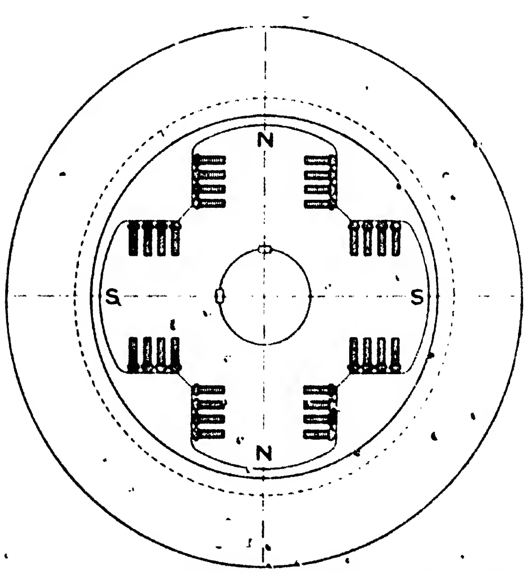


FIG. 233.—Four-pole internal magnet with salient-poles.

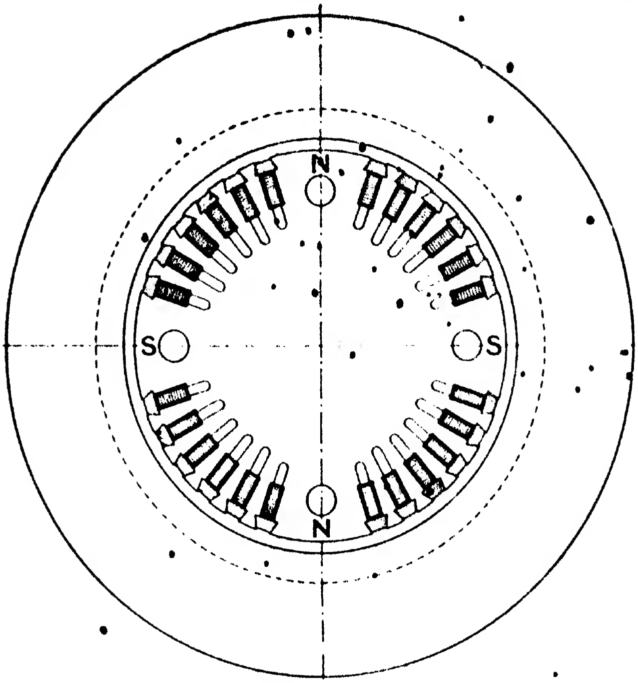


FIG. 234.—Four-pole internal magnet (non salient) with cylindrical rotor.

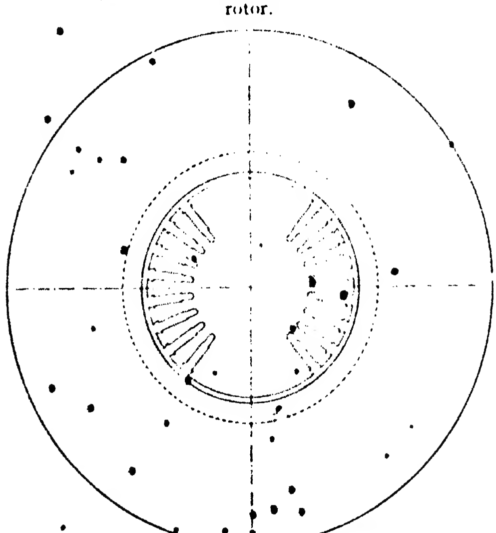


FIG. 235.—Two-pole internal magnet (non-salient) with cylindrical rotor.

4 or 2, as in modern steam turbo-alternators, the "salient-pole" type of magnet yields place to the "non-salient-pole" type with cylindrical rotor, having its exciting winding distributed in slots on the periphery, so that when finished a perfectly smooth surface is obtained.

For comparatively small loads of 100 to 750 kW. at such speeds that 4 or more poles are required, the "salient-pole" construction may still hold its own, but for small loads even so low as 100 kW if only 2 poles are required, there is great difficulty in finding room for a shaft of sufficiently large diameter, and the "smooth-cylinder" rotor becomes preferable since the field winding is disposed more nearly on the periphery, and a solid rotor in one with the shaft and of greater stiffness can be used. As soon as loads of 1,000 or more kW. per pole are reached the cylindrical rotor shows its superiority even with 4 and 6 poles.

An intermediate type which marks the transition between the salient and cylindrical types of Figs. 232 and 234 is found in the solid rotor with parallel slots which has been employed chiefly by the American Westinghouse Company.

In its 4-pole form a solid ingot-steel rotor has four salient poles, right round the edges of which are milled a few slots to receive a winding of thin flat copper strap (Fig. 233); the shaft may be forged in one with the core, or the core is composed of two forgings, each carrying one length of shaft and the two being bolted together. Although advantageous in the way of ventilation owing to the wedge-shaped spaces between the poles, this 4-pole type suffers to some extent from the limitations of the salient-pole type in that the possible sectional area of pole and of winding copper is very restricted. When the same parallel-slot type is applied to the 2-pole machine, the solid steel core can be made cylindrical, and a number of deep parallel slots are milled along its sides and also across its ends; in these are wound under tension the thin copper straps, two or more at a time, the cylinder being mounted on a turntable during the operation. To carry the wound core and at the same time to clear the end-windings, it then becomes necessary to bolt to either end a separate "head" or driving flange, which must be of bronze to avoid short-circuiting the magnetic flux, and to this again is bolted the required length of shaft. In both the 4- and 2-pole forms the exciting coils are entirely embedded in metal and well supported throughout their length, and the limiting conditions are the maximum stresses in the core, which occur chiefly in the overhanging strips of metal and the tips at the edges of the zone of winding.[1]

[1] See Miles Walker, *The Specification and Design of Dynamo-Electric Machinery*, pp. 366 and 373, and especially B. G. Lamme "High-Speed Turbo-alternators," *Trans. Amer. I.E.E.*, Vol. 32, Pt. I, p. 1.

§ 9. Mechanical strength of field-magnet frames.—Between the armature core and the polar surfaces of the magnet a very considerable attractive force acts; thus apart from the torque exerted on the stationary part of the dynamo by the rotating part when current flows through the armature, the magnet requires to be rigidly and firmly supported, so as to preserve the requisite clearance and avoid any liability of the poles to collapse upon the armature. The attractive force between opposite square centimetres of a pair of divided surfaces is proportional to the square of the number of lines that pass through from one sq. cm. to the other, *i.e.* to the square of the density in the gap B_g; and for a field of lines uniformly distributed over an area of A square centimetres the total pull[1] in dynes between the two surfaces is

$$\frac{B_g^2 A}{8\pi} = \left(\frac{B_g}{5000}\right)^2 A \text{ kilogrammes very closely}$$

or in pounds

$$P = \frac{B_g^2 A}{8\pi \times 981 \times 453 \cdot 6} = \frac{B_g^2 A}{11{,}183{,}000} \text{ lb.} \qquad (98)$$

acting along the direction of the lines, and tending to shorten their length. If A be in square inches, B_g being obtained in lines per sq. cm.,

$$P = \frac{B_g^2 A''}{1 \cdot 735 \times 10^6} = 5 \cdot 75 \times 10^{-7} \times B_g^2 A'' \text{ lb.}$$

If A_p'' be given the value in square inches of the area of one pole-face, the corresponding pull exists between each pole-pitch on the armature surface and the opposite pole-face, and has to be withstood by the mechanical rigidity of the magnet as a whole. A distinction is, however, to be drawn between such types of field-magnets as have a geometrical shape which is symmetrical about the armature, *e.g.* Figs. 226 and 227, and those which are not geometrically symmetrical. In the former when the armature is exactly central within the bore, the total radial pull round the whole circumference

$$P_m = P \, . \, 2p = \frac{B_g^2 A'' \, 2p}{1 \cdot 735 \times 10^6} \text{ lb.} \qquad (98a)$$

is symmetrically distributed, and leaves no unbalanced pull in any one direction. Such a condition is to be regarded as the normal

[1] As given by Clerk Maxwell; for the establishment of the expression, cp. Alexander Gray and J. G. Pertsch, *Trans. Amer. I.E.E.*, Vol. 37, Part II, p. 1417. It is strictly applicable only when the gap is at right angles to the direction of the lines of flux; and when the opposing surfaces are large as compared with the distance between them, so that the induction in the gap is appreciably uniform and there is no side leakage (cp. Du Bois, *The Magnetic Circuit*, Chap. VI, § 102); it is, however, sufficiently accurate for the ordinary purposes of dynamo calculations.

state, but in many cases it is not easily secured and maintained; the consequences of departure from it will be further considered in §§ 13 and 14 under the head of a closely connected subject, namely, the symmetrical distribution of the field.

But in field-magnets which are initially unsymmetrical with respect to the armature, as *e.g.* the two-pole horseshoe with magnet below or above the armature, there is a tendency for the lines to crowd into the lower or upper half of the armature, since the path which they then follow is shorter and of less reluctance than that of the lines which pass on into the upper or lower half of the armature. Thus in the former case even if placed symmetrically within the bore, the armature when the field is excited exerts a pressure on its bearings greater than that due to its mere weight; and if the inequality between the upper and lower halves of the fields be considerable, the downward pressure may reach such a value as to bend the shaft and perhaps cause serious heating of the journals. Especially is this likely to occur in a short-air-space dynamo, wherein any displacement will bear a large proportion to the total air-gap; for the slight bending of the shaft downwards due to the weight of the armature and any wear in the brasses of the bearings intensify the effect, by combining to shorten the length of the air-gaps in the lower quarters and to increase that of the upper quarters. On the other hand, if the magnet be above the armature, the unbalanced pull will be upwards, and will tend to lift the armature, so taking its weight off the bearings.

Generally speaking, the symmetrical types of field-magnet are to be preferred, but whether symmetrical or unsymmetrical, it is evident that in all cases a stationary magnet system must be securely bolted down, and its whole structure must be sufficiently rigid to withstand any mechanical stresses or vibration due to the working of the machine. All jointed surfaces must be securely bolted or screwed together, and the effect of a joint on the magnetic reluctance of an entire circuit may here be shortly mentioned.

§ 10. Influence of joints in the magnetic circuit.—If a bar of iron be divided transversely in two, and the two pieces be then placed in contact and magnetized, it is found that the total reluctance of the iron is slightly increased owing to the influence of the joint, and this increase in the reluctance may be expressed as that of an air-gap equal in area to the cross-section of the iron, and having a certain width depending on the exact conditions of the experiment. If the two surfaces of contact are such as are produced by ordinary planing or other machine-tools, and the joint may be regarded therefore as, comparatively speaking, rough, the width of the equivalent air-gap is equal to about 0·005 cm., which is only reduced to 0·004 cm. when the two pieces are squeezed together under a very considerable pressure. If, however, the two surfaces

are carefully scraped up, so that they are true planes, the equivalent width may be taken as 0·0035 cm. when the joint is not compressed, although under the influence of considerable compression the effect of the joint may be made almost entirely to disappear. It would seem that the increased reluctance due to a joint is in either case due partly to the interposition of a thin film of air between the surfaces, and partly to an alteration in the molecular structure of the iron at the joint, which decreases its permeability.[1] It will be seen, however, that the effect even of a comparatively rough joint, although appreciable, is not of great magnitude, and is especially unimportant in the case of the magnetic circuits of dynamos, which necessarily have air-gaps beside which the equivalent air-gaps of joints may be regarded as negligible. Absence or fewness of *joints in a magnetic circuit* must therefore be regarded as a minor consideration from an electrical point of view, and is only of importance as bearing on the mechanical strength of the design, and as lessening or increasing the first cost of the machine to the manufacturer.

§ 11. Proportioning of areas at different parts of magnetic circuit, and in different materials.—The question of joints where different portions of a magnet, possibly one of steel and the other of cast iron, are united, leads naturally to the third question—of the suitable proportioning of the sectional areas at different parts of the magnetic circuit. It is highly important that the lines of flux should never be "throttled," as it is termed, by having to traverse in the course of their path a portion which is of insufficient area; any such "throttling" implies a high density, and therefore a large number of ampere-turns for each inch in length of the contracted portion. If the disproportion between the areas at different parts of the magnetic path be carried to excess, the ampere-turns required to pass the lines through the area of the narrowest part may form so large a proportion of the whole number as almost to nullify the advantage of the larger section at other parts. In magnets composed throughout of iron of the same quality the area of section should be approximately identical at all parts of the circuit which carry nearly the same total number of lines; and, generally, in any portion of a circuit of the same magnetic quality the minimum number of ampere-turns is obtained if it be worked at a uniform density. Hence in designing, the distribution of the lines must be carefully studied.

When the lines pass from one material into another, as from forged iron or steel into cast iron, or *vice versâ*, as may be the case in Figs. 6, 226, the areas of the two parts should be so proportioned that both are worked at a suitable density; hence it follows from

[1] Vide Ewing, *Magnetic Induction in Iron and other Metals* § 168; and Du Bois, *The Magnetic Circuit*, Chap. IX, §§ 151–152.

§ 1 that the area of the cast iron should be more than double that of the forged iron or steel portion. Further, the nature of the joint should be such as to give ample area of contact, and afford an easy passage from the one metal into the other; *e.g.* in the dynamo of Fig. 227 the cast-steel cores may be drawn tightly up against machined facings on the yoke by means of screws, or magnet-cores of cast steel or of thin sheet-steel laminations may have a cast-iron yoke cast round them, suitable precautions being taken that the cast metal is not chilled as it flows round the cores, and that intimate adhesion of the two is secured.

In applying the above rules it must be borne in mind that "throttling" is a relative term, and under certain circumstances a high density and somewhat contracted area of certain parts may be legitimate; especially is this the case with the magnet-cores which are actually encircled by the magnetizing coils. On the other hand, portions of the magnetic circuit which are not overwound with copper, especially if of cast iron, which is comparatively inexpensive, may often be advantageously given somewhat lavish proportions and worked with a low density. Thus, in Fig. 226 in which the lines divide after passing through each core, half only passing in either direction through the yoke round to the next pole, the single cross-section of a cast-steel yoke may be more than equal to half the area of the steel magnet-core, so that the flux-density in the yoke is reduced to $B_y = 14{,}000$. When the material of the yoke is cast iron, and in this part gives strength and solidity to the whole structure, it may be economical to enlarge its area to considerably more than equality with the area of the steel pole, so as to further lessen the density and with it the ampere-turns on the field.

§ 12. Magnetic leakage.—In all calculations for the purpose of determining the proper area for any portion of the magnet, the question of *leakage lines* must not be left out of sight when estimating the density. As was pointed out in Chapter II, § 13, lines of flux will pass between any two points or surfaces between which there exists a difference of magnetic potential, and consequently in all dynamos there is a considerable number of lines that leak across from one pole to another without passing through the armature core, where alone their presence would be of use. Thus, in the multipolar dynamos of Figs. 226 and 227, lines will leak across from one pole-piece to the adjacent pole-pieces on either side, between the magnet cores, and even from the upper part of a core into the yoke; a few of these paths are traced out in Figs. 236 and 237. The presence of such leakage lines in the air is shown in a disagreeable manner by the magnetic effects which they produce in watches, electro-magnetic measuring instruments, or any piece of iron held in the vicinity of an excited dynamo.

If the magnet is to be worked with uniform density throughout

its length its area of cross-section should, strictly speaking, be continually varied as lines leak into or out of it. Since, however, the chief fall of magnetic potential occurs over the two air-gaps and the armature between them, it will be approximately sufficient (cp. Chapter II, § 14, and Chapter XVI, § 5) to regard all the leakage lines, ϕ_l, as flowing between pole-pieces of opposite sign, and then to add them to the lines through the armature in order to estimate the total number of lines which pass through the magnet. The density, therefore, in the magnet will be reckoned as equal to the sum of ϕ_l and Φ_a, *i.e.* Φ_m divided by the area of the magnet; and, if the density is to be approximately the same in the iron of the magnet and in that of the armature core, the area of the former must be larger than that of the armature.

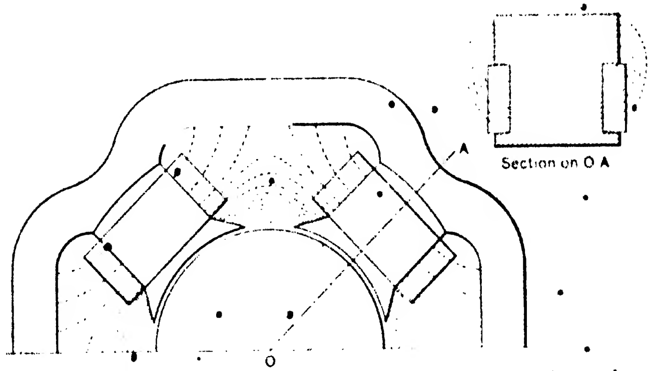


FIG. 236.—Magnetic leakage of multipolar magnet with salient poles external to armature.

Since it is the aim of the designer to reduce ϕ_l to a small proportion of the total number of lines Φ_m passing through the magnet, he will avoid bringing any large mass of iron into close proximity to the pole-pieces. In especial the magnetic leakage will be largely affected by the nearness or otherwise of the bed-plate, bearings, or the fly-wheel of the machine, since these by their comparatively high permeability lessen the total reluctance presented by any leakage path. All sharp edges conduce to magnetic leakage, especially from the poles, and are best avoided where practicable.

With commutating poles leakage takes place on one side of the commutating pole into a neighbouring main pole of opposite sign under a M.M.F. rising gradually to equality with the sum of the M.M.F.'s of one main exciting coil and one commutating field-coil, and also into the yoke at the root of the commutating pole under a portion of its own M.M.F. Owing to the greater surface of the commutating-pole in proportion to its section and the high M.M.F.'s,

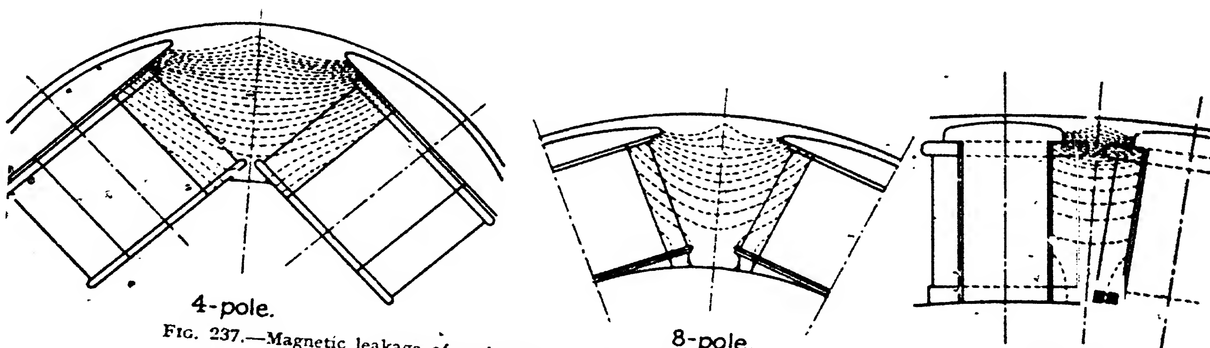


FIG. 237.—Magnetic leakage of multipolar magnets with salient poles internal to armature.[1]

[1] After Figs. 24 and 25 in "Reactance of Synchronous Machines and its Applications," by R. E. Doherty and O. E. Shirley, *Trans. Amer. I.E.E.*, Vol. 37, Part II, p. 1254, *qu. v.*

this leakage plays a much more important part than in the main poles, and if not properly calculated will lead to the iron of the auxiliary pole being made too small; it will then become very highly saturated, and proportionality of the reversing field to the current will be entirely lost.

With as many commutating poles as there are main poles it will be found from Fig. 231 that for the same useful main flux the average density in the main poles is practically unaffected by the addition of the commutating poles. In the armature the small portion of the path between a commutating pole and an adjacent main pole of opposite sign carries $\frac{\Phi_a + \phi_r}{2}$ lines, where ϕ_r is the flux of the reversing field, and the remainder of the path to a main pole of the same sign only carries $\frac{\Phi_a - \phi_r}{2}$ lines. Similarly, in the yoke-ring, if Φ_r is the total flux of the commutating pole including leakage, this number of lines is added to the flux carried by the sections of the yoke (B,B'), and deducted from the sections (A,A'), which complete the circuits from one main pole to another, so that the two densities are proportional to $\frac{\Phi_m + \Phi_r}{2}$ and $\frac{\Phi_m - \Phi_r}{2}$. If, therefore, the section of the armature core, and similarly that of the yoke, is sufficiently large so that they are far from saturation, in each portion the two changes largely counterbalance one another, and the percentage effect on the *total excitation* required for a given main flux is small; this is in practice usually arranged to be the case, but as will be seen later (Chapter XIX, § 13), it is a requirement that needs attention in the course of design.

§ 13. Symmetrical distribution of the fields in the air-gap.—Given a symmetrical type of field magnet to start with, any difference in the densities of the fields in the air-gaps will yield an unbalanced pull on the armature which may easily reach such an amount as to bend the shaft. This will result to a greater or lesser degree in such machines as those of Figs 226 and 227 when the bearings of the armature shaft wear and allow the core to sink slightly out of its true central position within the bore. The present requirement that the flux should be symmetrically distributed in the air-gaps must not, however, be interpreted to imply that the two halves of each field must be similarly distributed; in fact when the dynamo is at work on load (as will be explained in Chapter XIX), this condition is never secured, for in the "trailing" half of each pole-pitch, the flux will be denser than in the "leading" half, these terms having reference to the direction of rotation and the want of uniformity in the flux-distribution arising from the reaction of the armature current-turns upon the field. What is

implied is that if the flux is crowded more densely towards the trailing edge of one pole-face, every pole must be similarly affected. If a dividing plane be taken on any diameter, the distribution of the fields on the one side will then be the same as on the other side of it. The importance of this arises in the first place from mechanical reasons and secondly from the electrical effects due to inequality in the pole-strengths which have already been mentioned in Chapter XII. Assuming that there is no initial defect in the magnetic quality or setting of the poles, in a bipolar field displacement of the armature along the centre line or axis of the poles will produce but little effect, as compared with displacement along the neutral line dividing the interpolar gap between the two poles. But in a multipolar field, although displacement along the neutral line has the greatest effect, yet any displacement along the centre line of a pole will produce considerable resultant force, the two effects becoming more nearly equal the greater the number of poles. The magnitude of the unbalanced pull then remains to be estimated in the following section.

§ 14. Unbalanced pull from displacement of armature in a multipolar field.—If δ be the displacement of the armature from a central position in a vertical direction due to wear of the bearings or to incorrect mounting, and α be the angle which any radius line makes with the vertical, then if the magnet yoke-ring retains its true circular shape and suffers no deformation, the addition to or subtraction from the length of the air-gap l_g at the extremity of this radius is practically $= \delta \cdot \cos \alpha$.[1]

The guiding principle for the distribution of the fluxes between the several magnetic circuits under displacement is then that in each section of the yoke and armature core or of stator and rotor cores in which the flux bifurcates the loss of magnetic potential remains alike and the densities therein are similar. This is required in order that the flux may reach the maximum possible for the given number of ampere-turns of excitation, and it is in nature secured by a shifting of the bifurcation points until in each magnetic circuit through yoke and core the fluxes are alike. In the normal case without displacement the flux divides along the centre line of a magnet pole, and if this was retained under a downward displacement, the highest section of the yoke and armature core would carry less flux than the corresponding lowest section. But if the

[1] Through the centre of the displaced rotor draw any straight line inclined to the vertical at the angle α. From the true centre of the bore of the stator of radius R_p, drop a perpendicular on to the straight line, which is thereby divided into two equal halves, $x + r + \delta \cos \alpha = x' + r - \delta \cos \alpha$, where x and x' are the intercepts between the circle of the bore of the stator and the displaced circle of the rotor of radius r at the two ends of the straight line. Subtracting, $2\delta \cos \alpha = x - x'$ = the difference between the air-gaps. The only inaccuracy is that x and x' are not strictly radial to the circle of the stator bore as well as to the rotor, as the true air-gap lengths should be.

bifurcation of the flux instead of occurring opposite the centre of a pole is in every pole[1] shifted round nearer towards the shortest air-gap, the flux in the uppermost section of yoke and armature core is increased and that in the lowest section is decreased until equality again holds. But this alteration of the distribution has not altered the magnitudes of the fluxes in each of the pole-cores, gaps and armature teeth which form a series reluctance, save in so far that the total flux is thereby maintained at its maximum.

If, therefore, in each circuit[2] from AT_f is deducted $AT_c + AT_y$, the remainder $AT_f - AT_c - AT_y = AT_t + AT_g + AT_m$ is left for expenditure over the teeth, air-gap and magnet core of each and every pole.

What is required, therefore, is a set of partial magnetization curves for each of half the number of poles, such curves summing up the ampere-turns required over the series reluctance of teeth, air-gap and pole-core for different values of Φ_a in one pole-pitch, due allowance being also made for leakage in the magnet cores. Assuming then the uniform value of $AT_c + AT_y$ for each magnetic circuit (which must be afterwards checked), the ordinate to the partial magnetization curves for any value of $AT_f - AT_c - AT_y$, gives as many values for Φ_a as there are pole-pairs, and by traversing horizontally across to the air-line of each pole (Fig. 238) the corresponding values of AT_{gx}. The air-gap density at any point x under a pole is then given as

$$B_x = \frac{1{\cdot}257\ AT_{gx}}{l_g \pm \delta \cos \alpha}$$

This granted, it follows that unless the displacement be very large as compared with the air-gap, the final value of $AT_c + AT_y$ will diverge but little from its normal value, and we may at once use the latter to give $AT_f - AT_c - AT_y$ to be expended over the radial path of each pole.

Only half the number of poles need be so treated, since with a symmetrical field they will be representative of the other half, and the unbalanced pull thence deduced by resolution of the radial pull up or down the vertical line of displacement will only need to be multiplied by 2 to obtain the total. Thus Fig. 238 shows the pair of partial magnetization curves for a 4-pole machine with interpolar gaps on the vertical, in which as an extreme case with $l_g = 0{\cdot}34''$, δ has been taken as high as 50 per cent. or $= 0{\cdot}17''$. The air-gap permeance for any pole is then

$$L_f \int \frac{R_a \, . \, d\alpha}{l_g \{1 \mp (\delta / l_g) \cos \alpha\}}$$

integrated between the limits of the two edges reckoned from the

[1] When the displacement is along an interpolar line of symmetry.
[2] The symbols and results of Chap. XVI and XVIII are here anticipated.

vertical, where R_s is the radius to the pole-face, and L_f = its length along the axis of the armature. Since $L_f \cdot R_s \times$ the polar angle ϕ in circular measure = the area, A, of one pole-face in sq. cm., and $\phi = \beta\pi/p$, the permeance of any pole-face is thus

$$\frac{pA}{\beta\pi l_g}\int \frac{da}{1+(\delta/l_g)\cos a}$$

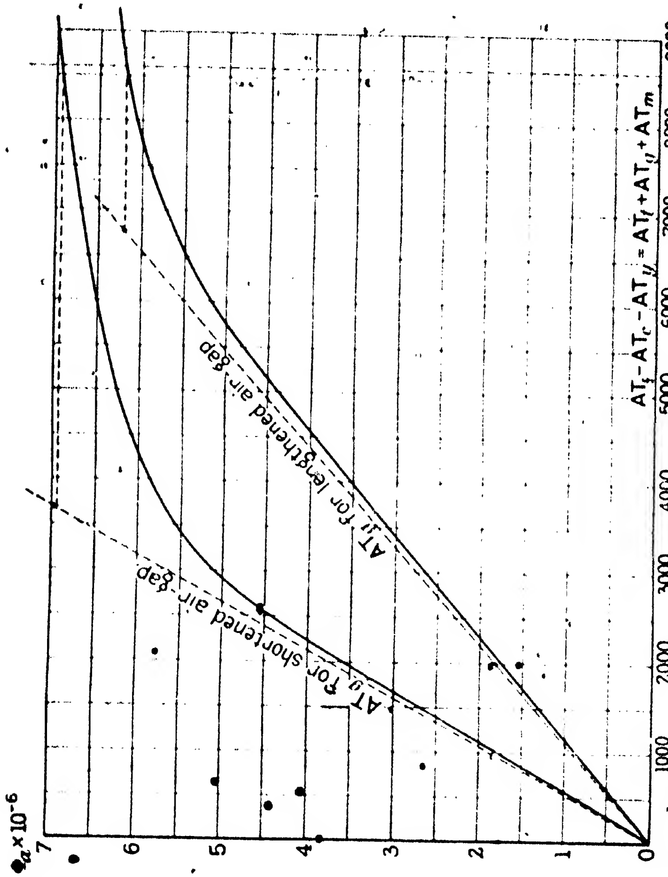


FIG. 238.—Partial magnetization curves for four-pole machine with rotor out of centre.

which in our 4-pole case with $\beta = 0{\cdot}7$ will be integrated between 76·5° and 13·5°.

* The length of air-gap when plotted in relation to a reckoned from the vertical gives a curve concave when viewed from the horizontal axis for the upper pole and a curve convex when viewed from the horizontal axis for the lower pole; the average value for l_g is therefore rather lower than the value of l_g at the centre of the pole in the former case, and higher in the latter case.

Let

$$v' = 2\tan^{-1}\sqrt{\frac{1+(\delta/l_g)}{1-(\delta/l_g)}}\tan\frac{a}{2}$$

$$\text{and } v = 2\tan^{-1}\sqrt{\frac{1-(\delta/l_g)}{1+(\delta/l_g)}}\tan\frac{a}{2}.$$

Then $\displaystyle\int\frac{da}{\{1-(\delta/l_g)\cos a\}} = \frac{v'}{\sqrt{1-(\delta/l_g)^2}}$ for a lower pole.

and

$\displaystyle\int\frac{da}{\{1+(\delta/l_g)\cos a} = \frac{v}{\sqrt{1-(\delta/l_g)^2}}$ for an upper pole.

The ratio of the true permeance to the normal permeance A/l_g is therefore in the four-pole case with $\beta = 0{\cdot}7$,

$$\frac{2}{0{\cdot}7\pi}\times\frac{v'_{76\cdot5}-v'_{13\cdot5}}{\sqrt{1-(\delta/l_g)^2}} \text{ for the lower pole,}$$

and

$$\frac{2}{0{\cdot}7\pi}\times\frac{v_{76\cdot5}-v_{13\cdot5}}{\sqrt{1-(\delta/l_g)^2}} \text{ for the upper pole.}$$

In our case for the assumed value of $\delta/l_g = 0{\cdot}5$

$$\sqrt{\frac{1+(\delta/l_g)}{1-(\delta/l_g)}} = 1{\cdot}7321 \qquad \sqrt{\frac{1-(\delta/l_g)}{1+(\delta/l_g)}} = 0{\cdot}5774$$

$$\sqrt{1-(\delta/l_g)^2} = 0{\cdot}866$$

For the two limits respectively, $\tan a/2 = \tan 38{\cdot}25°$ or $\tan 6{\cdot}75°$, $= 0{\cdot}7883$ or $0{\cdot}1184$. Hence for the lower pole,

$1{\cdot}7321 \times 0{\cdot}7883 = 1{\cdot}365$; $\tan^{-1} 1{\cdot}365 = 53{\cdot}76°$; $v'_{76\cdot5} = 107{\cdot}52° = 1{\cdot}88$
$1{\cdot}7321 \times 0{\cdot}1184 = 0{\cdot}2055$; $\tan^{-1} 0{\cdot}2055 = 11{\cdot}61°$; $v'_{13\cdot5} = 23{\cdot}22° = 0{\cdot}406$

$$v'_{76\cdot5} - v'_{13\cdot5} = 1{\cdot}474$$

and for the upper pole
$0{\cdot}5774 \times 0{\cdot}7883 = 0{\cdot}455$; $\tan^{-1} 0{\cdot}455 = 24{\cdot}46°$; $v_{76\cdot5} = 48{\cdot}92° = 0{\cdot}855$
$0{\cdot}5774 \times 0{\cdot}1184 = 0{\cdot}0685$; $\tan^{-1} 0{\cdot}0685 = 3{\cdot}91°$; $v_{13\cdot5} = 7{\cdot}82° = 0{\cdot}1365$

$$v_{76\cdot5} - v_{13\cdot5} = 0{\cdot}7185$$

From the relation

$$\Phi_a = 1{\cdot}257\, AT_g\,\frac{2A}{0{\cdot}7\pi\times2{\cdot}54\times0{\cdot}34}\times\frac{1{\cdot}474}{0{\cdot}866} \text{ for the lower pole}$$

$$= 1{\cdot}257\, AT_g\,\frac{2A}{0{\cdot}7\pi\times2{\cdot}54\times0{\cdot}34}\times\frac{0{\cdot}7185}{0{\cdot}866} \text{ for the upper pole}$$

are obtained the two dotted air-gap lines of Fig. 238, and by calculation for several values of Φ_a the full-line curves are completed.[1]

Carrying out, therefore, the above process for half the number of poles, for any given excitation $AT_f - AT_c - AT_v$, the air-gap AT_g for each pole is found,[2] and thence can be plotted a curve

[1] The value for k has here been thrown into the quantity l_g, but strictly speaking it would require to be worked out for the different actual lengths of air-gap when the displacement is great.

[2] With a rotor having a distributed winding or in an induction motor, the magnetic difference of potential across the air-gap increasing or decreasing in steps as the exciting slots are passed would need to be taken in each pole-pitch instead of the constant AT_g which holds for the pole-faces of wound salient poles.

for the flux-density B_x along the faces of the several poles in relation to angles measured from the vertical, as N, S in Fig. 239 for the lower and upper pole of the 4-pole machine.[1] This curve is a broken one, with as many portions as there are poles; the fringe of lines at the pole-edges is neglected, since the formula for the magnetic pull presupposes that the flux is radially directed and

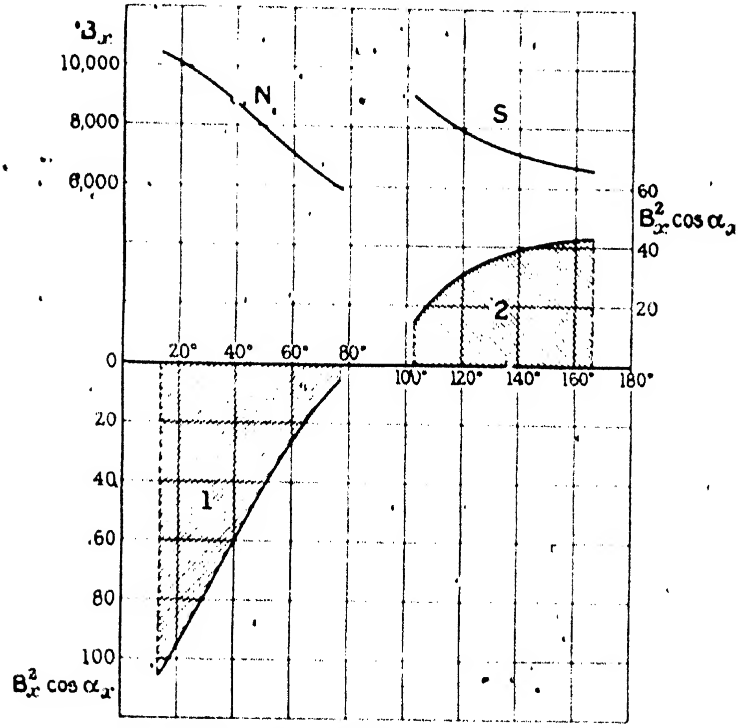


FIG. 239.—Air gap density and magnetic pull in 4 pole dynamo with armature out of centre.

confined between the area of the pole-face and the corresponding area of the armature surface. The total vertical pull due to any one pole is proportional to

$$\int_{\alpha_1}^{\alpha_2} B_x^2 L_f R_p \cos \alpha_x . d\alpha .$$

where α_2 and α_1 are the angles which radii to the pole-edges make

[1] For simplicity the varying values of AT_{tx} and AT_{cx} are here assumed to be replaced by a fair average value under a pole, as in the ordinary calculation of a magnetization curve.

with the vertical. A second broken curve must therefore be plotted graphically, which gives $B_x^2 \cos a$ in relation to a. Integrating the areas of this curve for all the poles on one side of the vertical, multiplying by $R_p . L_f$, and dividing by 11,183,000, we obtain half the total unbalanced vertical pull, P_u, acting downwards on the rotor or upwards on the lower half of the stator. P_u is thus proportional to $R_p . L_f$ (areas $1 + 2 + 3 + \ldots$) or considering half the areas up to p in number, since $L_f R_p \phi = A$, the area of a pole-face,

$$P_u = \frac{A \times 2}{11{,}183{,}000}\left(\frac{\text{areas } 1 + 2 + \ldots + p}{\phi}\right) \text{lb.}$$

where ϕ is the angle subtended by the polar arc in circular measure, and the expression in the bracket is virtually a mean square flux-density which if acting directly on one pole-face would give the same pull. It is this value then which would require to be taken into account in addition to the weight of the armature in calculating the deflection of the shaft, as in Chapter XIII, § 9.

It should be noted that a determination of B_x at several points from separate total magnetization-curves for a uniform air-gap of the length at each of the points does not meet the case; still less its determination by graphic construction or other method based on a single magnetization curve for the normal air-gap. When the squares of flux-density are in question, as much accuracy as is reasonably possible must be sought in their determination. The only constant quantity is $2\,AT_f$ on each magnetic circuit as a whole, but with little less truth $AT_f = AT_c + AT_g$ may be assumed as constant for each pole and air-gap separately. In calculating the partial magnetization-curves for the poles allowance must be made for the fact that the magnetic difference of potential between the pole-tips will be above the normal in the poles with lengthened air-gaps and below it in those with shortened air-gaps nearly in proportion to the divergence from the normal air-gap.

If the whole of the AT_f were expended over the air-gap—a condition only approached at very low densities and with iron of very high permeability—the broken curve of B_x would fall into a gradual sweep from a maximum at the lowest pole-edge to a minimum at the highest pole-edge; and for increasing values of AT_f, the unbalanced pull for the same displacement δ would continuously rise in amount. But the important point is that for increasing values of AT_f, and saturation of the iron, not only does the proportion which AT_g in the several poles bears to AT_f decrease, but it decreases more quickly in the more highly saturated pole with the shorter air-gap. Hence by the action of iron saturation, AT_{g1} for the pole with shortened air-gap is less than AT_{g2} for the pole with increased air-gap; in our case they are respectively 3,680 and 6,750 out of a

total of 10,400 = AT_f, and an average flux of 6,575,000 lines per pole. Thereby B_{x2} is maintained more nearly at the same average level as B_{x1}, and at the lower edge of the upper pole the density is actually much higher than at the corresponding upper edge of the lower pole (Fig. 240), producing an *upward* pull as far as these edges are concerned. Indeed, for small displacements, the densities at the similar corners of the two poles are not far different, and it is over their centres that the difference really takes effect. Thus since for increasing values of AT_f the effect of iron saturation acts powerfully to reduce the unbalanced pull that would result if the whole of the excitation were expended on the air-gap, the unbalanced pull reaches its maximum value for some particular value of the excitation. Trial confirms this and shows that it is reached at a comparatively low excitation[1] (about 5,500 AT_f only in our assumed case). Since during the process of excitation the ampere-turns per pole must pass through this value, it is evident that the machine must be sufficiently rigid in shaft and frame to withstand the displacement that will then arise.

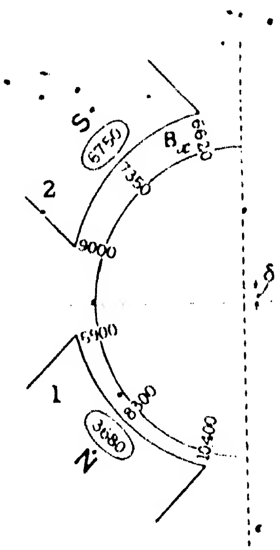


FIG. 240.—Distribution of flux-densities.

Let AT_g now be expressed as a certain fraction c of the total excitation per pole, *i.e.* $AT_g = c \, . \, AT_f$. Taking then any two points symmetrically situated in the lower and upper quadrants of a machine with 4 poles or a multiple of 4, the density at the two points will be respectively

$$B_{x1} = 1{\cdot}257 \frac{c_1 \, AT_f}{l_g - \delta \cos a} \quad \text{and} \quad B_{x2} = 1{\cdot}257 \frac{c_2 \, AT_f}{l_g + \delta \cos a}$$

or in terms of the normal $B_g = 1{\cdot}257 \dfrac{c \, AT_f}{l_g}$ before displacement,

$$B_{x1} = B_g \, . \, \frac{c_1}{c} \, . \, \frac{1}{1 - \dfrac{\delta}{l_g} \cos a} \quad \text{and} \quad B_{x2} = B_g \, . \, \frac{c_2}{c} \, . \, \frac{1}{1 + \dfrac{\delta}{l_g} \cos a} \, .$$

[1] As pointed out by C. R. Moore, *Electr. Review and Western Electrician*, Vol. 58, pp. 83–86, and by E. Rosenberg, *Trans. Amer. I.E.E.*, Vol. 37, Part II, p. 1435.

The unbalanced pull therefrom for similar points on one side of the vertical is proportional per sq. cm. in the 4-pole machine to

$$B_g^2 \left\{ \left(\frac{c_1/c}{1 - \frac{\delta}{l_g}\cos\alpha} \right)^2 - \left(\frac{c_2/c}{1 + \frac{\delta}{l_g}\cos\alpha} \right)^2 \right\} \cos\alpha \quad . \quad . \qquad (99)$$

and for the pair of pole-faces, each of A sq. cm. area, the total is equal to

$$\frac{B_g^2 A}{11,183,000} \times \frac{1}{\phi}\left[\left(\frac{c_1}{c}\right)^2 \int_{\alpha_1}^{\alpha_2} \frac{\cos\alpha \,.\, d\alpha}{\left(1 - \frac{\delta}{l_g}\cos\alpha\right)^2} - \left(\frac{c_2}{c}\right)^2 \int_{\alpha_1}^{\alpha_2} \frac{\cos\alpha \,.\, d\alpha}{\left(1 + \frac{\delta}{l_g}\cos\alpha\right)^2} \right] \quad . \quad . \qquad (100)$$

or double this amount for the two pairs of poles on the two sides of the vertical. Since the polar arc $\phi = \frac{2\pi}{2p} \times \beta$, the total unbalanced pull may be expressed as a fraction of the total pull in a radial direction on all pole-faces; thus

$$P_u = \frac{B_g^2 \times A \times 2p}{11,183,000} \times \frac{1}{\beta\pi} \times C = \frac{B_g^2 \times DL}{11,183,000} \times C \,. \qquad (101)$$

where C has the value within the square bracket for 4 poles and for more poles requires the addition of corresponding pairs of expressions for symmetrically situated pole-pairs.

Now $\displaystyle\int \frac{\cos\alpha \,.\, d\alpha}{\left(1 - \frac{\delta}{l_g}\cos\alpha\right)^2} = \frac{l_g}{\delta}\left[\int \frac{d\alpha}{\left(1 - \frac{\delta}{l_g}\cos\alpha\right)} - \int \frac{d\alpha}{\left(1 - \frac{\delta}{l_g}\cos\alpha\right)^2} \right]$

which is $\displaystyle = \frac{l_g}{\delta}\left[\frac{v' + \frac{\delta}{l_g}\sin v'}{\sqrt{\left\{1 - \left(\frac{\delta}{l_g}\right)^2\right\}^3}} - \frac{v'}{\sqrt{1 - \left(\frac{\delta}{l_g}\right)^2}} \right] = \frac{(\delta/l_g)\, v' + \sin v'}{\{1 - (\delta/l_g)^2\}^{3/2}}$

where v' has the same value as above (p. 457).

Similarly

$$\int \frac{\cos\alpha \, d\alpha}{\left(1 + \frac{\delta}{l_g}\cos\alpha\right)^2} = \frac{l_g}{\delta}\left[\int \frac{d\alpha}{\left(1 + \frac{\delta}{l_g}\cos\alpha\right)} - \int \frac{d\alpha}{\left(1 + \frac{\delta}{l_g}\cos\alpha\right)^2} \right]$$

which is

$$= \frac{l_g}{\delta}\left[\frac{v}{\sqrt{1 - (\delta/l_g)^2}} - \frac{v - \frac{\delta}{l_g}\sin v}{\sqrt{\{1 - (\delta/l_g)^2\}^3}} \right] = \frac{-(\delta/l_g)\, v + \sin v}{\{1 - (\delta/l_g)^2\}^{3/2}}$$

where v has the same value as above (p. 457).

With *e.g.* 4 poles and $\beta = 0{\cdot}7$, the limits for α are 76.5° and 13·5°, and

$$C = \left(\frac{c_1}{c}\right)^2 \left[\frac{(\delta/l_g)\,(v'_{76\cdot5} - v'_{13\cdot5}) + (\sin v'_{76\cdot5} - \sin v'_{13\cdot5})}{\{1 - (\delta/l_g)^2\}^{3/2}}\right]$$

$$+ \left(\frac{c_2}{c}\right)^2 \left[\frac{(\delta/l_g)\,(v_{76\cdot5} - v_{13\cdot5}) - (\sin v_{76\cdot5} - \sin v_{13\cdot5})}{\{1 - (\delta/l_g)^2\}^{3/2}}\right] \quad (102)$$

With 8 poles

$$C = \frac{1}{\{1 - (\delta/l_g)^2\}^{3/2}}\left[\left(\frac{c_1}{c}\right)^2 \left\{(\delta/l_g)\,(v'_{38\cdot25} - v'_{6\cdot75}) + (\sin v'_{38\cdot25} - \sin v'_{6\cdot75})\right\}\right.$$

$$+ \left(\frac{c_4}{c}\right)^2 \left\{(\delta/l_g)\,(v_{38\cdot25} - v_{6\cdot75}) - (\sin v_{38\cdot25} - \sin v_{6\cdot75})\right\}$$

$$+ \left(\frac{c_3}{c}\right)^2 \left\{(\delta/l_g)\,(v'_{83\cdot25} - v'_{51\cdot75}) + (\sin v'_{83\cdot25} - \sin v'_{51\cdot75})\right\}$$

$$\left. + \left(\frac{c_2}{c}\right)^2 \left\{(\delta/l_g)\,(v_{83\cdot25} - v_{51\cdot75}) - (\sin v_{83\cdot25} - \sin v_{51\cdot75})\right\}\right] \quad (102a)$$

In our case $\sqrt{\{1 - (\delta/l_g)^2\}^3} = 0{\cdot}65$, and

for the lower pole $v'_{76\cdot5} = 107{\cdot}52°$; $\sin v'_{76\cdot5} = 0{\cdot}9536$

$v'_{13\cdot5} = 23{\cdot}22°$; $\sin v'_{13\cdot5} = 0{\cdot}3846$

$\sin v'_{76\cdot5} - \sin v'_{13\cdot5} = 0{\cdot}569$

for the upper pole $v_{76\cdot5} = 48{\cdot}92°$ $\sin v_{76\cdot5} = 0{\cdot}7538$

$v_{13\cdot5} = 7{\cdot}82°$ $\sin v_{13\cdot5} = 0{\cdot}136$

$\sin v_{76\cdot5} - \sin v_{13\cdot5} = 0{\cdot}6178$

Thence

$$C = \left(\frac{c_1}{c}\right)^2 \left[\frac{\tfrac{1}{2} \times 1{\cdot}474 + 0{\cdot}569}{0{\cdot}65}\right] + \left(\frac{c_2}{c}\right)^2 \left[\frac{\tfrac{1}{2} \times 0{\cdot}7185 - 0{\cdot}6178}{0{\cdot}65}\right]$$

$$= \left(\frac{c_1}{c}\right)^2 \times 2{\cdot}01 - \left(\frac{c_2}{c}\right)^2 \times 0{\cdot}398 \quad (103)$$

From Fig. 238 when $AT_f = 10{,}400$, $AT_{g1} = 3{,}680$, and $AT_{g2} = 6{,}750$, while normally for the same excitation $AT_g = 5{,}450$; thence $c_1 = 0{\cdot}354$, $c_2 = 0{\cdot}649$, and $c = 0{\cdot}524$, so that $(c_1/c)^2 = (0{\cdot}354/0{\cdot}524)^2 = 0{\cdot}458$, and $(c_2/c)^2 = (0{\cdot}649/0{\cdot}524)^2 = 1{\cdot}44$. Thus for 10,400 AT per pole,

$$C = 0{\cdot}458 \times 2{\cdot}01 - 1{\cdot}44 \times 0{\cdot}398 = 0{\cdot}347,$$

and the unbalanced pull is proportional to $(7{,}880)^2 \times 0{\cdot}347 = 21{\cdot}5 \times 10^6$. It is evident how greatly the values of $(c_1/c)^2$ and $(c_2/c)^2$ equalize the vertical pull from each of the poles. If a low value of excitation was taken, so that in each pole nearly all the ampere-turns were expended on the air-gaps, $c_1/c = c_2/c = 1$ nearly, and C is increased to 1·612, but the normal B_g is itself reduced.

Since for a given machine and a given displacement, the numerical part of expression (103) remains constant, it is easy therefrom to calculate the excitation and normal B_g at which the pull reaches its maximum for that displacement. It is only necessary to take different values of AT_f and to obtain from Fig. 238 the appropriate values of c_1/c and c_2/c for insertion; it is thus found in our case that it occurs at about $AT_f = 5{,}500$, for which $AT_f - AT_c - AT_y = 5{,}000$, $B_g = 6{,}500$, and $AT_g = 4{,}450$, while $AT_{g1} = 3{,}320$, and $AT_{g2} = 4{,}600$. Thence $c_1/c = 0{\cdot}748$, $c_2/c = 1{\cdot}03$, and the unbalanced pull is proportional to $(6{,}500)^2 \times 0{\cdot}706 = 30 \times 10^6$.

But whether this maximum is reached or is exceeded in the given machine as its excitation passes through 5,500 AT entirely depends on the magnitude of the initial displacement δ_g when the machine is unexcited. Starting from this amount for any value of the excitation there will be an unbalanced pull of amount P_1 if rotor and stator are imagined to remain in their given initial

positions. From P_1 there results an elastic deflection of the shaft of the rotor and an elastic deflection of the stator frame, their magnitudes depending on the relative stiffnesses of shaft and frame at the point at which it acts. Let the sum of the two deflections be the first decrement δ_1 or the amount by which the air-gap between rotor and stator is further shortened. This gives rise to an additional increment P_2 to the pull, which in turn produces a second decrement δ_2, and so on. In the case of a horizontally supported rotor, apart from any error in centring the bearings with the stator frame, an initial displacement δ_g is always present, being the gravity deflection of the shaft under the weight W of the rotor, on which account the initial displacement has been marked with the suffix "g." Corresponding then to

$$W + P_1 + P_2 + \ldots$$

we have

$$\delta_g + \delta_1 + \delta_2 + \ldots = \text{final } \delta$$

If displacement and pull always retained strict proportionality, the final displacement would be the sum of a geometric series,[1] and would be finite

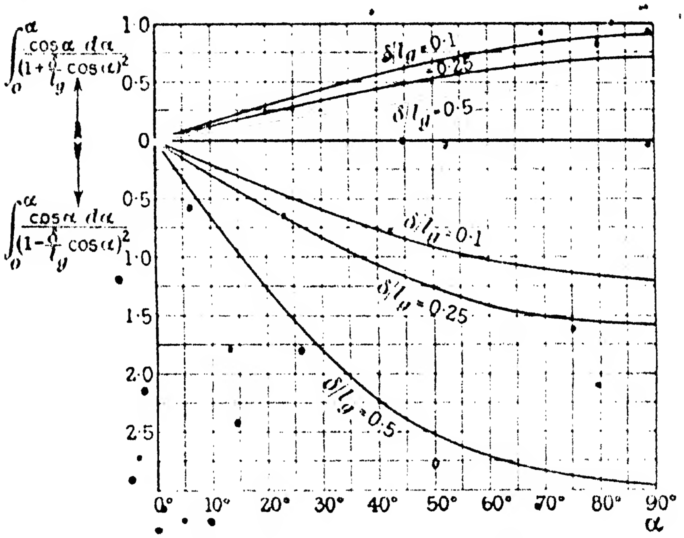


FIG. 241. Integrals for different ratios δ/l_g.

if the first decrement δ_1 was less than the initial displacement δ_g. Proportionality of pull and displacement is, however, very far from holding owing to the effect of iron saturation, as already explained. For a given *initial* displacement, therefore, the true growth of the unbalanced pull which rotor and stator experience when the machine is being excited and of the deflection can only be followed if a number of results such as (103) are calculated for different values of *final* displacement and are plotted to form a family of curves connecting pull and excitation, and the actual deflection is traced through them.

For any given ratio of δ/l_g, curves giving the integrals for values of α up to a complete quadrant or 90° for upper and lower poles respectively, as in Fig. 241 can be plotted once and for all, and are then available for any number of poles. It is only necessary to read off the integral for the one edge of a pole, to deduct that for the other edge, and to multiply the difference by

[1] Cp. E. Rosenberg, *loc. cit.*

$\left(\frac{c_2}{c}\right)^2$ to obtain the coefficient for the pole in question. The same process is repeated for the similarly situated pole in the other quadrant, and the sum of the differences for $p/2$ pole-pairs is the value required for C in expression (101). When the number of poles is large, it suffices to calculate $\left(\frac{c_1}{c}\right)$ and $\left(\frac{c_2}{c}\right)$ for the lowest and highest poles and to approximate the values for the ratio in the remaining pairs.

With a given value for β, the highest value is obtained with *e.g.* 4, 6, 8 poles when the displacement is in line with the axis of a pole, and the minimum when the displacement is along an interpolar line of symmetry.

For a given final displacement and constant values of normal B_g and β, as the number of poles is increased, apart from the values of $(c_x/c)^2$ the unbalanced pull in the former case declines and in the latter case increases, and in neither is there any great divergence from the value for $\alpha = 90°$, when multiplied by β. But in any such comparison account must also be taken of the changes in the values of c_x/c, unless the saturation is low. Further, as the number of poles is increased and the polar arc diminishes, the densities at the corners of each pole diverge less and less widely from the value at the centre, the curve of B_x for the several poles, therefore, falls more nearly into a gradual sweep. Finally with a very large number of poles we approach the case of two nearly co-axial cylinders, one of which is displaced by δ relatively to the other. If the displacement be very small as compared with l_g, the reluctance of the iron magnet may approximately be regarded as constant at its normal value $\mathcal{R}_m$ per pole. We then have

$$\frac{c}{c_x} = \frac{\mathcal{R}_g}{\dfrac{(\mathcal{R}_g + \mathcal{R}_m)\,\mathcal{R}_{gx}}{\mathcal{R}_{gx} + \mathcal{R}_m}} = \frac{\mathcal{R}_g \mathcal{R}_{gx} + \mathcal{R}_m \mathcal{R}_g}{\mathcal{R}_g \mathcal{R}_{gx} + \mathcal{R}_{gx} \mathcal{R}_m}.$$

$(\delta/l_g)\cos\alpha$ varies over each pole face, but as the number of poles is increased and the arc of each decreases, $\frac{\mathcal{R}_g - \mathcal{R}_{gx}}{\mathcal{R}_g}$ or $(\mathcal{R}_{gx} - \mathcal{R}_g)/\mathcal{R}_g$ for each pole-face becomes more nearly equal to $(\delta/l_g)\cos\alpha$. Hence with a large number of poles, the two expressions may approximately be identified. On this assumption of

$$\frac{c}{c_x}\left\{1 + (\delta/l_g)\cos\alpha\right\} = \frac{\mathcal{R}_g \mathcal{R}_{gx} + \mathcal{R}_g \mathcal{R}_m}{\mathcal{R}_g \mathcal{R}_{gx} + \mathcal{R}_{gx} \mathcal{R}_m} \times \frac{\mathcal{R}_{gx}}{\mathcal{R}_g}$$

$$= 1 - \frac{\mathcal{R}_g - \mathcal{R}_{gx}}{\mathcal{R}_g + \mathcal{R}_m} = 1 - \frac{\mathcal{R}_g}{\mathcal{R}_g + \mathcal{R}_m} \times \frac{\mathcal{R}_g - \mathcal{R}_x}{\mathcal{R}_g}$$

$$= 1 + \frac{c\,.\,\delta}{l_g}\cos\alpha$$

Here l_g/c may be regarded as an equivalent air-gap which makes allowance for a certain *constant* magnet reluctance. Expression (99) may then be written—

$$B_g^2\left\{\frac{1}{\left(1 - \frac{c\delta}{l_g}\cos\alpha\right)^2} - \frac{1}{\left(1 + \frac{c\delta}{l_g}\cos\alpha\right)^2}\right\}\cos\alpha$$

The value for C with the supposed large number of poles if covering the entire periphery (*i.e.* $\beta = 1$), as might be the case in a homopolar inductor generator, is then

$$\int_0^{\frac{\pi}{2}} \frac{\cos\alpha\,.\,d\alpha}{\left(1 - \frac{c\delta}{l_g}\cos\alpha\right)^2} - \int_0^{\frac{\pi}{2}} \frac{\cos\alpha\,.\,d\alpha}{\left(1 + \frac{c\delta}{l_g}\cos\alpha\right)^2}$$

Only the complete integrals for $\alpha = 90°$ are required, and these may be immediately combined. When $\alpha/2 = 45°$, and $\tan \alpha/2 = 1$, let $x' = \tan^{-1}\sqrt{\frac{1 + c\delta/l_g}{1 - c\delta/l_g}}$ and $x = \tan^{-1}\sqrt{\frac{1 - c\delta/l_g}{1 + c\delta/l_g}}$; then $\tan x' = \cot x$ and $x' = 90° - x$. Thus $v'_{90} = 180° - v_{90}$, and $v'_{90} + v_{90} = \pi$, the sine terms cancel out since $\sin (180° - v) = \sin v$, and

$$C = \frac{\pi c\delta}{l_g}\left(\frac{1}{(1 - (c\delta/l_g)^2)^{3/2}}\right) \quad . \quad . \quad . \quad (104)$$

δ having been assumed very small as compared with l_g, the denominator within the bracket approaches unity, and $C = \frac{\pi c\delta}{l_g}$ nearly.[1]

If the whole of the ampere-turns may be regarded as expended over the air-gap owing to the very low saturation of the iron, $c = 1$, and

$$P_u = \frac{B_g^2 \times \pi DL}{11{,}183{,}000} \times \frac{\delta}{l_g} \quad . \quad . \quad . \quad . \quad (105)$$

Or on the above supposition the same expression may also be derived from (102); since the ratio c_x/c is then unity, the effect of the upper and lower quadrants can be combined in one, as in (104). Whether there are 2 or many poles, so long as $\beta = 1$, the supposition implies that the difference of magnetic potential or AT_g between stator and rotor is uniform over the whole of the circle, although its sign changes as we pass from pole to pole. In these circumstances the density at any point is fixed by the air-gap at that point and by the uniform value of AT_g, not by the air gap AT_{gx} of the particular pole in which it falls. On this basis the expression in the form of (105) was given by B. A. Behrend[2] from the consideration of two nearly co-axial cylinders, one of which is displaced by δ relatively to the other, when $\left(\frac{\delta}{l_g}\cos\alpha\right)^2$ was neglected in comparison with $\frac{2\delta}{l_g}\cos\alpha$ in the denominator of $\frac{\cos\alpha \,.\, d\alpha}{\left(1 + \frac{\delta}{l_g}\cos\alpha\right)^2}$

§ 15. The electrical effect of eccentricity of the armature.—Apart from the mechanical effect, it has already been shown in Chapter XII, § 1, and the note appended to Chapter XII, that in a multipolar lap-wound armature, the result of any displacement of the armature out of the centre of the bore of the pole-pieces must be an unequal distribution of the current among the parallel branches of the armature winding and brush sets, attended with a lessened efficiency and perhaps sparking at the brushes.

Any considerable wear of the bearings must therefore be corrected by either raising the armature or lowering the field-magnet. For this purpose in large multipolar machines centring screws are often provided, by means of which the magnet frame may be adjusted horizontally and sideways. These may be used at the outset to erect the magnet so that its parallel circuits give very approximately equal voltages on open circuit as checked by a voltmeter applied to

[1] Cp. Fischer-Hinnen, *Dynamo Design* (1899), pp. 260-265, where an analogous expression was first given, and J. K. Sumec, *Zeitschrift für Elektrotechnik*, Vol. 22, pp. 727, 8.

[2] "On the Mechanical Forces in Dynamos caused by Magnetic Attraction," *Trans. Amer. I.E.E.*, Vol. 17, pp. 617-626. See also Miles Walker, *The Specification and Design of Dynamo-electric Machinery*, p. 59.

the several pairs of brushes; if the iron paths vary slightly in permeance, it may even be found advisable to set the magnet so that it is not quite concentric with the armature, yet so that the magnetic pulls of the poles and the electrical E.M.F.'s are closely balanced. At any later time the screws may again be called into use to follow up the effect of wear of the brasses.

But it has also been shown in the Note to Chapter XII, that owing to the circulating currents or unequal distribution of current in the multipolar lap-wound armature the inequality of the fluxes is less than would be due to the simple eccentricity, and this has the further effect of reducing the unbalanced pull and finally of lessening the actual amount of the eccentricity to which the resultant state corresponds. The addition of equalizing connections tends to reduce this beneficial action of lap-winding, but there still remains a considerable influence towards neutralization of the effect of the eccentricity.

In this respect then the lap-wound multipolar is more favourably situated than the wave-connected multipolar. The electrical objection from want of equality in the strengths of the fields is in the latter removed when there are only two sets of brushes, but on the other hand the mechanical disadvantage arising from the unbalanced pull persists to its fullest extent.

§ 16. Deflection and strength of external yoke-ring.—The formulæ of the above section have been based upon the supposition that the yoke-ring has retained its true circular shape, while the armature is displaced eccentrically from its true position by the amount δ. So long as this is assumed to be the case, the normal magnetic pull P_m from a uniform flux-density can be shown to have no effect upon the ring, and there is only left the total resultant *unbalanced* pull P_u assumed to act vertically. But the vertical components of the unbalanced pulls on the several poles, of which P_u is the sum, are, when considered in relation to the yoke-ring, not concentrated on the vertical axis, but are more or less distributed round the semicircle. The external ring would therefore be differently conditioned from the shaft, upon which the total unbalanced pull is really concentrated on the vertical diameter. The effect upon the ring might, however, still be expressed in terms of P_u. Thus if the semicircle be replaced by a straight beam freely supported at each end, and of length $l = 2r$, graphic construction of the bending-moment diagram for various numbers of poles would show that the same bending would in the 4-pole machine be given by a concentrated central weight equivalent approximately to $0{\cdot}66\ P_u$; and in a 6-pole machine, where one pole is situated immediately on the vertical axis, by an equivalent concentrated weight $= 0{\cdot}74\ P_u$ while above this number of poles it becomes $= 0{\cdot}72\ P_u$. To this would then have to be added the effect of the uniformly distributed

weight of the ring itself as acting over its upper semicircular half, and also the effect of the weight of half the poles, which is not far from uniformly distributed round the semicircle.

The necessary section which must for mechanical reasons be given to the multipolar yoke-ring is, however, a question not of the stress to which the material is subjected, but of the stiffness of the arch to resist deflection at the crown; the maximum deflection at the crown must, in fact, be limited to a small percentage of the normal air-gap, say, to $0{\cdot}05\ l_g$. But more than this, the deflection at the crown is accompanied by spreading outwards at the sides. Thus from the necessary fact of deflection the yoke-ring will no longer be circular, but will become more or less elliptical, and calculations based upon the above method with employment of P_u would be entirely fallacious. Hence for the above supposition of a circular yoke-ring and displaced armature must be substituted the assumption of a rigid armature shaft and a deformed external ring. The exact determination of either the maximum deflection or of the lateral spread by calculation is hardly possible owing to the intricacy of the various factors; if entirely prevented from expanding sideways, the deflection would be less than half of that due to the same concentrated load when the ends are quite free to spread; and in practice the partial constraint when the yoke is in one circular piece, or its two halves are strongly bolted together, introduces considerable indefiniteness into the problem. A comparative estimate can, however, be made which will err on the side of safety, if the semicircle be taken as free at its ends. The lateral extension of each end is then about 1·5 times the vertical deflection, but as an approximation it may be assumed that the two are equal. The air-gap at any angle α from the vertical, instead of being $l_g - \delta \,.\, \cos \alpha$ as in § 14, is then $l_g' = l_g - \delta \cos 2\alpha$; or if the maximum inward deflection δ at the crown be reckoned as negative, and α be reckoned from the horizontal which is here more convenient, it remains $l_g - \delta \,.\, \cos 2\alpha$.

Taking one or two values for l_g', the resulting B_x is calculated and thence can be found the corresponding pull per unit area of pole-face as proportional to B_x^2. The excess or deficiency of the latter as compared with the normal pull for the average air-gap l_g, and normal density B_g, can be approximated as proportional to λ, the decrease or increase of the air-gap, *i.e.* the excess pull in lb. per square inch $\dfrac{B_x^2 - B_g^2}{1{\cdot}753 \times 10^6} = -k \,.\, \lambda$, where $\lambda = l_g' - l_g$, and is negative when the air gap is reduced.

Hence if L_f be the breadth of the pole-face, the pull per unit length round the bore is $-k\lambda \,.\, L_f$. Owing to the broken surface presented by the poles with their intervening gaps, they may be replaced by a continuous surface in which the pull is, say, [illegible]ths of

its actual value at any spot. If in an alternator with rotating magnet the field varied as a sine curve, the pull at any spot would vary as $(B_{max} \sin 2\pi f \, . \, t)^2$, and when integrated over a period and averaged it would be $\propto \frac{1}{2} \, . \, B_{max}{}^2$, so that the above assumption is safe, for usual ratios of polar arc to pole-pitch. The abnormal pull per unit angle or radian may thus be taken as $-\frac{3}{4}k\lambda \, . \, R_p L_f = \frac{3}{4}k\delta \cos 2a$ lb. where R_p the radius to the pole-face, and L_f the breadth of the pole-face, are both in inches, when, as above, $k\lambda$ is reckoned in lb. per square inch.[1] The maximum value which the pull per radian assumes when $\lambda = \delta$ may be symbolized as

$$p_m'' = -\tfrac{3}{4}k\delta \, R_p \, . \, L_f$$

When the bending effect of both the normal and abnormal pulls is integrated, then, assuming the pole-faces to be replaced by an unbroken surface with the above proportion of the actual pulls, it can be shown[1] that the resultant bending moment at any point distant from the horizontal axis by the angle a is $B_m = \frac{2}{3} r \, . \, p_m'' \sin^2 a$, where r must now be made the radius to the neutral axis of the section of the ring. It is therefore independent of the average pull, and varies only with $\delta \sin^2 a$ for a given machine. The vertical deflection at the same point is

$$\Delta y = -\int_0^a r^2 (1 - \cos a) \frac{B_m}{EI} \, . \, da$$

and on the vertical axis this becomes

$$\delta = -0{\cdot}3 \, . \, \frac{r^3}{EI} \, . \, p_m''$$

where E is the modulus of elasticity of the material and I is the moment of inertia of the section of the ring. I and r must be in the same units in which δ is to be found, as also the unit area implied in E, while the stress E on this unit area must be in lb. if p_m'' is in lb. To this must be added the deflection due to the weight of the half ring and also to the weight of half the poles with their bobbins. The deflection of a semicircular beam of uniform cross-section due to its own weight W, when its ends are free, is at the centre[2] $-0{\cdot}0835 \frac{r^3}{EI} \, . \, W$, where r is the mean radius of the ring.

In a 4-pole machine, with the magnet-poles inclined at 45° to the horizontal, the additional weights are rather more, and in the 6-pole machine with one pole vertical, rather less favourably placed than under the condition of uniform distribution, and the coefficients

[1] As in Arnold, *Die Wechselstromtechnik*, Vol. 4, p. 258.

[2] This may be compared with the similar formula for a straight beam of length $l = 2r$, namely, $\delta = \frac{5}{48} \, . \, \frac{l^3}{EI} \, . \, \frac{1}{8}W = 0{\cdot}104 \frac{r^3}{EI} \, . \, W$, and, as might be expected, the deflection of the arched beam is the smaller.

become 0·0795 and 0·0856. The 8-pole and 10-pole machines approach more closely, with coefficients 0·083 and 0·0845. There is therefore in any case but little difference from the case of uniform distribution; and if W_y be the weight of the *complete* yoke-ring and W_p that of *all* the poles with their coils, the vertical deflection from the weight of ring and poles is closely

$$-0{\cdot}0835\,\frac{r^3}{EI}\cdot\frac{W_y+W_p}{2}.$$

The total deflection from the weight and from the magnetic pull is therefore

$$\delta = -\frac{r^3}{EI}\left(0{\cdot}0835\,\frac{W_y+W_p}{2}+0{\cdot}3\,.\,p_m''\right) \qquad . \quad (106)$$

Since p_m'' is $\propto k\delta$, and is therefore itself dependent upon δ, it is evident that some limiting value of δ must be assumed, such as $0{\cdot}05 l_g$, and the moment of inertia be given the required value, so that this limit is not exceeded. The above estimate, being based on the most unfavourable conditions, is little likely to be reached in practice.[1]

The modulus of elasticity E is for cast steel 30,000,000 lb. per square inch and for cast iron 14,000,000. The moment of inertia is $\frac{bh^3}{12}$ for a rectangular section, or, say, $\frac{bh^3}{15}$ for a slightly rounded section of maximum depth h, but in other cases requires to be calculated with accuracy.

In the lower half of the yoke-ring the effect from the unbalanced magnetic pull, if this is vertically upwards due to the armature having sunk, opposes that from the weight; and further, from the nature of its support on two projecting feet at the sides, the lower half possesses greater stiffness; it is only, therefore, in very large machines that a central support at the bottom may become advisable.

§ 17. Proportioning of multipolar machines.—A 2-pole dynamo of large output must have an armature core of considerable length as compared with its diameter, but such proportions give a section to the magnet limb which is inefficient in exciting copper. On the other hand, in the multipolar field the diameter can be larger, and its proportion to the length can be so chosen that the section of the magnet-core approximates to a square or to a circle, and is therefore economical in exciting wire. Thus, although the subdivision of the total flux between several magnetic circuits is in itself expensive in wire and exciting energy, the proportions of the magnet-cores in the multipolar case can be made so economical that it may actually take less weight of exciting wire than the 2-pole

[1] For a different treatment based on the assumption that the ends of the semicircular arch are held fast, see R. Livingstone, *The Mechanical Design and Construction of Generators*, p. 129.

massive horseshoe ; it then has the advantage not only in weight of iron in the magnet, but also in weight of copper. In order to obtain such advantageous proportions, the two dimensions A and B and therefore also MO and OP (Fig. 242) must not be far different from each other ; for, roughly speaking, the pole-core A, B must stand in the middle of the shoe MO, OP, with about the same amount of overhang all round, in order to avoid any undue thinning out of the lines towards any edge of the pole face. Since $\sin\frac{\phi}{2} = \frac{MN}{R}$, $MO = 2R\sin\frac{\phi}{2} =$

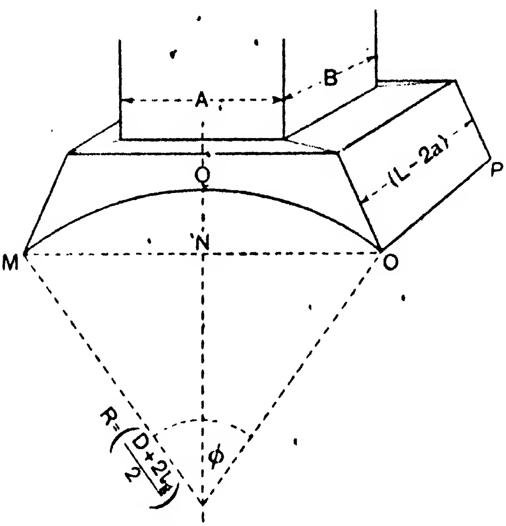


FIG. 242.

$(D + 2l_g)\sin\frac{\phi}{2}$, where D is the diameter of the armature core, and $(D + 2l_g)$ may be taken as approximately $= 1{\cdot}04\ D$. If a be the amount by which the length of the armature core exceeds the length of the pole-face at each end, $OP = L - 2a$, and this may be taken as equal to $0{\cdot}93\ L$. If then MO is to be equal to OP, $1{\cdot}04\ D\sin\frac{\phi}{2} = 0{\cdot}93\ L$. The polar angle bears an approximately constant proportion to the pitch of the poles, or $\phi = \frac{360^\circ \times \beta}{2p}$, where β is usually about 0·7. The favourable ratio of length to diameter of core for different numbers of poles on these assumptions would be

$$\frac{L}{D} = 1{\cdot}12\sin\frac{\phi}{2} = 1{\cdot}12\sin\frac{126^\circ}{2p}$$

A closer analysis may be made by taking into account the necessary reduction which there must be in the area AB carrying *useful* lines at a density of 16,800 per square centimetre, as compared with the area of the bored pole-face carrying lines at a density of, say, $B_g = 7{,}400$. Expressing the latter in terms of the chord MNO, we have

$$AB = MNO \times OP \times \frac{\pi}{126^{\circ} \sin\frac{126^{\circ}}{2p}} \times \frac{\beta}{2p} \times \frac{7{,}400}{16{,}800} = MNO \times OP \times xy$$

and a little calculation shows that the product xy decreases from 0·462 for four poles to 0·449 for six, and then becomes nearly constant for eight or more poles at 0·445. This product has now to be divided into the two factors x and y, being respectively the ratio of A to the chord $\left(= 1{\cdot}04 D \sin\frac{\phi}{2}\right)$, and of B to the length OP ($= 0{\cdot}93\,L$). Equating A and B, we thus obtain $\frac{L}{D} = \frac{x}{y} \cdot 1{\cdot}12 \cdot \sin\frac{\phi}{2}$. But though, as already stated, AB must, roughly speaking, stand in the middle of MO, OP, there is no necessity for the amount by which the width A is less than the chord to be precisely the same as the amount by which B is less than the pole-length. It is, in fact, better to make x somewhat less than y, the two values ranging from 0·64 and 0·72 in the 4-pole down to 0·63 and 0·71 in the 10-pole machine. The ratio $\frac{x}{y}$ is then a constant $= \frac{1}{1{\cdot}12}$, and

$$\frac{L}{D} = \sin\frac{\phi}{2} = \sin\frac{126^{\circ}}{2p}.$$

It is, however, advisable to keep the diameter on the small side, and consequently to make the ratio $\frac{L}{D}$ greater, even at the cost of less economy in the excitation. Hence the above expression may be regarded rather as giving the minima values which the ratio should take. The greatest departure from the most economical section of magnet-core which would be allowable in a multipolar machine would follow from the adoption of a ratio of $B : A =$ say, 1·5. We thus obtain as maxima values

$$\frac{L}{D} = 1{\cdot}5 \sin\frac{126^{\circ}}{2p}$$

While the above proportions admit of considerable variations according to different conditions of the design, the method by which they have been arrived at serves to show how they depend on the relative densities in the air-gap and magnet-core, and upon the

amount by which it is considered advisable for either edge of the pole-piece to project beyond the magnet-core.

No. of Poles.	L/D. Minimum.	$= \frac{3}{4} \times \frac{\pi}{2p}$	Maximum.
4	0·52	0·59	0·785
6	0·36	0·39	0·54
8	0·27	0·295	0·41
10	0·22	0·236	0·33
12	0·18	0·197	0·27

As a preliminary starting-point for an entirely new design L may conveniently be taken as equal to three-quarters of the pole-pitch, *i.e.*

$$\frac{L}{D} = 0{\cdot}75\frac{\pi}{2p} \quad . \quad . \quad . \quad . \quad (107)$$

This makes the pole-shoe roughly a square, and gives useful intermediate values which are tabulated above between the minima and maxima values obtained from the preceding expressions.

For a line of machines of small or moderate size, each yoke-ring or frame can be so proportioned as to admit of the employment within it of two or three armature cores of the same diameter but of different lengths, say, 0·6, 0·7 and 0·8 of the pole-pitch with rectangular field-coils, or 0·4, 0·5 and 0·6 of the pole-pitch with cylindrical coils.[1]

As explained in Chapter XII, § 17, with increasing sizes of machines the diameter must be increased, and for each diameter there is a minimum number of poles, below which the corresponding weight of yoke and also the reaction of the armature ampere-turns render it uneconomical to go. Such minimum numbers will be roughly as follows—

Armature Diameter.	*Number of Poles.*
Up to 25 in.	Not less than 4
Above 25 in.	,, 6
,, 36 in.	,, 8
,, 48 in.	,, 10, and so on.

In machines with commutating poles, assuming as a maximum 9,000 armature ampere-turns per pole or 18,000 armature ampere-conductors per pole, and 900 ampere-conductors per inch of periphery the maximum pole-pitch is given as 18,000/900 = 20 in. The length of armature core, if equal to three-quarters of the pole-pitch, is then 15 in., and it may be laid down in general that the over-all

[1] Cp. Miles Walker, *Specification and Design of Dynamo-electric Machinery*. p. 496.

length of the armature core should not exceed 15¾ in. or 40 cm., unless imperatively required by the peripheral speed with this length of core proving too high. If, on the other hand, the

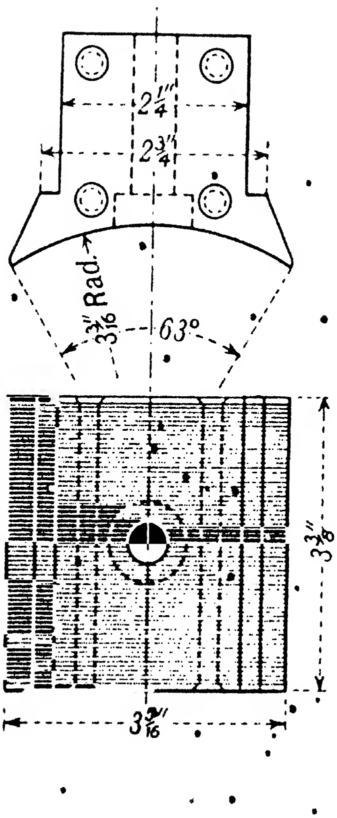


Fig. 243.—Complete laminated pole.

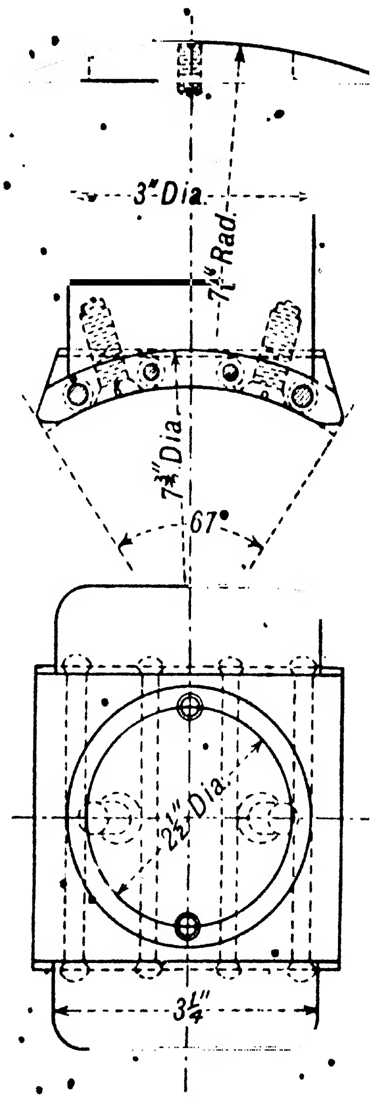


Fig. 244.—Magnet-core and laminated pole-shoe for small machine.

peripheral speed is low, it will usually be advisable to shorten the core to less than the above maximum, to raise the diameter and, if need be, increase the number of poles.

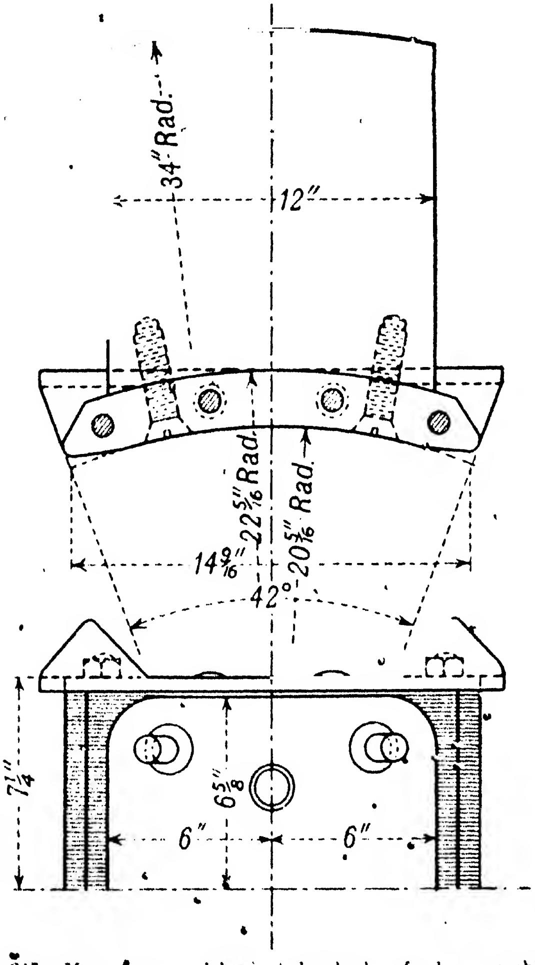


FIG. 245.—Magnet-core and laminated pole-shoe for large machine.

§ 18. Laminated pole-shoes.—The necessity for the employment of laminated pole-shoes in order to minimize eddy-currents due to the varying distribution of the flux as wide open slots sweep past the pole-face has already been mentioned in Chapter XIII, § 18. In small machines the whole pole is sometimes made up of sheet-steel

stampings (0·025" thick) riveted together, as in Fig. 243; the cross-section of the pole must then be square or rectangular. For attachment to the yoke-ring, if the pole is not too heavy, it is quite safe to drill and tap a laminated pole for screws, provided that the laminations are tightly squeezed together and well riveted between stout side cheeks. Or a bolt may be used to draw the pole up against its seating within the yoke-ring. Alternatively, a solid square bar

FIG. 246 Laminated pole shoe

is driven through a square hole in the laminations of the complete pole, and into this enter set screws through the yoke ring. In larger machines it is more usual to make up separate pole-shoes of sheet-steel stampings, and to fasten them to the solid pole-core by means of four screws. Flanges, projecting at right angles to the side-cheeks serve as supports for the magnet bobbins. Since the magnet pull on the pole-shoe is very considerable, especially at the tips, the side cheeks must be stiff, the rivets strong, and the screws fastening the whole to the pole-core well distributed over the face (Figs. 244 and 245). Fig. 246 shows a complete pole-shoe, and below it one of the component laminations.

CHAPTER XVI

THE AMPERE-TURNS OF THE FIELD

§ 1. The three divisions of the magnetic circuit.—In the previous equations for the internal E.M.F induced in an armature, one of the three determining factors has been Φ_a, or the number of lines of flux which enter into the armature-core from one pole or leave the core to pass into another pole. We have now to determine the ampere-turns of the excitation which must be placed on the magnet in order to produce Φ_a lines in that part of the magnetic circuit where they are required, namely, in the armature, where they are linked with the active wires. The process of calculation has been already illustrated in Chapter II, § 13, and consists in the subdivision of the entire magnetic circuit into such lengths as may be considered to be of the same sectional area and permeability, and to carry the same number of lines, the calculation of the magnetic differences of potential required to drive the lines through each of these lengths, and their subsequent summation as one magnetomotive force. The magnetic circuit in a dynamo may be divided into the three principal parts: (1) the *air-gaps*, (2) the iron of the armature, subdivided into (*a*) the *armature core* proper, and (*b*) the *teeth* in the case of a slotted armature, (3) the iron of the field-magnet; the latter may further require to be subdivided into two parts, namely, (*a*) the *magnet-cores* or *poles*, and (*b*) the *yoke*, and in some cases, where the pole-pieces differ much in area and quality from the rest of the magnet, a third subdivision becomes necessary, namely, (*c*) the *pole-pieces*. The subscript letters *g*, *c*, *t*, *m*, *y*, and *p* may be used to denote these several parts. Owing to leakage the total flux will vary in different parts of the circuit, but, as explained in Chapter II, on the supposition of the leakage paths being all in parallel with the armature and air-gaps, it will suffice to consider Φ_a lines as flowing through the armature, and a larger number, Φ_m, as flowing through the magnet and yoke.

§ 2. The equation of the magnetic circuit.—If Φ lines flow through a portion of a magnet circuit, having a total area of cross-section of a square centimetres normal to their path, the value which H must have is solely determined by the flux-density per square centimetre, *i.e.* by the induction $B = \Phi/a$; in other words, whatever be the total area normal to the flow, if a difference of magnetic potential, H, be maintained between two surfaces one centimetre apart, B lines will flow through each square centimetre of the cross-section. Each centimetre length of the substance

therefore requires a specific difference of magnetic potential, H (dependent on B), to be maintained between its ends; and if the same flux-density, B, be continued over a portion whose length is l centimetres, the total fall of magnetic potential between opposite faces of the portion will be $H \times l$, and that difference of potential maintained between its ends will cause a total flow of $\Phi = aB$ lines through it. Reckoning, therefore, the lengths of the five chief portions of the magnetic circuit in centimetres, l_g being the length of a single air-gap, l_t that of a tooth, and l_m that of a single pole, the fundamental equation for a complete circuit, with its two air-gaps, and its two poles in the usual multipolar types, may be expressed as—

$$\text{Total M.M.F.} = H_c l_c + H_t 2l_t + H_g 2l_g + H_m 2l_m + H_y . l_y$$

where H_c, H_t, H_g, H_m, and H_y are the magnetizing intensities required to produce B_c, B_t, B_g, B_m, and B_y in the armature core and teeth, air-gaps, pole-core and yoke respectively. Or, writing $f(B)$ for H, since H has been shown to be a function of the induction

$$\text{Total M.M.F.} = f(B_c) . l_c + f(B_t) 2l_t + f(B_g) . 2l_g + f(B_m) 2l_m + f(B_y) . l_y$$

And finally, since $B = \Phi/a$

$$\text{M.M.F.} = f\left(\frac{\Phi_a}{a_c}\right) . l_c + f\left(\frac{\Phi_a}{a_t}\right) . 2l_t + f\left(\frac{\Phi_a}{a_g}\right) . 2l_g$$
$$+ f\left(\frac{\Phi_m}{a_m}\right) 2l_m + f\left(\frac{\Phi_m}{a_y}\right) . l_y$$

The same flux Φ_a is here taken as passing alike through air-gaps, teeth and armature core, and a_c and a_y are the areas of the double sections of armature core and yoke, where the flux bifurcates.

In curves of flux-density, such as Figs. 205–209, are connected together corresponding values of B and H for various materials. Having calculated, therefore, the different values of B_c, B_t, B_g, B_m, B_y for a given value of Φ_a, all that is necessary is to look out in the flux-density curve of the material in question the particular value which B has in it, and then to read off on the scale of abscissae the corresponding particular value which $f(B) = H$ must have for that density; this value can then be substituted for the specific H in the above equation.

The above equation being expressed in C.G.S. units of magnetomotive force will require to be divided throughout by 1·257 in order to express it in ampere-turns of excitation on one complete magnetic circuit, as

$$X = X_c + X_t + X_g + X_m + X_y$$

But since the values of H in the flux-density curves were originally derived from a measured number of ampere-turns per centimetre length, it is still simpler to show on the horizontal axis of abscissae

a scale of ampere-turns per cm. length (as well as a scale of H in C.G.S. units), and thence read off the specific excitation *at* required per centimetre length to produce a flow of B lines per square centimetre. This has been done in the lower of the two horizontal scales of Figs. 205–209, whence, therefore, we can read off the new value which the function of the density takes when expressed in ampere-turns, or

$$f'(B) = \frac{f(B)}{1{\cdot}257} = 0{\cdot}8f(B) = a'$$

Since the permeability of air is strictly constant and on the C.G.S. system = 1, a density of one line per sq. cm. in air requires $\frac{1}{1{\cdot}257} = 0{\cdot}8$ of an ampere-turn, and any other flux density B_g requires $0{\cdot}8B_g$ ampere-turns, per cm. length. Even if the normal air-gap of the machine is partly filled with cotton-covered copper wire, the permeability of the latter two materials is sensibly the same as that of air. Hence for at_g may at once be substituted $0{\cdot}8B_g$, or $0{\cdot}8\ \Phi_a/a_g$.

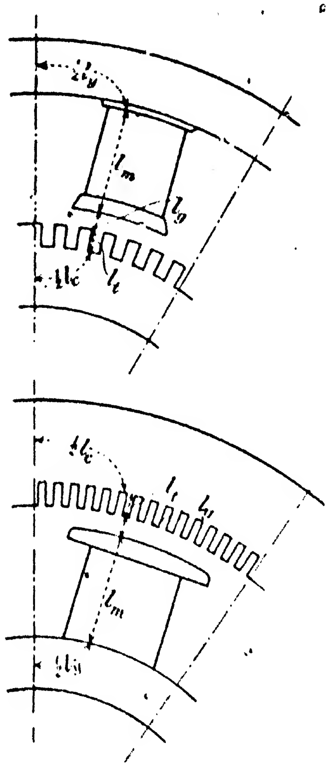


FIG. 247.—Half magnetic circuit of multipolar machines.

As foreshadowed in Chapter II, § 14, in calculating a_g a certain additional area must be reckoned over and above the actual area of the polar face in order to allow for the spreading out of the lines in a fringe, which increases the effective cross-section of the air-gap. On the other hand, some deduction must be made to allow for the presence of ventilating ducts reaching to the armature surface. The quantitative value for these adjustments will be treated in § 6.

Lastly, from considerations of symmetry it suffices in all types of symmetrical field-magnets as commonly used in practice to determine the ampere-turns required by a half magnetic circuit, *i.e.* by the sector (Fig. 247) corresponding to a pole (whether this be external or internal to the armature) since these turns must be

provided by the coil wound on one pole. If, therefore, in the present connection the symbol AT_f as contrasted with X is restricted to the ampere-turns on a *half* magnetic circuit, and with other subscripts is restricted to the ampere-turns required by a component part of the half circuit

$$\begin{aligned} AT_f &= AT_c + AT_t + AT_g + AT_m + AT_y \\ &= at_c \cdot \frac{l_c}{2} + at_t \cdot l_t + 0{\cdot}8B_g \cdot l_g + at_m \cdot l_m + at_y \cdot \frac{l_y}{2} \end{aligned} \quad (108)$$

The total number of ampere-turns on the whole machine is then $pX = 2pAT_f$. The actual process in designing will be to determine the densities, B_c, B_t, B_m, and B_y, and then to read off from a flux-density curve the corresponding values of at or the specific number of ampere-turns required per centimetre length; and even in the case of the air-gap to calculate B_g as a guide in comparing different machines.

§ 3. Use of flux-density or B-H curves. The flux-density curves from which the values of the specific ampere-turns are read must be suited to the exact nature of the materials of which the dynamo's magnetic circuit is composed. On referring to Fig. 205 it will be seen that if the magnet be worked with a low density, say, *e.g.* 10,000 in wrought iron or 6,000 in cast iron, the value of at for a given density is indeterminate, and may have any value lying between the ascending and descending curves; in the cases supposed the limits will be 1·5 or 4 ampere-turns per centimetre length for wrought iron, and 16 or 25 for cast iron. *Vice versâ*, with a given value for at, the density may vary between certain limits according as it has been reached from a higher or lower value. Hence a certain number of ampere-turns on the field may induce a slightly larger number of lines at one time than at another if the excitation be raised to a higher value and then decreased; and the student may therefore feel a doubt as to what value of exciting turns he is to select. In practice, however, any such effect of hysteresis resulting in a variation of the voltage given by a dynamo when running at a certain speed is hardly perceptible; not only is a dynamo seldom worked with a very low density in the iron of the field-magnets (and with higher values for the density the vertical difference between the ascending and descending curves becomes negligible), but also the exciting power expended over the reluctance of the field-magnet, or $AT_m + AT_y$, forms only a certain proportion, usually less than half, of the total number of ampere-turns, $AT_f = AT_c + AT_t + AT_g + AT_m + AT_y$, and therefore an increased number of ampere-turns is required for the armature and air-gaps, if an increased number of lines pass through the armature. Further, in actual working, the excitation would never be varied through so wide a range as in Fig. 205; in this latter the magnetizing intensity

per centimetre length of iron has been carried up to a high figure, and then gradually reduced, so producing the greatest possible difference between the ascending and descending curves. Such a condition would seldom occur, except in the conduct of workshop or laboratory experiments on dynamos; in the ordinary course, the field-magnets become gradually excited as the machine is set to work, and then the excitation is maintained at a more or less constant amount until, on the stoppage of the machine, the magnetism dies away. Hence in the design of dynamos it is sufficient to use invariably the ascending curve of induction; any difference, due to the excitation having been differently reached, may be neglected, since its only effect will be to reduce slightly the speed required to generate a given voltage.

§ 4. The back ampere-turns of armature.—There still remains one point on which the previous equation requires to be supplemented. When a current is passing through the armature of any machine its ampere-turns give rise to a magnetomotive force, and the armature *reacts* on the field, as will be explained in greater detail in Chapter XIX. Further, in a continuous-current dynamo it may be necessary to move the brushes forward through a small angle in the direction of rotation in order to prevent sparking at the commutator; the reason for this *shifting of the brushes* will also be more fully explained in Chapter XX. Suffice it here to say that if the brushes have to be given a forward *lead* in order to obviate sparking, the ampere-turns due to the current flowing in the belt of armature wires enclosed within twice the angle of lead λ are practically in direct opposition to the ampere-turns of the field. In Fig. 322 those wires in which the current, whether of the field or of the armature, is directed towards the observer are marked with a dot, and those in which the current is away from the observer are shown crossed; from this figure it will be apparent that the ampere-conductors within the angle 2λ, are opposed to the ampere-conductors of the field, and that the former if coupled up to form ampere-turns embracing the magnetic circuit tend to magnetize it in exactly the reverse direction to the latter; although it is not entirely an accurate description of the whole phenomenon, as will be shown in Chapter XIX, they may be regarded as *back ampere-turns*, and the result is that, if present, a certain number of extra ampere-turns have to be added to the field ampere-turns in order to neutralize the demagnetizing effect of the armature current. Let this additional number on a half circuit be AT_b; then the complete equation for the ampere-turns required from a field coil must contain this additional term, and thus becomes

$$AT_f = AT_c + AT_t + AT_g + AT_b + AT_m + AT_y \quad . \quad (109)$$

§ 5. The magnetic leakage due to the difference of potential between the poles.—The reason for the insertion of AT_b in the

fourth place needs further explanation. It has been said that the flow of leakage lines takes place in the main under the magnetic difference of potential existing between the poles of the dynamo, and even if this be not true of all the leakage, allowance can be made for the inaccuracy involved in the assumption as will be shown in § 11. Before Φ_m or B_m can be determined, the amount of leakage ϕ_l requires to be estimated. In order to do this, the joint permeance of the leakage paths is calculated; it is then only necessary to multiply the magnetic difference of potential at the poles by this joint permeance to determine ϕ_l. The magnetic difference of potential between the poles is equal to 1·257 times the ampere turns required to produce the flow of lines between the poles; let these be expressed as $2AT_p$, so that $+AT_p$ and $-AT_p$ represent the magnetic potentials of the N. and S. pole-faces respectively; then $\phi_l = 1{\cdot}257 \times 2AT_p \times \mathcal{P}_l$, where $\mathcal{P}_l$ is the joint permeance of the leakage paths. Thence $\Phi_m = \Phi_a + \phi_l$ from which the magnet density B_m and the ampere-turns required to produce it, AT_m, can be determined.[1]

Now AT_b, or the ampere-turns required to counterbalance the back ampere-turns of the armature, if such are present, take effect between the poles, and consequently must be included when estimating the difference of magnetic potential between the poles; thus

$$AT_p = AT_c + AT_t + AT_g + AT_b$$

and

$$\phi_l = 1{\cdot}257\,(AT_c + AT_t + AT_g + AT_b) \times 2\mathcal{P}_l \quad . \qquad (110)$$

Our final and complete *equation for the ampere-turns or excitation of a half magnetic circuit* will thus take the form

$$AT_f = at_c\frac{l_c}{2} + at_t l_t + 0{\cdot}8B_{g\,max}K\,l_g + AT_b + at_m l_m + at_y\frac{l_y}{2} \qquad (111)$$

the coefficient K remaining to be explained in § 7.

Simple though this equation is in its form, its application presents certain difficulties which preclude us from calculating the ampere-turns required with absolute and complete accuracy. Apart from the somewhat indefinite nature of the leakage paths, the lengths of the paths in the iron portions of the magnetic circuit can only be calculated approximately, a mean having to be struck between the longest and the shortest; or again, the area of cross-section normal to the flux may be continuously varying (as, for example, in the pole-piece). A mean value must then in practice be taken and in the estimation of this considerable judgment may be required. A second source of error is that our calculations are based on certain

[1] Cp. Chapter II, § 13.

flux-density curves, yet the particular class of iron used may not be exactly similar to that for which a curve has been obtained; cast iron in particular varies considerably in its magnetic properties. Still, when tested after completion, a dynamo should not require to be run at a speed differing by more than 5 per cent. from the designed speed in order to give the designed voltage.

FIG. 248.—Fringe of field at pole-edge with toothed armature.

§ 6. Calculation of air-gap permeance of smooth-core armature.—In the determination of the field ampere-turns two portions of the magnetic circuit require further discussion beyond the mere statement that the area and consequent flux-density must be calculated or measured. These are the air-gap of the armature whether smooth or slotted, and the teeth of the slotted armature.

In the calculation of the air-gap permeance or flux-density therein, it must be borne in mind that, owing to the spreading of the lines as they issue out of the polar face and pass into the armature core, the area of the air-gap is greater than the area of the bored face of the pole-shoe.

The actual paths of the flux curving out of the pole-tips into the tips of the armature teeth can be shown by the beautiful figures obtained from photographs of the stream-lines in a coloured viscous liquid forced between two sheets of glass. This method of investigation can be made to imitate with exactitude the analogous passage of lines through media of different permeability, and from the original paper on "Lines of Induction in a Magnetic Field"[1] by Professor Hele-Shaw and Alfred Hay, who first employed the method, Figs. 248 and 257 are reproduced by the kind permission of the authors

[1] *Phil. Trans*, series A, Vol. 195, pp. 303–327.

and of the Royal Society. Subsequently, Professor Hele-Shaw, Dr. Hay, and Mr. Powell published[1] a number of such stream-line photographs expressly designed to illustrate the phenomena occurring in a toothed armature, and to the accompanying papers the student is especially referred. They clearly show the refraction

(a) (b)

FIG. 249.—Stream-line photographs representing flux distribution in (a) smooth, (b) toothed armature (Prof. W. M. Thornton).

of the lines as they pass from the air, a medium of low permeability, into the iron, a medium of high permeability, and also that owing to the low value of the ratio $1/\mu$, even when the lines within the iron are approaching the bounding surface with a very small inclination thereto, their exit into the air is made practically at right angles to the bounding surface.

Further stream-line photographs, which show the interpolar

[1] *Journ. I.E.E.*, Vol. 34, p. 21, and Vol. 40, p. 228.

fringe well are reproduced in Fig. 249 from originals kindly supplied by Prof. W. M. Thornton.[1]

Lastly, to complete the pictures of Fig. 237 for the leakage of internal poles, there is added Fig. 250 showing the nature of the useful lines of the interpolar fringe in the three cases. It will be seen that they fill up the spaces left blank in Fig. 237 which only included the true leakage between the pole-cores and pole-shoes.

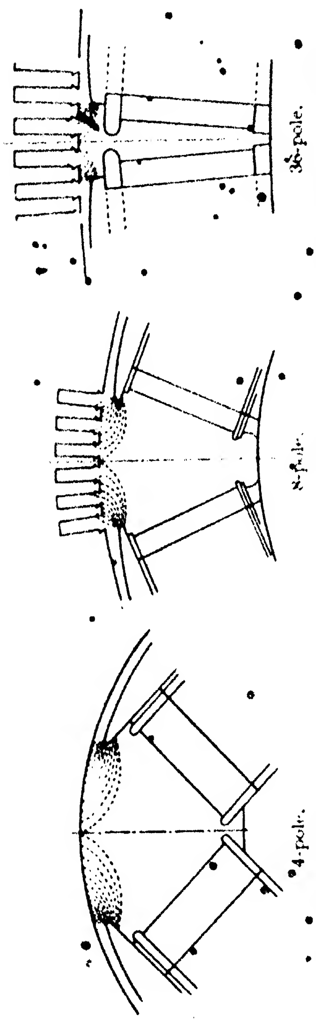


Fig. 250.—Useful interpolar fringes of multipolar machines with internal field-magnets on no load.

Not only do lines pass by a curving path into the core from the sloping or radial edges of the pole-tips as shown above, but some also enter into the armature from the outer flanks of the pole-pieces at the edge of the bore, and this is especially the case when, as usual, the length of the armature core is slightly greater than the width of the poles parallel to it (Fig. 251).

(*a*) *The interpolar fringe.*

The distribution of the magnetic potential and of the flux at points in the interpolar gap even on no load when only the field M.M.F. is present cannot be expressed by any simple formula with mathematical accuracy, nor by any empirical formula that is of universal application. Still less can it be so expressed under load when one of the M.M.F. systems (that of the armature) is not symmetrically placed in reference to the polar system. One ideal case can,

[1] See also Prof. Thornton's paper, "The Distribution of Magnetic Induction and Hysteresis Loss in Armatures," *Journ. I.E.E.*, Vol. 37, p. 125, Figs. 10–24, for the above and many others in addition.

however, be exactly solved on no load, viz., that of a flat armature core with smooth surface at zero potential throughout to which is presented a pair of poles whose edges are planes making a right angle to the core, these poles being at potentials + 1·257 AT_p and − 1·257 AT_p respectively throughout their entire surface. By use of the theory of conjugate functions Mr. F. W. Carter[1] has shown that in such a case the normal value which the flux-density has under the greater portion of the polar area when the air-gap length is uniform is not maintained right up to the edges of the pole-face, and that the decrease begins at a distance from the pole-edge practically equal to the air-gap; at the extreme edge its value is 0·84 of the normal, and thence if the opposite pole is absent it rapidly falls, as shown by the upper curve of Fig. 252.

In the above case of a pole making a right angle with the armature core and apart from the presence of a second pole, the curve for the flux-density of the fringe is closely approximated between the limits of $x = 2\,l_g$ and $x = 13\,l_g$ if it is assumed that the length of path of the lines entering the armature from a N. pole or leaving it for a S. pole, at any point distant x centimetres (measured on the armature surface) from the pole-tip is made up of a straight portion l_g *plus* a curved arc of length ξx, where $\xi = 0{\cdot}9 \times \pi/2$; the flux-density is then $\dfrac{1{\cdot}257\ AT_p}{\xi x + l_g}$. This approximate expression fails however close to the pole-edge, so that for greater accuracy in a numerical calculation an additional multiplying factor m may be used, which has the following values—

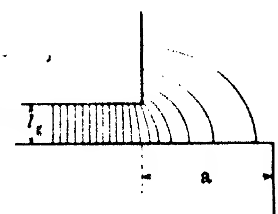


FIG. 251.—Fringe from pole-flank.

TABLE VIII.

x/l_g	− 1	0·75	− 0·5	− 0·25	0	0·25	0·5	0·75	1
m	0·99	0·973	0·952	0·91	0·84	1·00	1·12	1·14	1·15
x/l_g	1·5	2	3	4	6	8	10	12	
m	1·11	1·08	1·06	1·04	1·00	0·97	0·955	0·94	

But now in the case of the actual interpolar fringe the effect of the adjacent pole of opposite sign must be taken into account. This causes the flux entering or leaving the armature core to become zero within the interpolar gap, and on no load it is evident from considerations of symmetry that the zero point occurs on the line

[1] "Note on Air-gap Induction," *Journ. I.E.E.*, Vol. 29, p. 929. See also J. F. H. Douglas, "The Reluctance of Some Irregular Fields," *Trans. Amer. I.E.E.*, Vol. 34, Pt. I, p. 1067, and F. M. Roeterink, *Archiv für Elektrotechnik*, Vol. 7, 1919, p. 292.

of symmetry between the two poles. At this which may be called the "tangent point" of the field flux, the resultant flux as it passes across from the N. pole-tip to the S. pole-tip only just touches the armature surface without entering it. If c be the distance measured on the surface of the armature between a radius drawn to the pole-tip and the interpolar line of symmetry, it might be thought that the case would be truly represented by taking the upper curve (Fig. 250) of the flux-density from an assumed single pole from

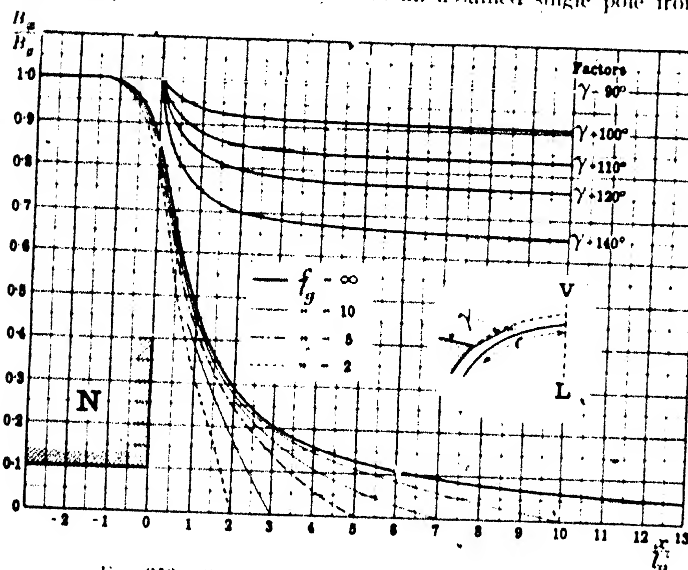


FIG. 252. Curves of distribution of interpolar fringe.

$x = 0$ to $x = c$, and deducting from it the reversed continuation of the same curve from $x = c$ to $x = 2c$. But this does not make the flux-density assume its right value at the pole-tip which formed the starting point, and a characteristic of the true case is that all interpolar curves, whatever the value of c/l_g, at the pole-edge very closely approach the value for $c/l_g = \infty$. An exact solution for the same ideal case when both pole-edges make a right angle with a flat armature surface which is throughout at zero potential, has been given by Mr. F. W. Carter in the above-mentioned paper, and the results are reproduced in the lower curves of Fig. 252 for various values of c/l_g.

The edges of the pole-shoes facing the interpolar gap are, however, usually inclined to the cylindrical surface of the armature at some angle other than a right angle. To meet this practical case, the value of the flux-density in the fringe as calculated for a flat armature

and a pole-edge at right angles to it may be multiplied by a factor $\dfrac{0{\cdot}9\frac{\pi}{2}x + l_g}{0{\cdot}9\,\gamma\,x + l_g}$, where γ is the angle in circular measure made by the pole-edge with a line joining the pole-tip to a point midway towards the interpolar line of symmetry. Such approximate correction factors are added in the upper part of Fig. 252.

The above ideal case, even when corrected for the inclination of the pole-edges, fails to represent strictly the true case of the actual armature for reasons to be explained in Chapter XVIII, but the results of Fig. 252 are sufficiently near to the truth to warrant their application to practical use. By their aid with increasing values of γ from 90° to 140°, the curves of Fig. 253 have been plotted, which have now to be explained.

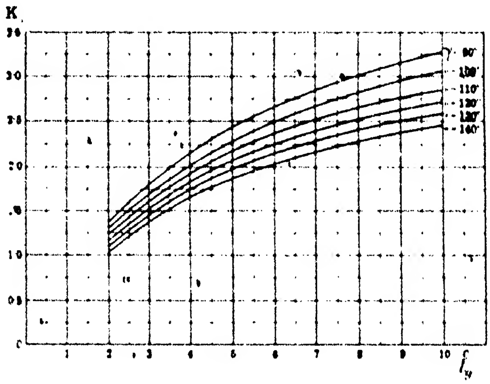


Fig. 253. Coefficient K_1 for interpolar fringes.

The additional number of useful lines entering the armature core as a *fringe* along any edge may be taken into account by assuming the normal density B_g to hold over the entire pole-face, and by then adding along the edge in question the permeance of an additional strip of air. This strip must be of such width that with a length of path through it equal to the normal l_g, and with the normal density B_g assumed to be caused in it by $1{\cdot}257\ AT_g$, the required addition to the flux is obtained. The effective width of the equivalent strip of air along the edge parallel to the axis of rotation will in general be different from its width along the flanks of the pole-shoes. Thus the width of the strips producing the equivalent of the interpolar fringe will depend upon the ratio c/l_g. On the other hand, if a be the amount by which at each end the armature length exceeds that of the pole-piece (Fig. 251), the width of the strip producing the equivalent of the flank-fringe will

depend upon the ratio a/l_g. In both cases the width of the strip may be expressed as a certain multiple of l_g. In Fig. 253, therefore, are given the values of the co-efficient K_1 by which in a smooth-core armature l_g must be multiplied to obtain the joint effective width of the *two* strips, one along each interpolar edge of a single pole, as determined from Fig. 252.

(*b*) *The flank-fringe.*

In Fig. 254 is given an analogous curve for the value of the coefficient K_2 by which l_g must be multiplied in order to obtain the joint width of the *two* equivalent strips, one on each side-flank of the pole-piece, for various values of a/l_g. For small values of this ratio below 1 the figures for K_2 are rather small, since in such cases, when the pole-piece is nearly as long as the armature core, some lines do not enter on the circumference of the core, but curve round into the ends, yet these lines hardly need to be taken specially into account.

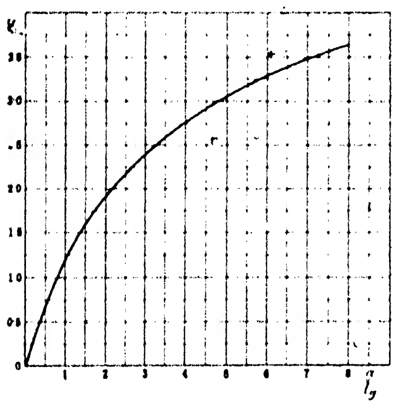


FIG. 254. Coefficient K_2 for flank-fringes.

(*c*) *Ventilating ducts.*

There remains to be taken into account the effect of ventilating ducts in reducing the effective area of the air-gap. As shown by the spacing of the lines in Fig. 255, the density of the flux diminishes towards the centre of each duct, so that if it be assumed that the normal induction holds throughout the entire area, a certain proportion of the width of each duct w_d must be subtracted. This proportion, as dependent upon the ratio $\frac{w_d}{l_g}$, has been given in a paper[1] by Mr. Carter, from which is derived the curve of Fig. 256. From the value of L_f the axial length of the pole-face must therefore be deducted an amount equal to $K_3 \times w_d$ multiplied by the number n_d of similar ducts which are present.

If D be the diameter of the armature core, and l_g the length of one air-gap, the length of arc subtending the polar angle at the mean radius of the air-gap is $A = \pi(D + l_g)\,\frac{\text{polar angle}}{360^\circ}$.

[1] *Electrical World and Engineer* of New York, Vol. 38, p. 884.

In order to take into account the fringe at each of the four corners of the pole-piece, the strip along each flank may approximately be extended past the edge for a distance of $\frac{K_1 . l_g}{2}$ in either direction;

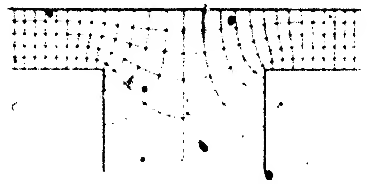


FIG. 255. Field within an duct.

the total permeance is thus the sum of the three portions in parallel, or if L_f = the actual width of the pole-face along the axis of the armature,

$$\frac{A'(L_f - K_3 w_d n_d)}{l_g} + \frac{K_1 l_g (L_f - K_3 w_d n_d)}{l_g} + \frac{K_2 l_g (A' + K_1 l_g)}{l_g}$$

$$= \frac{(A' + K_1 l_g)(L_f + K_2 l_g - K_3 w_d n_d)}{l_g}$$

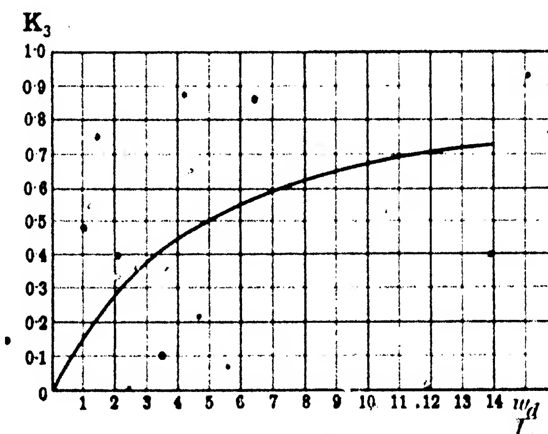


FIG. 256.—Deduction for ventilating ducts.

Hence the effective area of the air-gap of the smooth-core armature is

$$a_g = (A' + K_1 l_g)(L_f + K_2 l_g - K_3 w_d n_d)$$

and from this is calculated the normal air-gap density $B_g = \Phi_a / a_g$, whence

$$AT_g \doteq 0{\cdot}8 B_g . l_g.$$

§ 7. Calculation of air-gap permeance of toothed armature.— Owing to the presence of open slots in a toothed armature the direction of the lines in the interferric gap is no longer strictly radial from pole to armature core, nor is their density within the bore of the pole-piece uniform. Along the edges of the slots the lines curve round and enter the iron through the sides or flanks of the teeth, and as their path is then longer than that of the lines which enter directly through the face of the teeth their density is less. Fig. 255 will, in fact, equally well represent the field about a slot of which the width w_s is three times that of the gap. Thus the total flux is divided into bands of dense and weak density corresponding to the teeth and slots, as may be made visible by iron filings introduced into the gap after the method described by Dettmar (*E.T.Z.*, vol. 21, p. 944). The flux in the air-gap and the actual curved paths of the fringe at the slot-edges are also well shown by the stream-line photographs (Fig. 257) described at the beginning of § 6. The gradual broadening of each stream-line passing into the slot indicates the weakening of the density, or conversely the density is proportional to the number of stream-lines cut per unit area in a plane normal to their direction.

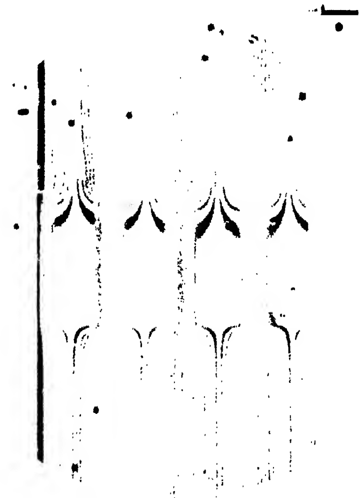


FIG. 257.—Lines of flux in air gap of toothed armature. Permeability of teeth = 100.

The permeance of the air-gap of the smooth armature (apart from the fringes) being equal to the area under the pole-face divided by the normal length of the air-gap, the permeance in the case of a toothed armature is evidently less by some amount depending mainly upon the relative proportion of the width of tooth to the opening of the slot and of the width of the slot to the air-gap. The question is, however, still more complicated, since with any given values for the ratios of the width of the tooth at its top to the width of the slot opening and of the latter to the air-gap, *i.e.* with fixed values of w_{t1}/w_s and w_s/l_g (Fig 258) the ratio of the permeance of the slotted armature to that of the smooth armature of the same dimensions may not always be the same if other conditions are much varied.

In the smooth-surface armature the bored face of the pole-piece and the portion of the armature cylinder corresponding thereto at once give two equipotential surfaces between which there is a certain uniform loss of magnetic potential corresponding to the air-gap excitation. But in the case of the toothed armature the air-gap and teeth project, as it were, into one another, so that the mapping out of the boundaries that are to be assigned to the air-gap and teeth excitation respectively is not so easy a matter. In Fig. 258 there have been drawn a number of equipotential surfaces cutting the flux at right angles, and from this figure it will be seen that if an equipotential surface be taken which coincides with the iron surface along the tops of the teeth it will be crenellated round the periphery of the toothed iron core, since it dips downwards into the slots between the teeth.

Let a point x be considered in the middle of the air-gap, fixed relatively to the pole, in a line with the centre of one of the packets of which the armature is composed and opposite or nearly opposite to the centre of the pole-face. The induction at this point x will then vary in time, as the toothed armature rotates, and the opening of a slot comes opposite to it: its instantaneous value at time t may be symbolised as $B_{x,t}$, and it will pass through a cycle of values in the time required for a tooth-pitch to pass by the point x. When the centre of the crown of a tooth in the packet is exactly opposite to the point x, the induction $B_{x,t}$ will reach its maximum value,[1] which may be symbolized as $\mathbf{B}_{g\,max}$. Corresponding thereto, there will be a certain difference of magnetic potential across from one side to the other of the air-gap, viz., $\mathbf{B}_{g\,max}\, l_g = 1{\cdot}257\, AT_g$, and this provides us with a measure for the ampere-turns of air-gap excitation expended between the two equipotential surfaces that are to be used, the one being the bored face of the pole, and the other coinciding with the tips of the teeth but dipping downwards into the slots. Now the average value of $\mathbf{B}_{g\,max}$ over a tooth-cycle

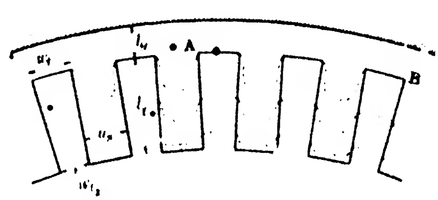


FIG. 258.—Equipotential magnetic surfaces of toothed armature.

[1] The point of maximum induction on the iron surface of a tooth occurs along the corner at its extreme edge; this local effect does not spread to any great distance through the air-gap from the iron edge towards the pole-face, but it makes it necessary to define $B_{g\,max}$ as the mean between the density at the centre of a tooth and the maximum on the pole-surface which occurs opposite to it. The increased densities at the edges of the teeth are well shown in Dr. T. F. Wall's experiments (*Journ. I.E.E.*, Vol. 40, p. 550).

will also be the maximum average value of B_g over a tooth-pitch between the lines CD, EF (Fig. 259), and since it does not vary in time, will be written $B_{g\ max}$: although an average over a tooth-pitch, it is a maximum owing to our choice of the position to be considered, viz., nearly opposite to the centre of a pole and along the central line of a packet of discs. It is this constant quantity which has been intended in previous equations containing $B_{g\ max}$, and the use of which it is proposed to continue, its great serviceableness in design being due to the fact that from it can quickly be found the uncorrected density B'_t at any section of the tooth. Next, let $1{\cdot}257\ AT_a = \mathbf{B}_{g\ max} \times Kl_g$. Then $K = \dfrac{\mathbf{B}_{g\ max}}{B_{g\ max}}$ and Kl_g is the effective length of the air-path between the two equipotential surfaces if they were concentric as they would be with a smooth core, but with the same ventilating ducts. A definition of the air-gap of the toothed armature of which the permeance is to be found is thus obtained. The difficulty still remains that K is not strictly a constant even with the same values of w_g/w_s and w_s/l_g; it is also, although to a less degree, dependent upon the depth of the slot, upon the taper of the teeth, and upon their permeability, since upon these latter factors depends the exact curvature of the lines within the slot, and therefore the depth of the crenellations of the equipotential surface. Why this should be so is rendered more evident if we consider the extreme cases of a very shallow slot, and secondly of a highly tapered tooth.

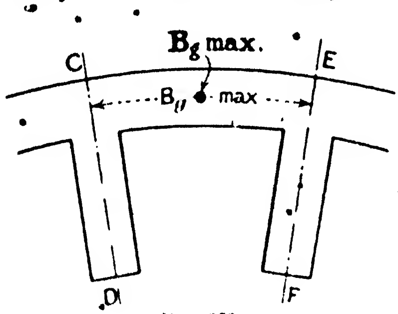


FIG. 259.

In the actual armature there is always present a tendency for some flux to pass straight down the centre line of a slot into its bottom surface with a density inversely proportional to the direct distance from the pole-face to the bottom. Such a tendency is not very evident in Figs. 257 and 248, since here the sides of the teeth are parallel, and their permeability is constant and high, so that little potential is absorbed over their length, even though the depth of the slot is considerable. Such a tendency is, however, well shown in other stream-line diagrams given in the above quoted paper of Professor Hele-Shaw, Dr. Hay, and Mr. Powell, and it would be strengthened if the reluctance of the teeth were increased; the large number of ampere-turns expended over the teeth would

then be available for assisting towards such a field, the lines within the slot would be slightly straightened, and the equipotential surface defining the air-gap would fall at a higher level within the slot. Still more would this be the case when the slots are very shallow, as in Fig. 260, when the lines would be further straightened, and the proportion of the flux entering the faces of the teeth would be higher than in an armature with the same values of w_{ti}, w_s, and l_g, but with deep slots.

Again, if the taper of the tooth is considerable (Fig. 261), the same straightening effect is produced: the density at the root would rise to such a high value that lines begin again in appreciable numbers to issue forth into the slot as there shown. The taper of the tooth is itself related to the ratio of the area at the root to the area at the top, which has been already discussed in Chapter XIII, § 39.

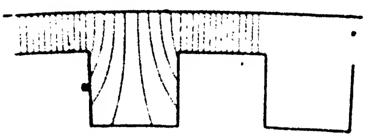


Fig. 260.—Shallow slot.

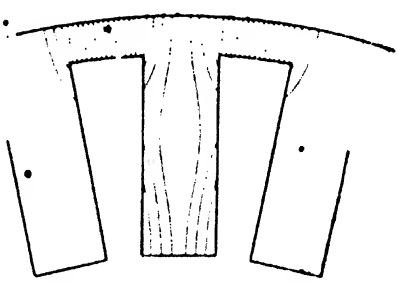


Fig. 261. Effect of tooth with considerable taper.

But enough has been said to show that K is not strictly determined solely from the comparative values of air-gap and widths of slot and teeth, but is also dependent upon the magnetic condition of the teeth and the depth of the slot.

In order, therefore, to simplify the problem, some of the less important factors which enter into it must be discarded, and only such variables be retained as suffice to determine with reasonable accuracy the cases which occur in ordinary practice. The assumptions may at once be made that the iron teeth are very permeable as compared with the air, that the sides of each tooth are sensibly parallel, so far at least as the entrances of the slot are concerned; and, further, that the depth of the slot is by comparison so great that no lines penetrate to the bottom of it. With these simplifications the mathematical solution of Mr. Carter, on which is based Fig. 256, is also applicable to the determination of the effective length of the air-gap, and has been so applied by him in the above quoted paper.[1] The validity of these assumptions, and the accuracy of the

[1] "Air-Gap Induction," *Elect. World and Engineer*, 30th November, 1901.

mathematical results, when applied to the practical cases of slotted armatures, are amply borne out by the stream-line figures obtained by Professor Hele-Shaw, Dr. Hay, and Mr. Powell, and by the experiments of Dr. T. F. Wall (*Journal I.E.E.*, vol. 40, p. 550).[1] While the case of an air-duct is most conveniently treated as reducing

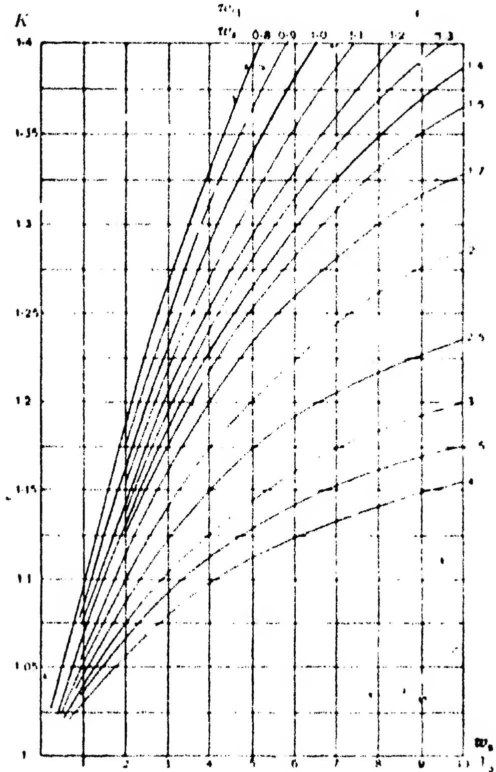


FIG. 262.—Extension coefficient for effective air-gap in toothed armatures.

the effective area of the air-gap, the case of teeth and slots spread over the whole area is best treated for the reason given above as increasing the effective length of the gap. The contraction coefficient upon the supposition that $B_{g\,max}$ held over the peripheral length of a tooth-pitch when this is reduced by subtracting $K_3 w_s$ from it

[1] For the effect of tooth reluctance and for the small influence of the depth of slot, if not very shallow, see especially F. M. Roeterink, *Archiv für Elektrotechnik*, Vol. 7, 1919, p. 292. Cp. also E. Gaas, *Archiv für Elektrotechnik*, Vol. 9, 1920, p. 231.

would be $C = \frac{w_{t1} + w_s(1 - K_3)}{w_{t1} + w_s}$. The extension coefficient is then

$$K = \frac{1}{C} = \frac{w_{t1} + w_s}{w_{t1} + w_s(1 - K_3)}.$$

Since $B_{g\,max} = C \times \mathbf{B}_{g\,max}$, we thus reach the same value for AT_g, either as $0{\cdot}8\mathbf{B}_{g\,max} \cdot l_g$ or as $0{\cdot}8B_{g\,max} \cdot Kl_g$. In this way, therefore, is constructed the set of curves given in Fig. 262 for the coefficient K by which the direct distance from the tips of the teeth to the pole-face must be multiplied in order to obtain the effective length of air-gap, or the AT_g as calculated for a smooth core must be multiplied to obtain the real AT_g of the toothed armature.[1]

For the case of half-closed slots the same curves may again be used, but with the values of K increased by 2 or 3 per cent., w_s being replaced by the actual slot opening w_3. Comparison of Mr. Powell's figures[2] with Mr. Carter's results shows that the widening of the slot below the mouth increases K, as might be expected, but in practical cases only by a small percentage, unless either the overhang be very great or the lip be very thin, in which cases the increase in the value of K may rise to 8 or 9 per cent.

When the pole-face is slotted as well as the armature surface, *i.e.* when both rotor and stator are toothed, as in turbo-alternators with non-salient poles, an empirical formula[3] gives for the combined result

$$K = \tfrac{1}{2}(k_1k_2 + k_1 + k_2 - 1). \qquad (112)$$

where k_1 and k_2 are the calculated extension coefficients for the whole air-gap length for rotor and stator separately with the opposite face assumed to be unslotted.

Given the value of AT_g and of $B_{g\,max}$ in the toothed armature, the determination of the total air-gap permeance and of the total flux of a pole-pitch Φ_a, at least to a close degree of approximation, does not present any serious difficulty. The flux of a tooth-pitch right across the axial length of the armature is closely

$$B_{g\,max}(w_{t1} + w_3)(L_f + K_2l_g - K_3w_d n_d).$$

This assumes that for the same axial projection in the toothed as in the smooth armature beyond the pole-piece edge, the effective joint width of the flank-fringes K_2l_g is the same in spite of the surface of the armature surface being partially cut away by the slots. But since this width is only reckoned as being filled with flux at the density $B_{g\,max}$ whereas at the centre of a tooth the width

[1] For the same values plotted on a different method, see T. C. Baillie, *Electr.*, 8th Jan., 1909. [2] *Journ. I.E.E.*, Vol. 40, p. 230.

[3] For which the writer is indebted to Mr. B. Hague, of the City and Guilds (Engineering) College. For a different formula, see F. W. Carter, *Trans. Amer. I.E.E.*, Vol. 34, Part I, p. 1132, and *Electn.*, Vol. 81, p. 400.

is actually filled at the density $B_{g\,max}$, the error involved is inappreciable.

As regards the interpolar fringe, the influence of the dentated surface becomes less important towards the centre of the interpolar gap, where the path of the lines from the pole to the armature core is in any case of considerable length. It is only therefore near to the pole-edge that the density of the fringe is much reduced owing to the broken surface of the core. It therefore suffices, as in the previous case, to assume the same peripheral width of fringe as in the smooth armature, viz. $K_1 l_g$, filled with the density $B_{g\,max}$.

The same expression for the effective area of the air-gap, a_g, as given in § 6 for the smooth armature, will thus be retained for the toothed armature, viz.,

$$a_g = (A' + K_1 l_g)(L_f + K_2 l_g - K_3 w_d n_d) \quad . \quad . \quad (113)$$

but

$$AT_g = 0{\cdot}8\, B_{g\,max} \cdot K l_g \quad . \quad . \quad . \quad . \quad . \quad . \quad (114)$$

The total air-gap reluctance of one pole-pitch is

$$\mathcal{R}_g = \frac{K l_g}{(A' + K_1 l_g)(L_f + K_2 l_g - K_3 w_d \cdot n_d)}$$

$$\Phi_a = \frac{1{\cdot}257\, AT_g}{K l_g} \times (A' + K_1 l_g)(L_f + K_2 l_g - K_3 w_d n_d)$$

$$= B_{g\,max} \cdot a_g$$

The mean density over the whole of the area of a pole-pitch, i.e. $kB_{g\,max}$

$$= \frac{\Phi_a}{\pi(D + l_g)L/2p}$$

whence

$$k = \frac{(A' + K_1 l_g)(L_f + K_2 l_g - K_3 w_d n_d)}{\pi(D + l_g)L/2p} \quad . \quad . \quad (115)$$

For a given type of construction in which the number of ventilating ducts is proportional to L_f or L, k becomes practically a constant, and for continuous-current machines may be taken for the preliminary processes of design with commutating poles as 0·66 to 0·7.

It has above been assumed in both the smooth and toothed cases that the bore of the poles is concentric with the armature. When it is larger or the pole-edges are much chamfered off, so that from the centre of the pole l_g increases considerably, $B_{g\,max}$ should strictly be calculated separately for each tooth-pitch, and the mean of the values be derived therefrom, but it often suffices to take a mean value for l_g under the pole-face.

The width $K_3 w_d n_d$ always exceeds $K_2 l_g$ and the net amount by

which L_f is reduced as a general rule more or less nearly balances $K_1 l_s$, so that roughly

$$a_s = \frac{\pi (D + l_s)\beta}{2p} \times L_f \quad \ldots \quad (116)$$

With commutating poles $\beta = 0{\cdot}66$ to $0{\cdot}7$, and therefore also k.

§ 8. Calculation of reluctance of teeth in toothed armatures.—The next question is the determination of the ampere-turns necessary to carry the lines through the teeth and slots from the equipotential surface AB to the unbroken circle of the core level with the bottom of the teeth (Fig. 258). In order to reach a suitable density in the air-gap and to limit the reaction of the armature ampere-turns on the field it is advisable to work the teeth at a high density, so that it becomes necessary to calculate closely the ampere-turns required over the teeth, especially when they are deep and much reduced in area at the bottom owing to their tapering shape.

The commonly accepted method[1] of calculating the ampere-turns of excitation expended over the teeth of the slotted armature presupposes that teeth and slots may be divided into a number of corresponding transverse sections from tip to root, and that in parallel with each section of iron across a tooth is a corresponding strip of air extending across the width of half the slot on either side, and of the same radial depth; next, that the total flux corresponding to the width of the tooth-pitch is in the case of each small section divided between the two parallel paths of air and iron in proportion to their respective permeances. On such an assumption, as the section of a tooth tapers, the sides of the slot being in most cases parallel, the proportion of the total flux which is carried by the iron progressively decreases as the root of the tooth is approached; the density in the tooth progressively rises and may reach a very high figure, yet from this very fact the reluctance of the iron increases, and lines are, as it were, squeezed out into the slot. Although the method in question is far from correctly representing the actual facts in the region near the crown of the teeth, it becomes increasingly true as the root of the tooth is approached, and on this account as will be seen later, it yields practical results of value. The method, strictly speaking, assumes that the equipotential surfaces divide tooth and slot into a number of concentric layers; in reality they consist of a number of ridges and depressions, somewhat as shown in Fig. 258, and these gradually become more and more square-cornered but of less depth as we approach the bottom of the slots, until they finally become nearly concentric circles. The experiments of Professor Hele-Shaw, Dr. Hay, and Mr. Powell show, as might be expected, that the flux in each tooth situated well under a pole,

[1] Due to Parshall and Hobart, "Electric Generators," *Engineering*, Vol. 66, p. 130.

so far from diminishing progressively as we move from tip to root, increases at first for about half the length of the tooth, but shows a reduction again towards the root. This, however, only occurs when the flux-density, approximately speaking, begins to rise above $B_t = 20{,}000$, and it is then that the method finds its application in practice. If the percentage which the flux carried by a tooth bears to the total flux corresponding to the tooth-pitch is plotted in relation to distance along the teeth from tip to root, such typical

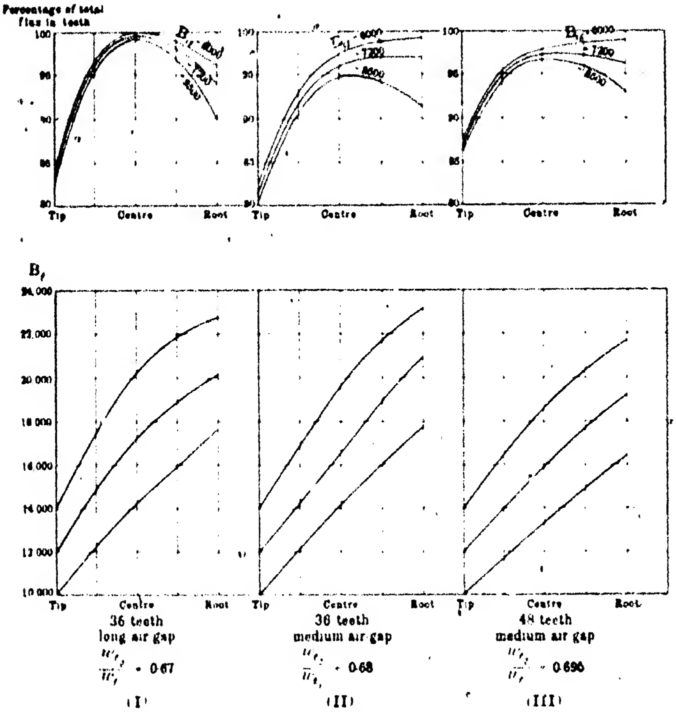


FIG. 263.—Flux-density in teeth of slotted armature.

curves as those in the upper part of Fig. 263 are obtained, which show that for moderate densities in the air-gap and teeth nearly all the flux finds its way into the teeth, and the percentage may rise to close upon 100 per cent. even at the root. But as the flux-densities in the air-gap are increased the curves gradually bend over; the maximum occurs nearer to the tip, and further, this maximum itself usually bears progressively lower and lower ratios to the total flux.

If all the flux passed through the iron tooth from tip to root, the curve of flux-density plotted in relation to distance from the tip towards the root, being proportional to the reciprocal of the width, would be convex when viewed from the axis of distance along the tooth, while, if lines passed out of the tooth owing to its increasing reluctance into the slot, the curves would be straightened and finally become concave. When the experimentally obtained densities are plotted the curves are found always to be concave, and towards the crown of the tooth fall below the curves which

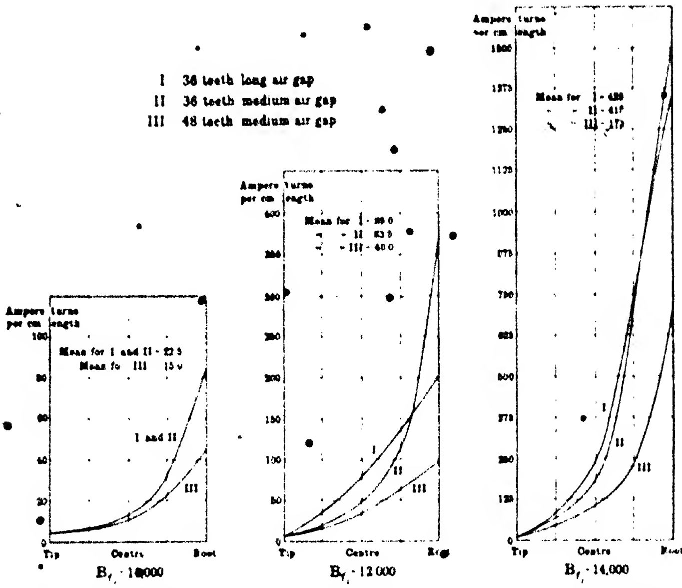


FIG. 264.—Ampere turns per centimetre length of tooth, corresponding to Fig. 263

result from supposing the flux to be divided between iron and air strips in proportion to their relative permeances. Thus the tooth gains a considerable number of lines at first, and the increase is rapid; as the reluctance rises, the increase falls off, yet on the whole there are always more lines in the tooth at its root than at its tip. Thus corresponding to the upper curves of Fig. 263 are the curves of the lower part, and it is from such curves as these that the ampere-turns of excitation over the teeth must be estimated. Owing to the bunching of the lines at the corners of the teeth the density is there somewhat in excess of the density at the centre of the crown of the tooth—an effect which may be seen in the stream-line figures illustrating the paper of Professor

Hele-Shaw and Dr. Hay. The ratio w_d/w_{t1} chosen for illustration being about 0·8, and the flux within the teeth being assumed to be concentrated entirely in the iron without any allowance for the paper or other insulation between the discs, the flux-density close up to the tips of the teeth is about 1·66 times the mean density of the air-gap, so that the tip densities of 10,000, 12,000, and 14,000 correspond roughly to the air-gap densities of $B_g = 6{,}000$, 7,200, and 8,500 of the upper part of the figure.

If the specific ampere-turns required per centimetre length, in accordance with Figs. 209 and 210, are now plotted in relation to distance along the tooth for each of the curves of Fig. 263, we obtain the three sets of curves of Fig. 264, corresponding respectively to the tip densities of 10,000, 12,000, and 14,000. An examination of these suffices to emphasize the great divergencies produced by even comparatively small differences in the amount of reduction of area at the root as compared with the area at the tip of the tooth; and also how complex is the effect of differences of air-gap and depth of slot in causing entirely different shapes of the curve. The mean number of ampere-turns per centimetre has been added to each curve, and it will be seen that on the whole the most decisive factor is the degree of reduction of the area at the root, e.g. the curves (III) for 48 teeth, with which the reduction of area is less, fall uniformly much below those (I and II) for 36 teeth.

It is evident that the specific ampere-turns per centimetre towards the root of the tooth have a much greater influence than those towards the crown, and it is for this reason that the method which is based upon the assumption of corresponding strips of iron and air in parallel affords in practice a reliable guide to the ampere-turns expended over the teeth, especially over the narrower part of each tooth, since it enables us to check the greatest density likely to arise at the root, when the total flux is divided between a small strip of iron at the root of the tooth and small strips of air across the slot and in the ducts in proportion to their relative permeances.

We are thus led finally to a division of the tooth into two halves—from the crown to the centre and from the centre to the root—and their separate treatment, the former on a basis of 80 to 85 per cent. of the total flux of a tooth-pitch as present in the iron at the crown, increasing to 95 to 100 per cent. at the centre as indicated by the curves of Fig. 263 or as calculated from the proportions of iron and air, and the latter on the basis of the corrected densities calculated on the assumption of strips of air and iron in parallel.

The total flux of a tooth-pitch taken right across the armature core under the centre of a pole is

$$B_{g\,max}\,(w_{t1} + w_3)\,(L_i + K_2 l_g - K_3 w_d n_d)$$

At any height up the tooth, the area of iron in the section of a tooth-pitch across the over-all length L of the core is

$$w_t \, \eta \, (L - w_d \, . \, n_d)$$

where η = the efficiency coefficient of the packing of the insulated discs, say, 90 per cent.

The apparent or uncorrected flux-density at any section of a tooth, is therefore

$$B'_t = B_{g\,max} \times \frac{w_{t1} + w_3}{w_t} \times \frac{L_f + K_2 l_g - K_3 w_d n_d}{\eta \, (L - w_d \, . \, n_d)}$$

$$= Q \, . \, \frac{B_{g\,max}}{w_t}$$

where $Q = (w_{t1} + w_3) \dfrac{L_f + K_2 l_g - K_3 w_d n_d}{\eta \, (L - w_d \, . \, n_d)}$

It will be seen that by the use of $B_{g\,max}$ it becomes unnecessary to calculate the number of teeth under the pole carrying the major part of the flux Φ_a. Further, for a machine of given dimensions Q can be calculated once for all, and we then have

Uncorrected density at the crown, $B'_{t1} = \dfrac{Q \, . \, B_{g\,max}}{w_{t1}}$

" " " centre, $B'_{tc} = \dfrac{Q \, . \, B_{g\,max}}{w_{tc}}$

" " " root, $B'_{t2} = \dfrac{Q \, . \, B_{g\,max}}{w_{t2}}$

It may also in passing be mentioned that if $L_f + K_2 l_g$ be identified with L from which it differs but little, the last fraction in the expression for Q becomes $\dfrac{1}{\eta}\left\{1 + \dfrac{w_d \, (1 - K_3)}{l\left(1 + \dfrac{1}{n_d}\right)}\right\}$ where l is the axial length of an iron packet and $L = (n_d + 1) l + w_d n_d$. Since there is, say, a $\frac{1}{2}$-inch ventilating duct between adjacent iron packets, each 3 inches wide, this reduces in average cases approximately to $\dfrac{1}{0{\cdot}9} \times \dfrac{1}{0{\cdot}9} = \dfrac{1}{0{\cdot}81}$ as given in equation (95).

Next, on the assumption of strips of iron and air in parallel, at any section across a tooth-pitch let K_s = the ratio of the total combined section of iron and air to the net section of iron alone. Its value, which is unity for solid iron without air-spaces will rise according to the amount of air in slots, ducts and insulating laminæ between the discs. It cannot be less than $1/\eta$ = say, 1·11, and it may rise to as much as 4 at the roots of the teeth of a two-pole turbo rotor, when there are many and wide ducts and wide slots. Then if H be the magnetizing intensity acting over 1 cm. length of

iron and giving therein a real density of B_t, the apparent density if the total flux in both iron and air be imagined to be concentrated within the iron[1] is

$$B_t' = B_t + H\,(K_s - 1)$$
$$= B_t + 1{\cdot}257\ at\ (K_s - 1)$$

where at is the specific number of ampere-turns per cm. length required to give B_t in the iron of the teeth. It is therefore easy to construct from an original B_t, at curve a set of curves connecting B'_t and B_t for different values of K_s as shown in Fig. 265.

Considering the width of a tooth-pitch, its total section at any height up the tooth and across the length L of the core is $(w_t + w_s)L$, and the area of iron being $w_t\,\eta\,(L - w_d\,.\,n_d)$,

$$K_s = \frac{w_t + w_s}{w_t} \times \frac{L}{\eta(L - w_d\,.\,n_d)} \quad . \quad . \quad (117)$$
$$= \frac{w_t + w_s}{w_t} \times Q'$$

The value of $Q' = \dfrac{L}{\eta\,(L - w_d\,n_d)}$ can also be worked out once and for all for a given machine, and we have

$$\text{at the centre of a tooth, } K_{sc} = \frac{w_{tc} + w_s}{w_{tc}} \times Q'$$

$$\text{and} \quad ,, \quad \text{root} \quad ,, \quad ,, \quad K_{s2} = \frac{w_{t2} + w_s}{w_{t2}} \times Q'$$

The method recommended for calculating the ampere-turns required over the whole length of a tooth is then as follows:—

(1) *From the crown to the centre.* Assuming 85 and 95 per cent. of the total flux of a tooth-pitch as present in the iron at the crown and at the centre respectively, the corrected densities are $B_{t1} = B'_{t1} \times 0{\cdot}85$ and $B_{tc} = B'_{tc} \times 0{\cdot}95$, for which the corresponding values at_1 and at_{tc} for the specific ampere-turns per cm. length are found from Fig. 211. The value for B_{tc} may also be checked by comparison with Fig. 265 for the given value of K_{sc}, and the value for at_{tc} thence found. Up to this point since the densities do not, as a rule, much exceed 19,000, the difference between the apparent and real densities is small, and further the average value of the ampere-turns per cm. length does not exceed greatly the arithmetical mean of the maximum and minimum values. The ampère-turns required from crown to centre of tooth are thus

$$\tfrac{1}{2}\,(at_1 + at_{tc})\,l_t/2.$$

The actual curve of at being convex when viewed from the horizontal

[1] Throughout the following, see S. Neville on "Magnetic Calculations for Tapered Teeth" in Hawkins, Smith, and Neville, *Papers on the Design of Alternating-Current Machinery*, p. 255.

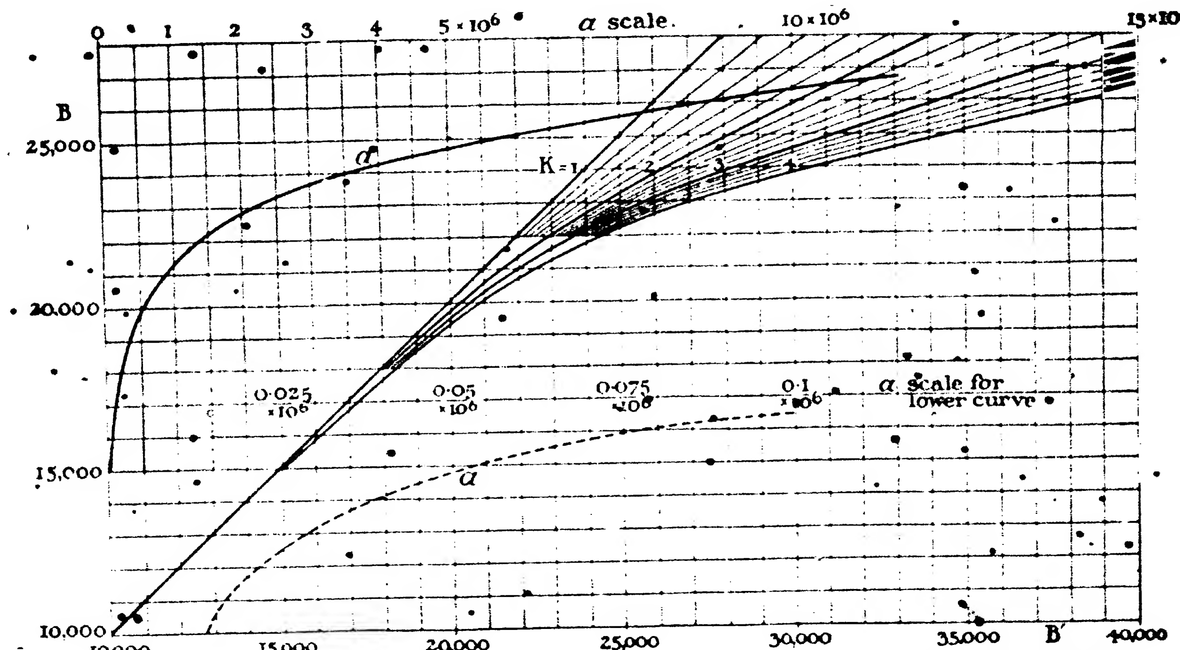


FIG. 265.—Calculation of ampere-turns required by tooth reluctance (Based on Fig. 2 in "The Calculation of Tooth Reluctance," by S. Neville, B.Sc., *Journ. I.E.E.*, Vol. 58, p. 61.)

axis of abscissae, the ampere-turns for the wider part of the tooth are thus over-estimated, but by comparison with those for the narrower part of the tooth the inaccuracy is not usually of any serious account.[1]

(2) *From the centre to the root.* The simplest and best procedure for calculating the ampere-turns for tapering teeth when highly saturated at the root is the modified form of Hird's method which has been worked out by Mr. Neville,[2] requiring the curves reproduced in Fig. 265. The values for the uncorrected density at the root and for the corresponding slot ratio, K_{s2}, have already been given. Taking then the uncorrected density B_{t2}' on the lower horizontal scale, the value of the real B_{t2} for the particular value of K_{s2} is read off from the vertical scale and also along the same horizontal line the corresponding value of the integral, viz. a_2. The same process is repeated for a lower value of B_t at a wider part of the tooth, and a smaller value of a is found. The average excitation per cm. length of the portion of the tooth considered is then

$$at_{av} = \frac{a_2 - a}{B_{t2} - B_t}$$

From the centre of the tooth onwards to its root, the method finds its application, so that in our case for the narrower half of the tooth

$$at_{av} = \frac{a_2 - a_c}{B_{t2} - B_{tc}} \qquad . \quad . \quad . \quad (118)$$

Thus from Fig. 265 is found B_{t2}, and the two values of a for B_{t2} and B_{tc} respectively for insertion in (118). The total ampere-turns expended over the length of a tooth are then

$$AT_t = \{\tfrac{1}{2}(at_1 + at_c) + at_{av}\} l_t / 2 \qquad . \quad . \quad (119)$$

If the exact equipotential surfaces in tooth and slot and the exact distribution of the flux were known, a more accurate determination could be made graphically by plotting the specific ampere-turns as in Fig. 264 and integrating the area by planimeter to find the mean ordinate.[3] But the relative proportions in which the flux should be assigned to tooth and slot are seldom so accurately known as to warrant any great degree of refinement in the method of calculation, although for irregularly shaped teeth the graphical method is invaluable.

[1] If greater accuracy is desirable, when the tooth is highly tapered and the uncorrected densities are high, a better division of the tooth would be into thirds, one-third from the crown being dealt with as above, leaving two-thirds to be dealt with by the method now to be described.

[2] "The Calculation of Tooth Reluctance," *Journ. I.E.E.*, Vol. 58, p. 61. Cp. also Mr. Neville's paper on "Magnetic Calculations for Tapered Teeth" in *Papers on the Design of Alternating Current Machinery*, by Hawkins, Smith and Neville (Sir Isaac Pitman & Sons), p. 255.

[3] Cp. Hawkins, Smith and Neville, *Papers on the Design of Alternating Current Machinery*, p. 258, and Figs. 20, and 5, pp. 99, 158.

§ 8. Calculation of ampere-turns for a multipolar field-magnet.—The above methods of calculation will now be applied to the design of an 80-kilowatt 4-pole dynamo with slotted drum armature, giving an output of 350 amperes at a pressure of 230 volts at the dynamo terminals when running at 500 revolutions per minute. The chief dimensions of the iron carcase which forms the magnetic circuit are given in Fig. 266, from which it will be seen that it has an external circular yoke-ring of cast steel with circular poles of rolled ingot iron bolted to the inside of the ring. The armature core is 21″ in diameter × 11″ long, and a preliminary estimate will lead us to allow an internal loss of volts of about 6 volts over the resistance of the armature winding, to which must be added a loss of 2 volts over the two sets of carbon brushes and of 1 volt over the series winding of the field, assuming it to be compound-wound. The total E.M.F. which it must generate is therefore 239 volts, and it is to have 450 active conductors arranged as a simple lap winding in 75 slots each containing 6 wires, in two layers of three abreast. Hence by equation (44*a*)

$$E_a = 239 \text{ volts} = \Phi_a \times 450 \times \frac{500}{60} \times 10^{-8}$$

and $\Phi_a = 6{\cdot}38 \times 10^6$ C.G.S. lines per pole.

The single air-gap is $0{\cdot}3125'' = 0{\cdot}794$ centimetre, and the length of the armature core exceeds that of the pole-face by $\frac{1}{4}''$ on each side. The width of the polar arc of 63° at the mean diameter of $21{\cdot}3125''$ is $A' = 11{\cdot}7''$, and the half interpolar gap measured on the circumference of the armature $c = \pi \times 21'' \times \frac{13{\cdot}5}{360} = 2{\cdot}47$, whence $c/l_g = 7{\cdot}9$. From the lowest curve of Fig. 253, $K_1 = 2{\cdot}27$, so that the effective breadth of the air-gap is $A' + K_1 . l_g = 11{\cdot}7 + 0{\cdot}71 = 12{\cdot}41'' = 31{\cdot}5$ centimetres.

The ratio $\frac{a}{l_g}$ being $\frac{0{\cdot}25}{0{\cdot}3125} = 0{\cdot}8$, the value of K_2 from Fig. 254 is 1, and $K_2 . l_g = 0{\cdot}3125''$. The ratio of the width of an air-duct to the air-gap being $\frac{w_d}{l_g} = \frac{0{\cdot}5}{0{\cdot}3125} = 1{\cdot}6$, K_3 from Fig. 256 is $0{\cdot}24$, so that the amount to be deducted is $K_3\, w_d\, n_d = 0{\cdot}24 \times 0{\cdot}5 \times 3 = 0{\cdot}36''$. The effective width of the air-gap across the core is therefore

$$L_f + K_2\, l_g - K_3\, w_d\, n = 10{\cdot}5 + 0{\cdot}3125 - 0{\cdot}36 = 10{\cdot}4525'' = 26{\cdot}6 \text{ cm.}$$

The effective area of the air-gap is therefore by equation (113) $26{\cdot}6 \times 31{\cdot}5 = 838$ sq. cm., and the density over it which is also the maximum density is

$$B_{g\,max} = \frac{6{,}380{,}000}{838} = 7{,}610$$

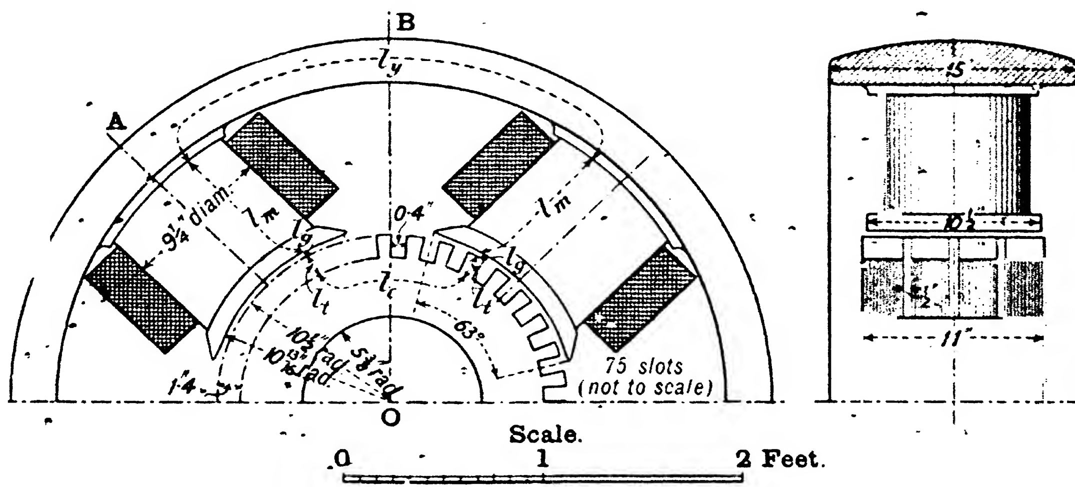


FIG. 266.—Dimensions of magnetic circuit of 4-pole armature, 21 in. diameter × 11 in. long.

Taking the separate elements of the magnetic circuit in the order of equation (108), we have—

(1) *The armature core* below the teeth, requiring AT_c ampere-turns. The depth of a slot being 1·4″, the diameter of the armature core below the slots is 18·2″, and the internal diameter of the discs where they begin to be cut away by ventilating apertures is 10·75″. The radial depth of iron between these two limits is therefore on either side $h_c = 3·725″$. Along the length of the core there are three air-ducts each ½″ wide; after deducting their total width from the gross length of the core, and also after allowing 10 per cent. for the space occupied by the insulation between the discs, the net length of iron is $L_i = 9·5″ \times 0·9 = 8·55″$. The net area of iron in the two cross-sections between which the flux issuing from a pole is divided is therefore $2h_cL_i = 2 \times 3·725 \times 8·55$ square inches = 412 square centimetres. The maximum density over a cross-section between the poles is thus

$$B_c = \frac{6,380,000}{412} = 15,500$$

The mean length of path through the armature is shown in Fig. 266 by the dotted line marked l_c.

By reference to the curves of Fig. 211 for stamped armature discs, it will be seen that for a density of 15,500 lines per square centimetre an excitation of about 25 ampere-turns per centimetre length of path is required. It is, however, only between the pole-tips that the maximum density obtains; across a section under the centre of a pole B_c is zero, and midway between these extremes the necessary excitation is only about 5 ampere-turns per centimetre length of path. On the other hand, it must be remembered that, as pointed out in Chapter XIV, § 12, and Chapter XIII, § 38, the above calculated maximum B_c is averaged over the whole of an interpolar cross-section, and that, as shown in Fig. 221, the local density may be higher over a short length of path. A fair value will therefore be obtained for the ampere-turns over the core if we assume the above maximum B_c to hold over a length of path shorter than that shown in Fig. 266, but longer than the interpolar space and equal approximately to 0·4 of the pole-pitch, say 6·6″ = 16·75 centimetres. Hence

$$AT_c = at_c \cdot l_c 2 = 25 \times 8·375 = 210 \text{ ampere-turns.}$$

Even if such an estimate is not quite correct, an error here will produce but little effect on the total result.

(2) *The armature teeth*, requiring AT_t ampere-turns.

The width of a tooth-pitch at the surface, centre and root of the teeth respectively being $\frac{\pi \times 21}{75} = 0·88$ in., $\frac{\pi \times 19·6}{75} = 0·821$ in.,

and $\frac{\pi \times 18 \cdot 2}{75} = 0 \cdot 763$ in., and the width of slot being 0·4 in., the widths of a tooth at its crown, centre and root are 0·48, 0·421 and 0·363 in. The value of Q is

$$(w_{t1} + w_s)\frac{L_i + K_1 l_g - K_2 w_d n_d}{\eta(L - w_d \cdot n_d)} = 0 \cdot 88 \times \frac{10 \cdot 45}{8 \cdot 55} = 1 \cdot 075$$

and $B_{g\,max} Q = 7{,}610 \times 1 \cdot 075 = 8{,}195$, while $Q' = 11/8 \cdot 55 = 1 \cdot 29$. Hence

Uncorrected Density.	*Slot Ratios.*	*Corrected density.*	
At the crown –			
$B'_{tc} = \frac{8{,}195}{0 \cdot 48} = 17{,}000$		$B_{t1} = 17{,}000 \times 0 \cdot 85 = 14{,}450$	$at = 14 \cdot 8$
At the centre –			
$B'_{tc} = \frac{8{,}195}{0 \cdot 421} = 19{,}500$	$K_{sc} = 1 \cdot 29 \frac{0 \cdot 821}{0 \cdot 421} = 2 \cdot 52$	From Fig. 265, $B_{tc} = 19{,}280$	$a_c = 0 \cdot 4$
		From Fig. 211	$at_{tc} = 180$
At the root –			
$B'_{t2} = \frac{8{,}195}{0 \cdot 363} = 22{,}550$	$K_{s2} = 1 \cdot 29 \frac{0 \cdot 763}{0 \cdot 363} = 2 \cdot 71$	From Fig. 265, $B_{t2} = 21{,}500$	$a_2 = 1 \cdot 1$

In the lower half of the tooth, therefore

$$at_{av} = \frac{(1 \cdot 1 - 0 \cdot 4)10^6}{21{,}500 - 19{,}280} = \frac{700}{2 \cdot 22} = 316$$

By equation (119) the length of a tooth being $l_t = 1 \cdot 4'' = 3 \cdot 56$ cm.,

$$AT_t = \left\{\tfrac{1}{2}(14 \cdot 8 + 180) + 316\right\} \times 1 \cdot 78$$
$$= 173 + 562 = 735.$$

(3) *The air-gaps*, requiring AT_g ampere-turns.

For the ratios $\frac{w_s}{l_g} = \frac{0 \cdot 4}{0 \cdot 3125} = 1 \cdot 28$, and $\frac{w_{t1}}{w_s} = \frac{0 \cdot 48}{0 \cdot 4} = 1 \cdot 2$, the value of K from Fig. 262 is 1·09. Thence by equation (114)

$$AT_g = 0 \cdot 8 B_{g\,max} \cdot K l_g = 0 \cdot 8 \times 7610 \times 1 \cdot 09 \times 0 \cdot 794 = 5275$$

The extreme edges of the pole-tips are slightly opened out, but the percentage effect will be so small that it may be neglected, or, say, $AT_g = 5{,}300$.

(4) *The back ampere-turns, AT_b.*

The method by which these are calculated will be given in Chapter XIX. For the machine now under consideration in the absence of commutating poles they may be reckoned at full load as = 555.

The result so far is

$$AT_c = 210$$
$$AT_t = 735$$
$$AT_g = 5{,}300$$
$$AT_b = 555$$
$$\text{Sum} = 6{,}800 = AT_r$$

and the excitation acting between the poles of the machine is twice this amount. The preponderance of the item due to the air-gap is evident at a glance.

The next step necessitates a knowledge of $\mathcal{S}_l$, the leakage permeance corresponding to a magnetic circuit of the machine in question. For the present this may be taken as = 60, the method by which this numerical value is obtained being reserved for explanation in the ensuing sections. Thence the leakage flux is by equation (110),

$$\phi_l = 1{\cdot}257 \times 2AT_r \,.\, \mathcal{S}_l = 1{\cdot}257 \times 13{,}600 \times 60 = 1{,}025{,}000$$
$$\text{and } \Phi_m = \Phi_a + \phi_l = 7{,}405{,}000$$

the increase of the flux being 16 per cent.

(5) *The magnet-cores and pole-pieces*, requiring AT_m ampere-turns.

The area of the rolled ingot-iron cores forming the poles $9\frac{1}{4}''$ in diameter is 67 square inches = 432 square centimetres, and the average density in them is

$$B_m = \frac{\Phi_m}{a_m} = \frac{7{,}405{,}000}{432} = 17{,}150$$

In order to secure a thoroughly stable machine and even compounding, as will be more fully described in the next chapter, a high density has been chosen for the pole-core, and a reliable material is employed such as may be counted on to give uniform results in practice. Since at high degrees of saturation the specific ampere-turns increase very rapidly, it will be advisable, in order to be on the safe side, to take an intermediate value between curves 2 and 3 of Fig. 206, which has reference to ingot-iron forgings, and to assume not less than 70 ampere-turns per centimetre length, so that any slight inferiority of the material may be guarded against. The length of path in each magnet-core under the exciting coil and through the pole-shoe is $l_m = 9\frac{1}{2}'' = 24{\cdot}1$ centimetres. Within the pole-shoe the density decreases, but against this may be set the fact that it is laminated, which to some extent checks the spreading outwards of the lines to the extreme flanks and corners. We thus have

$$AT_m = at_m \,.\, l_m = 70 \times 24{\cdot}1 = 1{,}690$$

(6) *The yoke*, requiring AT_y ampere-turns.

The section of the yoke is 36 square inches, but as the flux is

divided between two such sections, their joint area is double or 464 sq. cm. The density is

$$B_y = \frac{7,405,000}{464} = 16,000$$

The mean length of path is $l_y = 32'' = 81{\cdot}4$ centimetres, and by reference to Fig. 207 for steel castings it is found that the ampere-turns will be approximately 31 per centimetre. Thence

$$AT_y = at_y \,.\, l_y/2 = 31 \times 40{\cdot}7 = 1,260$$

The excitation required is therefore

$$AT_f = 6,800 + 1,690 + 1,260 = 9,750 \text{ ampere-turns per pole.}$$

§ 10. Formulæ of the leakage permeances.—We have now to return to the question of the magnetic leakage ϕ_l, and in the first place to the calculation of the *leakage permeance*, $\mathfrak{P}_l$, of any dynamo. Since we have to do with a large number of paths, which are assumed to be all in parallel with one another, it is simplest to deal with their permeances, for these can be immediately added together to discover the joint permeance of the leakage paths. The lines are assumed to follow certain directions in the air according to the situations and distances of the two surfaces between which they flow, and to meet these different cases three general propositions are, as a rule, sufficient.

(i) In the case of two parallel surfaces facing each other, the areas of which are approximately equal, the lines of flux may all be assumed to pass straight across from the one surface to the other. The permeance of the air-gap between the two surfaces is then, on the C.G.S. system, equal to the mean of their areas divided by their perpendicular distance apart (Fig. 267); or

$$\mathfrak{P} = \frac{\frac{1}{2}(A_1 + A_2)}{l} \quad . \quad . \quad . \quad . \quad (120)$$

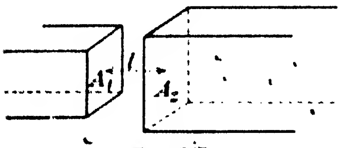


FIG. 267.

This does not, however, take into account the transverse pressure on the tubes of flux which causes them to curve outwards at the edges of the air gap, so that the permeance is under-estimated by an amount which increases with an increase in the distance l.

(ii) In the case of two equal rectangular surfaces situated near each other in the same plane, with their neighbouring edges parallel, the lines of flow may be assumed to be semicircles described about a central line c, c drawn between the two surfaces (Fig. 268); the

permeance of the air-path from one to the other is then, on the C.G.S. system—

$$\mathcal{P} = \frac{a}{\pi} \log_\epsilon \frac{r_2}{r_1} \quad . \quad . \quad . \quad . \quad . \quad (121)$$

where r_1 and r_2 are respectively the distances from the central line to the nearest and farthest edges of either rectangle, and a is the depth of each rectangle at right angles to r, *i.e.* along the parallel edge.

(iii) In the case of two equal rectangular surfaces situated similarly in one plane, but at some distance apart, the lines of flow may be assumed to be quadrants connected by straight lines, the quadrants being described from the neighbouring edges of the rectangles as

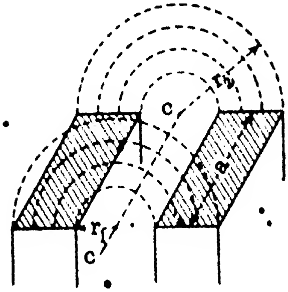


FIG. 268.

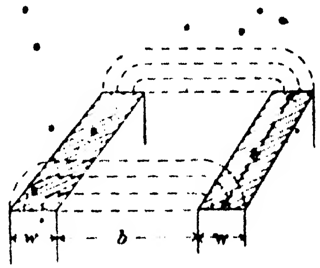


FIG. 269.

lines of centres; the permeance of the air between the two surfaces is then, on the C.G.S. system—

$$\mathcal{P} = \frac{a}{\pi} \log_\epsilon \frac{\pi w + b}{b} \quad . \quad . \quad . \quad . \quad (122)$$

where a is again the depth of each rectangle, w its width, and b is their distance apart (Fig. 269).

If the two surfaces of Fig. 268 are rotated about the centre line $c.c$, or in Fig. 269 about their inner edges, so that they do not lie in the same plane but are inclined to one another at some angle other than π, then the value of this angle in circular measure is to be inserted in each place instead of π in equations (121) and (122).

While convenient for calculation in the case of surfaces not in the same plane, propositions (ii) and (iii) like (i) fail to allow for the tubes of flux being bent outwards by the transverse pressure. A nearer approach to the facts is therefore made when the surfaces lie in one plane by assuming that the paths are ellipses having their foci at the inner edges of the gap. The permeance[1] is then

$$\mathcal{P} = \frac{a}{\pi} \log_\epsilon \left\{ 1 + 2 \frac{w + \sqrt{w^2 + wb}}{b} \right\}$$

[1] The formula, due to Mr. Finnis, is given in "Permanent Magnets in Theory and Practice," by S. Evershed, *Journ. I.E.E.*, Vol. 58, p. 821.

To these may be added other extensions of the same principles,[1] but in general the three main propositions above considered will meet the most common cases, or in default of any other guidance it must suffice to map out a probable course for the leakage on the above lines, and thence by scaling the mean areas of the two surfaces and the mean length of path between them to deduce the permeance $\frac{\text{area}}{\text{length}}$.

It must be remembered that in all the foregoing equations the dimensions are in centimetres and the logarithms are to the Napierian base ϵ; hence, for English dynamo designers, it may be useful to give the equivalent equations when the dimensions are in inches, and the areas in square inches; they are—

$$\text{(i)}\quad \mathcal{P} = 2{\cdot}54 \cdot \frac{\frac{1}{2}(A_1'' + A_2'')}{l''} = \frac{1{\cdot}27\,(A_1'' + A_2'')}{l''} \quad . \quad (120a)$$

$$\text{(ii)}\quad \mathcal{P} = \frac{2{\cdot}54\,a''}{\pi} \log_{10} \frac{r_2''}{r_1''} \times 2{\cdot}3 = 1{\cdot}86 \times a'' \times \log_{10} \frac{r_2''}{r_1''} \quad . \quad (121a)$$

$$\text{(iii)}\quad \mathcal{P} = 1{\cdot}86 \times a'' \times \log_{10} \frac{\pi a'' + b''}{b''} \quad . \quad . \quad . \quad . \quad (122a)$$

and alternatively, when the surfaces lie in the same plane

$$\mathcal{P} = 1{\cdot}86 \times a'' \times \log_{10} \left\{ 1 + 2\,\frac{w + \sqrt{w^2 + wb}}{b} \right\} \quad . \quad (122b)$$

§ 11. Calculation of the leakage permeance.—By the aid of the foregoing equations an approximate calculation of the leakage permeance of a dynamo can be made when a drawing showing the main outlines of its magnetic field system is to hand. In the case of a multipolar field-magnet such as Fig. 266 the component leakage paths fall naturally into four groups, namely, (1) between the tips of the pole-shoes across the interpolar gap, (2) between the flanks of the pole-shoes, (3) from the sides of the pole-cores partly (*a*) across the interpolar gap, and partly (*b*) into the under-surface of the yoke, and (4) from the flanks of the pole-cores, partly (*a*) across to the neighbouring poles and partly (*b*) into the yoke-ring. Taking the permeance of these several paths in succession, and assuming them to be approximately as shown in Fig. 270, we proceed as follows, the machine in question having the dimensions there given.

(1) Between the edges of the pole-shoes in the interpolar gap. By scaling off from the drawing the mean width w_1 and length l_1

[1] See especially J. F. H. Douglas, "The Reluctance of Some Irregular Magnetic Fields," *Trans. Amer. I.E.E.*, Vol. 34, Part I, p. 1067.

of the air-path, and bearing in mind that there are two such interpolar gaps corresponding to each pole

$$\mathcal{F}_1 = 2.54 \frac{L_f'' \times w_1}{l_1} \times 2 = \frac{2.54 \times 10.5 \times 1}{6.25} \times 2 = 8.5$$

(2) Between the flanks of the pole-shoes.

Each half of a pole-shoe may be regarded as a rectangle of mean width $w_2 = 6''$ and of 1″ depth, from which lines curve round into

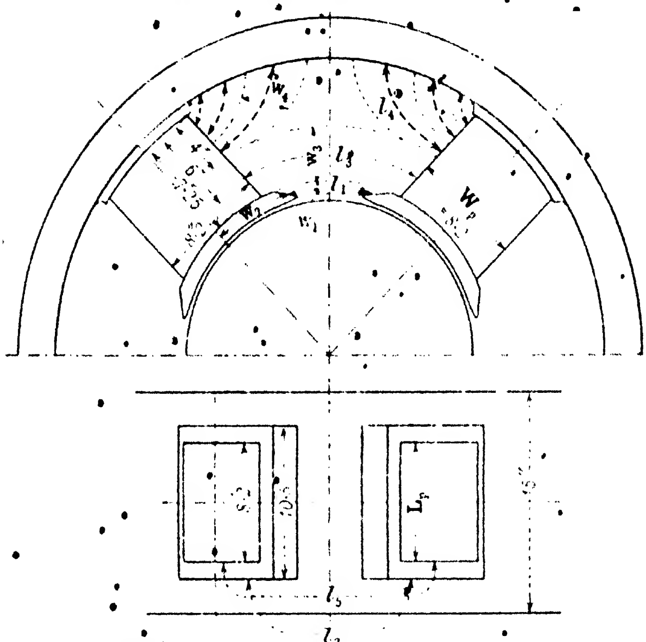


FIG. 270.—Leakage paths of multipolar magnet.

the corresponding half pole-shoe of a neighbouring pole, as indicated by the line l_2. The direct distance separating the edges of the two pole-shoes is again practically equal to l_1. Thence by proposition (iii), taking the paths as partly quadrants and partly straight lines, and with four such paths, two at the front and two at the back

$$\mathcal{F}_2 = 1.86 \times 1 \times \log \frac{\pi \times 6 + 6.25}{6.25} \times 2 \times 2 = 4.48$$

(3) From the pole-cores.

(*a*) Across the interpolar gap into the opposite side of a neighbouring pole. The distance along the pole-core (reckoning inwards from the yoke) at which the leakage passes across into the adjacent

pole rather than into the yoke, is determined by the point at which the reluctance of the latter path is half that of the former. It thus diminishes with an increase in the number of poles as fixing the angle by which the opposite pole-sides diverge from one another; and also depends upon the length of the bobbin or pole (cp. Fig. 237 for internal poles). It is further affected by the shape of the yoke-ring, and would, *e.g.*, be lower down the bobbin-length if the yoke-ring were octagonal, so as to bring its under surface nearer to the pole-side. In the present case the division of the leakage flux is estimated to take place at a distance of 4″ down the bobbin out of a total bobbin-length of $7\frac{1}{4}$″.

Next, the leakage out of the pole-cores takes place under varying differences of magnetic potential, as we pass from the yoke to the pole-shoes. The yoke-ring midway between the poles is at zero potential, and the magnetomotive forces of the ampere-turns are uniformly distributed along the length of the two bobbins, the potential reaching a positive and negative maximum respectively at their inner ends. The difference of magnetic potential between the pole-sides is thus the gradually increasing sum of the magnetomotive forces, less the fall of potential over the iron, and rapidly rises in value as we proceed from the zero potential of the yoke towards a pole-shoe. The loss of potential over the iron may be approximately assumed to be confined to the limits of the bobbins and to be uniform in value over each inch of their length; the difference of potential between the pole-sides is then at any point $= 1{\cdot}257\ X_r \cdot \frac{x}{l}$, where x is the distance of the point in question from the root of the pole at the yoke, and l is the bobbin length.[1]

The permeance of the two areas, one on each side of the pole, is thus $2{\cdot}54 \frac{L_p \times w_3}{l_3} \times 2$, where L_p is the transverse length of the rectangular pole, and this is acted upon by a difference of potential of $1{\cdot}257\ X_r \cdot \frac{6}{7{\cdot}25}$, the mean distance of the surface in question from the root of the pole being estimated to be 6″. The total leakage flux resulting therefrom is

$$\phi_3 = 2{\cdot}54 \times \frac{L_p \times w_3}{l_3} \times 2 \times 1{\cdot}257\ X_r \times \frac{6}{7{\cdot}25}$$

In order, then, to rank the permeance of (3a) with those of (1) and

[1] For a more accurate graphical method, see Miles Walker, *The Specification and Design of Dynamo-electric Machinery*, p. 326, and for "stepped" coils see Dr. R. Pohl, *Journ. I.E.E.*, Vol. 52, p. 173. For measurement of difference of magnetic potential by Chattock's magnetic potentiometer, see Appendix to "The Magnetic Testing of Bars of Straight or Curved Form," by A. Campbell and D. W. Dye, *Journ. I.E.E.*, Vol. 54, p. 43.

(2), it must be reduced in the proportion of $\frac{6}{7 \cdot 25}$ before it can be regarded as in parallel with the preceding permeances, and the leakage lines be deduced on the assumption that they are all due to the full difference of potential X_p at the poles. For our purpose, therefore, it will be necessary to take the value

$$\mathscr{P}_3 = 2 \cdot 54 \times \frac{L_p \times w_3}{l_3} \times 2 \times \frac{6}{7 \cdot 25} = \frac{2 \cdot 54 \times 8 \cdot 2 \times 3 \cdot 8 \times 2 \times 6}{13 \cdot 75 \times 7 \cdot 25} = 9 \cdot 5$$

(b) Into the under side of the yoke. The permeance may here be estimated by an approximate division of the space in question into, say, three tubes, and adding together the results deduced from the mean cross-section and length of each of the three tubes, such as the one marked w_4, l_4 (Fig. 270). Again, allowance must be made for the varying difference of potential, but as opposed to the preceding case the difference of potential under which this leakage flows is $1 \cdot 257 \frac{X_p}{2} \cdot \frac{x}{l}$.

Since there is the same distribution of flux on each side of the pole, the permeance of a pair of similar tubes, one on each side, in relation to X_p is $2 \cdot 54 \frac{L_p \times w_4}{l_4} \times \frac{x}{l}$, this quantity when multiplied by $1 \cdot 257\ X_p$ giving the required number of leakage lines through the pole-core. The permeance of the three pairs of tubes into which the space has been divided is thus—

$$2 \cdot 54 \times \frac{8 \cdot 2 \times 3 \cdot 5}{7 \cdot 9} \times \frac{3 \cdot 5}{7 \cdot 35} = 4 \cdot 45$$

$$2 \cdot 54 \times \frac{8 \cdot 2 \times 2}{4 \cdot 5} \times \frac{2}{7 \cdot 25} = 2 \cdot 55$$

$$2 \cdot 54 \times \frac{8 \cdot 2 \times 1 \cdot 1}{2 \cdot 25} \times \frac{0 \cdot 75}{7 \cdot 25} = 1 \cdot 05$$

$$\mathscr{P}_4 = 8 \cdot 05$$

The true permeance of each tube is practically constant, since as the area is increased so also is the length of path, but the equivalent permeance for our purpose decreases owing to the diminishing number of ampere-turns which act as the root of the pole is approached.

(4) From the flanks of the pole-cores.

The leakage into a neighbouring pole-flank and into the yoke respectively may again be divided along a line similar to that which separated the corresponding fluxes from the pole-side, *i.e.* at a distance of 4″ from the top of the bobbin out of the total length of 7¼″.

(*a*) Across into the flank of a neighbouring pole on either side. Treating each half-flank as a rectangle $\frac{W_p''}{2}$ in width, the two adjacent halves being separated by a direct distance equal to l_8, the equivalent permeance of the four paths such as l_8 which are concerned is by proposition (iii).

$$1{\cdot}86 \times 3{\cdot}8 \times \log \frac{\pi \times W_p/2 + l_8}{l_8} \times 2 \times 2 \times \frac{6}{7{\cdot}25} =$$

$$1{\cdot}86 \times 3{\cdot}8 \times \log \frac{\pi \times 4{\cdot}1 + 13{\cdot}75}{13{\cdot}75} \times 4 \times \frac{6}{7{\cdot}25}$$

$$\mathcal{P}_5 = 6{\cdot}7$$

The area to which the lines have here been confined is in reality too small, since they can spread out widely over a longer path :[1] the present item will, therefore, be increased to $7{\cdot}5 = \mathcal{P}_5$.

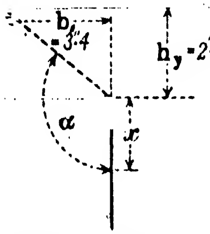


FIG. 271. Leakage from pole-flank into overhanging yoke.

(*b*) Into the yoke-ring. The area of the pole-flank may here be regarded as repeated in the edge of the yoke, since, although this is usually of less radial depth, it is of greater width and spreads out on either side of the pole. Further, owing to the edge of the yoke overhanging the pole-core, the angle α may be approximately reckoned from the relation (Fig. 271)—

$$\sin (180° - \alpha) = \frac{b_1}{\sqrt{h_y^2 + b_1^2}} = \sin \alpha$$

in our case $= \frac{3{\cdot}4}{\sqrt{2{\cdot}4^2 + 3{\cdot}4^2}} = 0{\cdot}816$, whence $\alpha = 125°{\cdot}3$. Expressing α in circular measure, the permeance of a small strip of air of width dx, and stretching across the pole-flank, is $\frac{dx \times W_p}{x \times \alpha}$, and this is acted upon by $1{\cdot}257 \frac{X_p}{2} \cdot \frac{x}{l}$. The flux on either side is therefore $1{\cdot}257 \frac{X_p}{2} \cdot \frac{W_p}{l.\alpha} \int dx$, and since $\int dx = 4''$, the equivalent permeance on both sides is

$$\mathcal{P}_6 = 2{\cdot}54 \frac{W_p}{\alpha} \times \frac{4}{7{\cdot}25} = 2{\cdot}54 \frac{8{\cdot}2}{2{\cdot}19} \times \frac{4}{7{\cdot}25} = 5{\cdot}25$$

This item also should be increased, say, to $\mathcal{P}_6 = 6$.

[1] Cp. J. F. H. Douglas, "Leakage Flux Calculations," *Electr.*, 17th Sept., 1915.

All the permeances have now been reduced to values which may be regarded as in parallel with the armature and air-gaps, and which may be immediately added together.[1] The leakage from items (3) and (4), strictly speaking, does not all pass through the entire length of pole-core and yoke, yet for the purpose of an approximate calculation this may be assumed. The joint leakage permeance is then

$$\mathfrak{P}_l = 8{\cdot}5 + 4{\cdot}48 + 9{\cdot}5 + 8{\cdot}05 + 7{\cdot}5 + 6 = 44{\cdot}03$$

and the number of lines passing through the magnet is $\Phi_a + \phi_l$, where

$$\phi_l = 1{\cdot}257\, X_p \,.\, \mathfrak{P}_l = 1{\cdot}257 \times 2AT_p' \times \mathfrak{P}_l$$

In addition to the several paths already considered, there are others which would have to be taken into account if greater exactitude were required, such as, for instance, from the pole-flanks into the top of the yoke-ring or into the flywheel if this is in close proximity to the magnet. Especially does the presence of sharp edges tend to concentrate the leakage, and increase its amount. But in their actual distribution the lines never cross one another or intersect, so that if new paths or surfaces are added a fresh re-arrangement of the lines must be made different from that assumed in our first approximation; certain areas must then be contracted, so that on the whole recalculation will not lead to a very different result. The estimate may, indeed, seem to err in being too high, since, as already mentioned, all the lines that may be found at the particular section of maximum flux, *e.g.* at the root of the pole, do not flow through its entire length; the total flux is in fact varying all along the length of magnet and yoke, as lines leak into or out of them. Yet in spite of this, such calculations as the above usually err in being too low, and their approximate character must be fully recognized. Since maxima values are required for the purposes of designing, rather than minima, in order that the error if any may be on the safe side, it is very necessary to allow an ample margin by increasing the calculated permeance by some 30 to 40 per cent. The comparative figures obtained by calculation when so increased are justified by the experimental results of practice, and by their use a very close approximation may be made to the ampere-turns actually found to be required when the machine is tested.

[1] So far as the average ampere-turns per cm. length of pole-core are concerned, Mr. J. F. H. Douglas (*loc. cit.*) has shown that they are closely reproduced in normal cases by imagining 72 per cent. of the total (*i.e.* maximum) distributed leakage flux to be concentrated at the pole-tips, or else the whole of it to occur at a point half-way between pole-tip and yoke. But in order to make use of this fact it would appear to be necessary first to determine the total leakage flux in its true distributed state, which can seldom be done immediately by a simple mathematical formula but only by a somewhat tedious graphical process.

Thus, in the case considered above, the leakage permeance will for the purposes of design be taken as $\mathcal{P}_l = 60$ instead of 44·3 as calculated. The machine sketched in Fig. 270 has a rectangular section of pole and yoke. If the section of the pole is circular, a sufficiently close approximation is obtained by substituting for it a square pole of equivalent area. Each side of the square is then $\frac{d}{2}\sqrt{\pi} = 0{\cdot}885d$, and it is for this reason that in Fig. 270 a pole 8·2″ square has been chosen as being the equivalent of the pole of Fig. 266 of diameter $9\frac{1}{4}$″; similarly an equivalent for the rounded section of the yoke, namely, 24″ × 15″, has been taken, so that the assumed permeance of 60 has been used in § 9 in the determination of the ampere-turns of the magnetic circuit of Fig. 266.

A certain amount of judgment is required in the first instance in selecting the direction of the several paths, none of which intersect; but to guide us in this, actual experiment may be called to our aid, if a machine with the required type of field-magnet is at hand. If a needle be fastened to a thread passing up the centre of a hollow tube of wood or cardboard, and the end of the thread be held so as to prevent the needle from being drawn to a pole, it can be used as an exploring magnet, and on plotting the directions in which it sets itself a good idea can be obtained of the distribution of the leakage flux.

The effect of a false estimate of $\mathcal{P}_l$ will mainly depend upon the degree of saturation of the field magnet, since, the higher the induction, the greater is the difference between the ampere-turns required to pass the supposed and the actual number of lines Φ_m through the magnet. But a considerable percentage error in its determination, does not, under ordinary conditions, give rise to nearly so large an error in the total number of ampere-turns required to produce the useful field through the armature; it only affects the ampere-turns required for the magnet and yoke, and the final error thus introduced will depend on the relative amount of $AT_m + AT_y$ as compared with the total AT_f.

§ 12. Empirical formulæ for leakage permeances.—If the value of $\mathcal{P}_l$ for a particular machine of a given type has once been determined by calculation and checked by experiment, certain further conclusions may be drawn applicable to other machines of the same type, but differing in their dimensions. If the linear dimensions of one machine were simply magnified n times in the design of another, the new value for $\mathcal{P}_l$ would be n times its former value; the lengths of the leakage paths would be increased n times, but their cross-sections would be increased n^2 times. More often, however, the ratio of the two chief dimensions of the dynamo, namely, the diameter and length of the armature core, is altered, and this calls for a more detailed consideration of the way in which

he leakage permeances are affected by any change in these dimensions.

Taking the four items of the leakage permeance seriatim, the first or that between the pole-tips varies directly as the length of armature core, and inversely as the pole-pitch. The radial depth of the pole-shoe does not vary greatly in machines of the same construction but of different size, so that the sectional area of the path varies only with the length of core, while the interpolar gap which fixes the length of path bears almost a constant relation to the pole-pitch $\pi D/2p$. Hence this permeance is proportional to $\frac{L}{D/2p}$, or with the same number of poles is determined by the ratio of the length to the diameter of the armature core.

The leakage (2) between the flanks of the pole shoes is independent of the length, and is also practically independent of the diameter. The radial depth of the pole-shoes being assumed constant, the half width of the pole-shoe flank increases with the pole pitch or with the diameter for a given number of poles, but so also does the length of the path between the neighbouring surfaces. The permeance is therefore practically a constant quantity, independent of the dimensions and of the number of poles.

The leakage (3a) across the interpolar gap between the sides of the pole-cores varies directly as the length of the pole core parallel to the axis of the armature, and this length itself bears a ratio to the length of the armature core which varies but little even with different numbers of poles if the machines are of similar type. The proportion of the radial length of the exciting coil over which this leakage extends remains approximately the same for a given number of poles, so that the absolute radial depth of the area across which the lines in question flow varies with the absolute length of the exciting bobbin. The length of this latter depends chiefly upon the air gap and flux-density therein, yet, roughly speaking, for the same number of poles it increases proportionately to an increased diameter of armature. But so too does the length of path across the interpolar gaps, since the ratio of the polar arc to the pole-pitch is unaltered. The length of path and radial depth of area thus neutralize one another, so that for the same number of poles the permeance in question is dependent only upon the length of armature core. With an increase in the number of poles the proportion of the radial depth of bobbin from which the lines pass immediately across to a neighbouring pole is increased, and for the same diameter the distance to be traversed is reduced; but on the other hand the greater number of poles has the effect of reducing the length of bobbin, so that the latter with large machines, which usually have many poles, becomes more nearly constant. The leakage (3a) thus increases but slowly with an increase in the number of poles. The leakage (3b) into the yoke on its under side is independent of the diameter, and conversely to (3a) decreases with an increase in the number of poles, since the proportion of the radial depth over which it extends is reduced. When, therefore, the two are grouped together, they become almost solely dependent upon the length of the armature core. This may be investigated on the lines of Fig. 272, by means of which the dimensions become expressible in terms of $D/2p$ on the assumption that the bobbin-length ranges from $1{\cdot}5\,\frac{D}{2p}$ to $2\,\frac{D}{2p}$. When the relative division of the fluxes, the lower part crossing the interpolar gap, and the upper part returning immediately into the

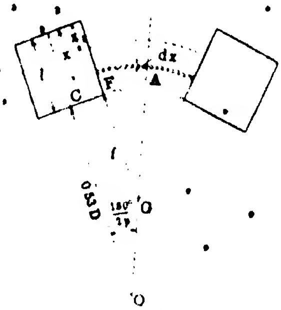


Fig. 272 Leakage between pole cores

Hence the proportion which the leakage lines bear either to the useful or to the total number of lines through the magnet will continuously increase, and for each degree of magnetization of the dynamo $\frac{\Phi_a + \varphi_l}{\Phi_a} = \frac{\Phi_m}{\Phi_a}$ will have a different value. In the case of the drum multipolar dynamo above considered, $\Phi_m = 1{\cdot}16\Phi_a$ when it is magnetized as it would be under ordinary working conditions, so as to obtain a density of about 7,610 in the air-gap; in other words, the leakage is about ·16 per cent of the useful lines, or 13·85 per cent of the total flux. Strictly speaking, the factor by which the useful lines through the armature must be multiplied in order to obtain the greater number of lines flowing through any other part of the circuit will vary at different parts of the magnet, as lines leak into or out of it. With sufficient accuracy, however, the flux may be assumed to be constant over each of our main subdivisions of the magnetic circuit, so that $\Phi_m = \nu_m \Phi_a$, $\Phi_y = \nu_y \Phi_a$, $\Phi_p = \nu_p \Phi_a$; the different parts of the magnet being denoted by the subscript letters as before, each part will have its appropriate factor, ν, greater than unity. Such was the basis of the original form of the equation for the ampere-turns of a magnetic circuit as first published by Drs. J. and E. Hopkinson in their classical paper on *Dynamo-Electric Machinery*. Having established the theoretical equation, they further measured experimentally the number of stray lines that leaked through different portions of the air space about the field-magnets of two dynamos of different types, each of which was excited with its normal magnetizing current. Thence they were enabled to deduce the values which the factors ν_m, ν_y, etc., have for the given dynamos when magnetized to their working degree of saturation.

In most cases we may without much error assume the different factors for the different parts of the magnet to be identical, and reckon the number of lines through any part of it to be the same, namely, $\Phi_m = \nu\Phi_a$ where ν may be called the *leakage coefficient* of the dynamo. The greatest number of lines flows through a section either at the centre or near to the root of each magnet core, the value gradually decreasing from this point up to its end. But, as already stated, in designing the field-magnet what is required is rather the *mean* number of lines Φ_m flowing through the entire magnet, from which we can approximate closely the total ampere-turns required to overcome the reluctance of the magnet.

The actual value of the flux passing any section of the magnetic circuit can be determined by surrounding it with a coil of several turns and measuring the throw upon a ballistic galvanometer when the flux is reversed.[1] By comparison with the flux passing through the armature core the leakage coefficient is determined. But as the ballistic galvanometer requires skilful manipulation and care to obtain correct results, the leakage coefficient may be directly measured by a differential method devised by R. Goldschmidt;[2] two search coils are wound on the magnet core and armature respectively, and are connected in series through a millivoltmeter, so as to oppose one another; their respective numbers of turns are then adjusted until no deflection is observed upon the voltmeter when the field excitation is suddenly varied. A convenient extension of this method has been given by Dr. R. Pohl,[3] in which exact balancing even when the two coils each contain one or the same number of turns is secured by means of the adjustable resistances of a Wheatstone bridge of Post Office pattern.

The values of ν and the percentage distribution of the leakage for different types of dynamos have been experimentally determined and recorded by

with increasing magnetomotive force is one in which the analogy of the magnetic and electric circuits entirely breaks down. For the same reason it is impossible to apply the methods of Kirchhoff's laws to branch magnetic circuits with any real scientific accuracy. This objection does not, however, prevent the analogy from being of great service to the dynamo designer, since the conflicting cases seldom occur in practice.

[1] For certain points relating to leakage testing with a ballistic gavanometer, see (Giles) *Electr. Eng.*, Vol. 8, p. 343.

[2] *E.T.Z.*, Vol. 23, 1902, p. 314.

[3] *Electrician*, Vol. 50, p. 215.

various observers[1]; but, unfortunately, in many cases little or no information has at the same time been given of the degree of saturation of the armature and magnet, beyond the fact that they were magnetized to their normal working extent.

Such recorded values of ν may be used for the purpose of approximately calculating the area and ampere-turns required by the field-magnet. Thus in multipolar dynamos of the type of Figs. 226 7 with external poles, ν ranges from 1·1 at no load to 1·25 at full load. Implicit reliance cannot, however, be placed on any such values of ν. As already mentioned, it varies decisively if the degree of saturation of the armature core be much altered; hence it varies with different values of the "back ampere-turns," *i.e.* with different values of the armature current, even though the total number of ampere-turns on the field remains unchanged; or again, if the transition is made from a smooth-surface armature with long air-gap to one with a short air gap and toothed core. On this account the somewhat laborious task of calculating $\mathfrak{F}_l$ and ϕ_l, even if at best they are only approximations, is decidedly to be recommended (especially when a new type of field-magnet is to be adopted), in preference to the use of a factor ν. Unless this has been experimentally determined, we are more liable to be misled by taking a false value of ν than by errors in determining $\mathfrak{F}_l$.

Finally, it may be remarked that a certain amount of leakage is an inevitable necessity, and is but a small evil. If the energy required to magnetize a dynamo is from 3 to 5 per cent. of its output, as it is in most modern dynamos, magnetic leakage cannot very greatly affect the efficiency and cost of working, for a supposed complete absence of leakage would but slightly decrease this percentage.

§ 14. Leakage permeance and flux-distribution with commutating poles.—When the main poles are excited and the commutating poles are unexcited, the latter, being situated in the neutral planes between the main poles and of no great width, do not greatly affect the leakage flux of the machine, although they increase it by shortening the air-path from main pole to main pole to the extent of the peripheral width of the iron. This effect is of chief account in items (1) and (3*a*), and necessitates their increase beyond the values given in § 12 for the first and third terms of equation (123).

But when the excitation of the commutating pole begins and gradually increases with the load, an additional M.M.F. is brought into action, and the sign of the resultant magnetic potential of the commutating pole agrees with that of the main pole on the one side but is opposite to that of the main pole on the other side. In consequence the difference of magnetic potential across the air-path is as much reduced on the former side as it is increased on the latter.

Experiment confirms that the increase and reduction of the stray flux balance one another,[2] so that to determine the leakage

[1] *Vide* Esson, "Some Points in Dynamo and Motor Design," *Journ. I.E.E.*, Vol. 19, Part 85; "Magnetic Leakage in Dynamos and Motors" (Ives), *Electr. Review*, 22nd and 29th January, 1892; "Magnetic Data of Sprague Motor" (Parshall), *Electr. Eng.*, 13th June, 1890; (Puffer) *Electr. Review*, 15th April, 1892; and especially (Thornton) *Electr. Eng.*, Vol. 29, p. 523 ff.

[2] "Die Wendepolstreuung, etc.," by F. Schimrigk, *Arbeiten aus dem Elektrotechnische Institut zu Karlsruhe*, Vol. 1, p. 225.

flux that must be carried by the commutating pole it is only necessary to calculate the leakage from all of its sides, under its own excitation when the main poles are unexcited, and this amount must be added to the useful reversing flux ϕ_r to obtain Φ_r, the total flux in the commutating pole.

The leakage permeance must, therefore, be estimated by drawing tubes of flux after the fashion shown in Fig. 273 for both sides and flanks and correcting for the difference of magnetic potential which increases from zero at the yoke to the full value at the pole-shoe. Its value will not be far different from $1\frac{1}{2}\,\mathcal{P}_l$, where $\mathcal{P}_l$ is the leakage permeance of the same machine with the commutating pole unexcited, so that if the magnetic potential at its face, AT_{rc}, happened to be equal to AT_p of the main pole, the leakage flux of the commutating pole would be $1{\cdot}257\, AT_{rc} \times 1\frac{1}{2}\,\mathcal{P}_l$ as compared with ϕ_l of the main pole $= 1{\cdot}257\, AT_p \times 2\mathcal{P}_l$, *i.e.* it is about $\frac{3}{4}$ths of ϕ_l. But any such approximate estimate is liable to much variation according to the different proportions of axial length to peripheral width which may occur in practical designs, and should be checked. Owing to the greater surface of the commutating-pole in proportion to its section and the high M.M.F.'s, leakage plays a much more important part than in the main poles, and if not properly calculated will lead to the iron of the auxiliary pole being made too small, especially near the yoke; it will then become very highly saturated, and proportionality of the reversing field to the current will be entirely lost.

With as many commutating poles as there are main poles it has already been stated that for the same useful main flux the average density in the main poles is unaffected by the addition of the commutating poles. But whether the useful flux obtained by a given excitation of the main poles when commutating poles are absent will be maintained practically unaffected when such poles are added and excited, turns entirely upon the degree of saturation of the armature core and yoke, and chiefly of the latter.

It was shown in Chapter XV, § 12, that in the yoke-ring, if Φ_r is the total flux of the commutating pole including leakage, $\Phi_r/2$ is added to the flux carried by each of the sections of the yoke B, B' (Fig. 231), and deducted from each of the other sections A, A', which complete the circuit from one main pole to another, so that the two densities are proportional to $\dfrac{\Phi_m + \Phi_r}{2}$ and $\dfrac{\Phi_m - \Phi_r}{2}$. The very considerable increase of the useful reversing flux ϕ_r by the leakage up to the total flux of the commutating pole Φ_r, is therefore an important factor in the case. Apart from a similar requirement in the section of the armature core, it is only when the section of the yoke is sufficiently large so that it is far from saturation, that in each portion of the path the two changes practically

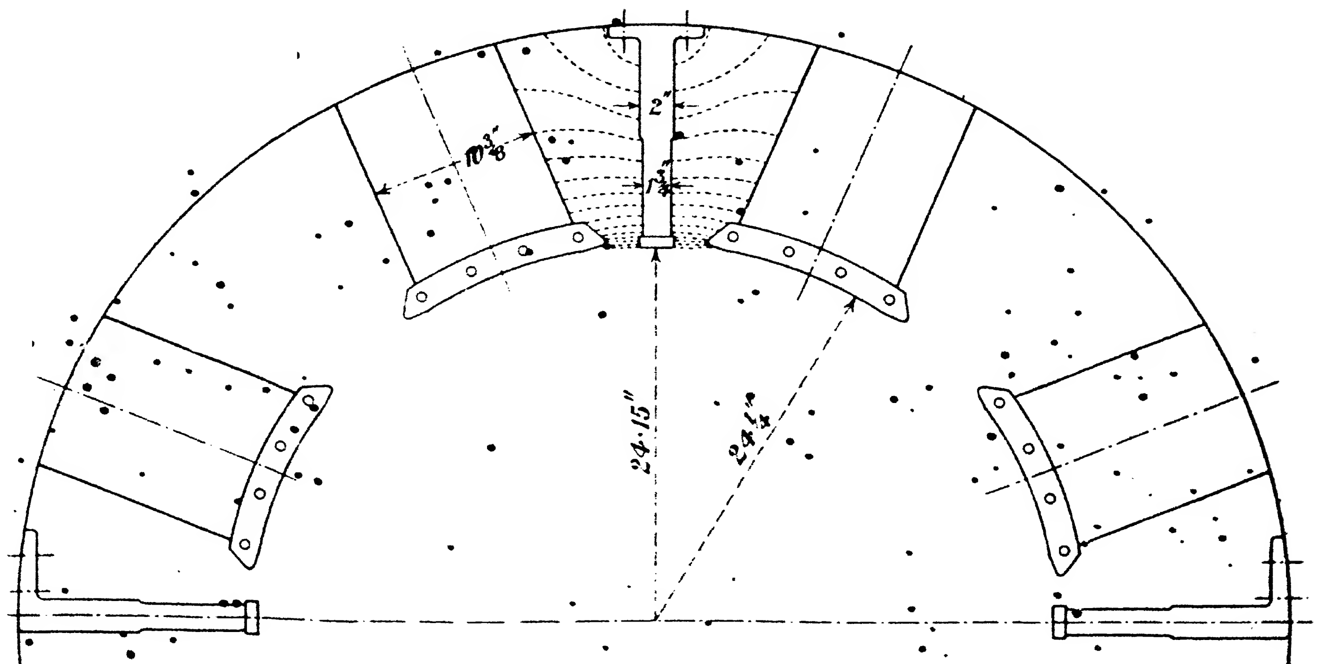


FIG. 273.—Commutating-pole leakage.

counterbalance one another, and the percentage effect on the *total excitation* required for a given main flux remains small; this is in practice usually secured, but the consequences of insufficient attention being paid to this point are traced in Chapter XIX, § 13. The effect of the commutating poles on the main flux is chiefly to cause it to be somewhat unequally distributed over the main pole-faces, the trailing edge being more crowded, so that if the excitation of the commutating poles is annulled the main flux swings back to an equal distribution, and the amount of throw on a ballistic galvanometer caused by the swinging backwards of the flux across the centre line of the main pole is proportional to the commutating flux.[1]

When there are only half as many commutating poles as there are main poles (Fig. 273), there is greater danger from saturation of the magnetic circuit. In the armature and yoke the difference between the more saturated and the less saturated portions is again equal to the reversing flux, *i.e.* the densities are severally proportional to $\frac{\Phi_a + \phi_r}{2}$, $\frac{\Phi_a - \phi_r}{2}$, $\frac{\Phi_m + \Phi_r}{2}$, $\frac{\Phi_m - \Phi_r}{2}$, where Φ_a and Φ_m are the normal fluxes before the addition of commutating poles, but ϕ_r and Φ_r are greater. More than this, an inequality arises between the total fluxes carried by the magnet-cores of each pair of poles; the one of opposite sign to a commutating pole must carry a total flux roughly proportional to $\Phi_m + \frac{\Phi_r}{2}$ and that of the same sign $\Phi_m - \frac{\Phi_r}{2}$. Since each armature loop is acted upon by a pair of poles, this does not affect the E.M.F.'s of the various parallel paths, but the difference of density equal to Φ_r may lead to an appreciable reduction in the normal main flux owing to the greater number of ampere-turns required by the highly saturated pole-core. Further, it may become difficult to dispose of the necessary ampere-turns on the single commutating pole without overheating, so that this arrangement requires care in its application.

§ 15. Flux-curves of dynamo.—It has been shown in § 9 that the number of ampere-turns required per pole to give 6,280,000 lines through the armature of a particular dynamo is 9,750 when the armature current has its full value (*i.e.* the sum of 350 amperes in the external circuit and about 6 amperes in the shunt circuit of its field-magnet, or 356 amperes in all). If a number of different values be assigned to Φ_a, and the ampere-turns required in each case be determined (always on the assumption that the armature current has its full value), a curve may be plotted, connecting together the corresponding values of Φ_a and AT_f (or $2AT_f = X_f$),

[1] Professor E. Arnold, *E.T.Z.*, 15th March, 1906.

TABLE IX.—MAGNETIZATION CURVE.

Φ_a in 10^6 C.G.S. lines	3		4		4·5		5		5·5		6		6·38		6·6	
$AT_c = at_c \times 8{\cdot}375$		25		34		42		67		92		150		210		335
$AT_t = at_t \times 3{\cdot}56$.		20		30		71		118		211		475		735		950
$AT_g = \Phi_a \times 830 \times 10^{-6}$		2487		3320		3730		4150		4565		4975		5300		5475
$AT_b = \frac{356}{4} \cdot 450 \cdot \frac{5}{360}$		555		555		555		555		555		555		555		555
AT_r		3087		3939		4398		4890		5423		6155		6800		7315
$\phi_l = 1{\cdot}257 \times 2\,AT_r \times 60$ . .	0·467		0·595		0·664		0·74		0·82		0·93		1·025		1·105	
$\Phi_m = \Phi_a + \phi_l$ . .	3.467		4·595		5·164		5·74		6·32		6·93		7·405		7·705	
$B_m = \frac{\Phi_m}{432}$ . .	8040		10620		11950		13300		14620		16050		17150		17850	
at_m	2·4		4		4·5		5·5		8·5		22		70		130	
$AT_m = at_m \times 24{\cdot}1$		58		96		109		133		205		530		1690		3135
$B_y = \frac{\Phi_m}{464}$ . . .	7500		9900		11110		12370		13600		14950		16000		16600	
at_y	2·3		5·2		6·4		8·2		11		16		31		44	
$AT_y = at_y \times 40{\cdot}7$.		94		212		261		335		446		650		1260		1790
AT per pole . .		3239		4247		4768		5358		6076		7335		9750		12240

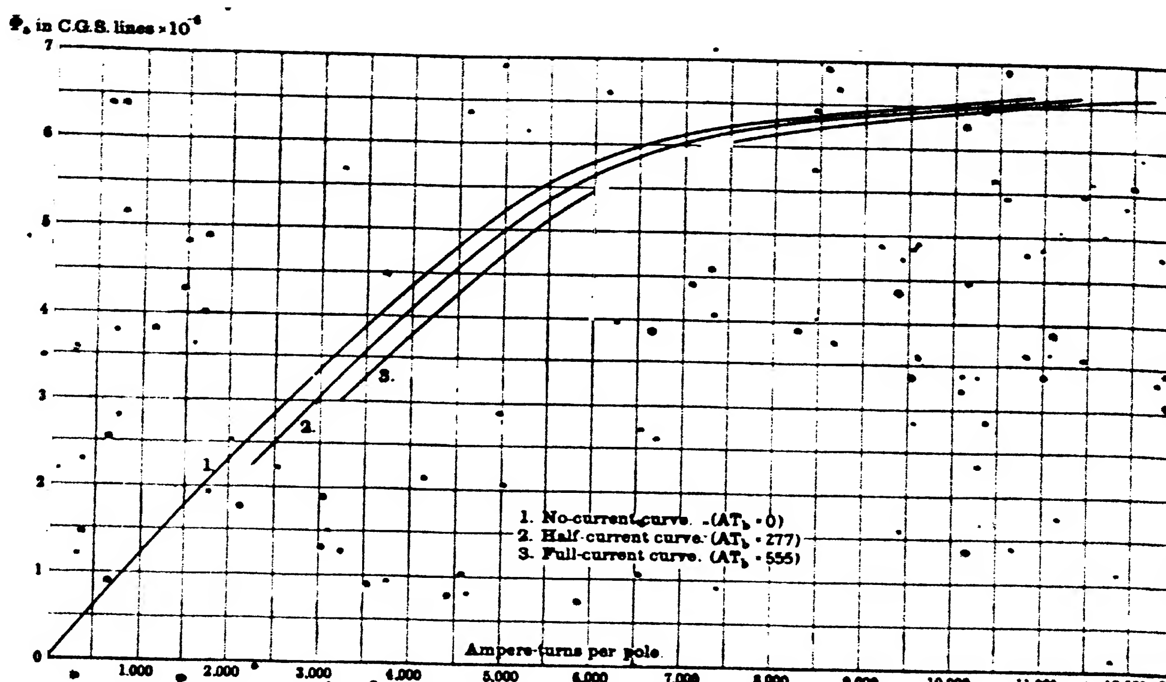


FIG. 274.—Curves of useful flux of 21 in. × 11 in. dynamo.

and this curve may be called the *flux-curve* of the particular dynamo for its full current, or more strictly for the full value of its armature ampere-turns. Curve 3 in Fig. 274 shows such a full-current curve

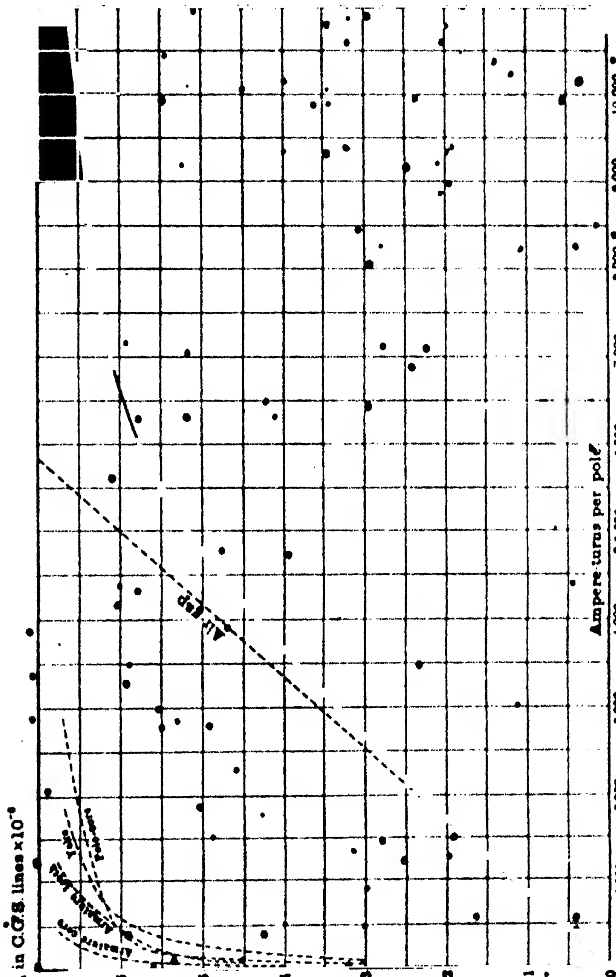


FIG. 275.—No-load flux curve of 21 in. × 11 in. dynamo, and its components.

of flux for our 21″ × 11″ dynamo, I_a being maintained throughout at its full value of 356 amperes. The corresponding values of Φ_a and AT_f may conveniently be worked out in a tabulated scheme after the pattern of Table IX, shown on page 527, which shows the

steps of the calculation in a slightly abbreviated form. Or, if the machine be already built, they may be determined experimentally, as will be shown in Chapter XVII. In either case the brushes are throughout assumed to be adjusted to the point of minimum sparking under normal conditions with the current in question, and the abrupt termination of the curve marks the point below which it is impossible to reduce the magnetization without causing excessive sparking at the brushes. If we again carry out the same process for a different and smaller value of I_a, say half its maximum value, or 178 amperes, a second curve of flux will be obtained for half-current; in the absence of commutating poles this will fall higher than curve 3, inasmuch as the back ampere-turns are less, and therefore fewer ampere-turns are required on the field to produce a given number of lines through the armature.

Finally, if I_a be taken as 0, there is no reaction of the armature current on the field, the back ampere-turns, AT_b or $2AT_b = X_b$ are zero, and the highest or "no-current" curve of flux (1, in Fig. 274, p. 528) is obtained.

The use and importance of these curves will be more apparent in the next chapter. In Fig. 275 is repeated the no-current curve with its various component items shown separately by the dotted curves; such a separation is instructive as illustrating the relative importance of the several portions of the magnetic circuit under different degrees of saturation. A tangent to the initial part of the curve 1 may be called the "air-line," since it gives the ampere-turns required by the air-gap for any value of the flux, and it is of service to determine this experimentally in order to check the calculated values, especially in toothed armatures. It should be observed that the horizontal distance between the curves for no-current and full-current for any particular value of Φ_a is more than the direct value of the back ampere-turns AT_b, namely, 555, inasmuch as these latter increase the leakage, and therefore the ampere-turns required over the iron of the magnet. The horizontal divergence of the curves becomes, in fact, increasingly marked as the magnet approaches saturation.

§ 16. Determination of size of field-wire.—The different ways in which the field-magnets of dynamos are excited, according to the sources whence the magnetizing current is derived, will be explained in the following chapter. Apart, however, from such differences of source, it remains to determine the necessary gauge and weight of copper wire required for the winding of magnetizing coils which are to give a certain number of ampere-turns, AT; these may form either the whole or a part of the total excitation required by the machine, the following being a general solution of the problem applicable to all cases. The data that form the starting-points may vary, but usually they are a knowledge, direct or indirect,

of (1) the voltage that will be applied to the ends of the wire, and (2) the mean length of a turn. Given the thickness and width of a rectangular magnet core, or the diameter of a round core, it is easy to estimate fairly closely the mean length of one turn of a coil encircling either the one or the other; a certain allowance must be made for the depth of the winding, by reason of which the mean length of a turn in the central of several layers of winding will be greater than the actual perimeter of the magnet itself; and the correctness of this allowance must be subsequently checked when the winding has been determined. Experience, however, forms a ready guide on this point; and it may therefore be assumed that the mean length of one exciting turn is known when the dimensions of the iron encircled by the coils are known. Let this mean length $= l_x$, and let $V_x =$ the voltage which will be applied to the ends of the exciting coils. Let $\omega =$ the resistance in ohms of unit length of the required wire at a certain standard temperature, say 60° F.; and let A and T_f be the two factors, the magnetizing current in amperes, and the number of turns per magnet coil, so that if N_c be the number of coils in question $AT_f N_c$ is the total number of their ampere-turns. The resistance of the magnetizing turns (assumed to be all in series) at the standard temperature is $= N_c T_f \times l_x \times \omega$; but when a current passes through them, then, as explained in Chapter XV, § 5, their temperature will gradually rise, and, in consequence, their resistance after a run of some hours will be higher than at starting. It will thus be necessary to multiply their resistance at 60° F. by some coefficient, k, dependent upon their rise of temperature in working, and on the temperature of the surrounding air from which that rise is reckoned; or $R_x = N_c T_f \times l_x \times \omega \times k$.

The final temperature of the coils will be attained when the rate at which the heat is carried away by radiation, convection, and conduction is equal to the rate at which it is generated. The maximum rise of the temperature of the outside of the coils above that of the surrounding air will depend upon the ratio which their cooling surface bears to the rate of generation of heat in them, provided that the coils which are compared are under similar conditions in regard to their effective ventilation. Hence from a knowledge of this ratio, and also of the temperature of the surrounding air, the maximum temperature which will be attained by the outside of the coils in continuous work can be predicted; or, in the case of a finished machine, it can be measured by a thermometer. But with a large number of layers wound closely over one another, as in a field-magnet coil, an appreciable difference of temperature is required to produce the flow of heat from the central layers to the outer surfaces, by conduction partly through the length of the wire itself, and partly from layer to layer across the

intervening cotton covering, this latter being a bad thermal conductor. The temperature of the central layers of the winding is, therefore, considerably higher than that of the outer layer as measured by a thermometer applied to the external surface. The value of k is fixed by the mean temperature of the whole mass of the coil, and is thus dependent not only on the maximum temperature of the external surface, but also upon the depth of the winding. For field-magnet coils when divided into sections and well ventilated, the ratio of the mean rise of temperature to the surface rise as measured by a thermometer varies from about 1·25 for a depth of winding of $\frac{3}{4}$ inch to about 1·6 for a depth of $2\frac{1}{2}$ inches, or 1·75 for 3 inches. If not divided into sections, the ratio will be even higher, and may be 2 or over. For each degree Fahrenheit that the temperature of a piece of copper rises between the limits of 60° and 150° F., its electrical resistance rises 0·222 per cent. of its resistance at 60° F. or 0·22 per cent. of its resistance at 68° F. Hence, to give an idea of the practical value of k, if the field-coils are $2\frac{1}{2}$ inches deep, and the maximum temperature attained by their outside be 45° F. above that of the surrounding air, the mean rise of the temperature of the whole mass will be, say, $1{\cdot}6 \times 45° = 72°$ F.; the value of k, or the ratio of the resistance at the new temperature to the resistance at 60° F. will then be $1 + (72 \times 0{\cdot}00222) = 1{\cdot}16$, when the temperature of the surrounding air is assumed to be 60° F., or $1 + (72 \times 0{\cdot}0022) = 1{\cdot}158$ for the same rise of temperature reckoned from 68° F. (20° C.) as the initial temperature of the engine-room, or as the standard temperature given in many tables of copper conductors.

Now $A = V_s/R_s$; therefore $AT_fN_c = \dfrac{V_s T_f N_c^2}{T_f N_c^2 \times l_s \times \omega \times k}$

whence
$$\omega = \frac{V_s}{AT_fN_c \times l_s \times k}$$

or if l_s be reckoned in yards, and ω' be the resistance of 1,000 yds. of the required wire,

$$\omega' = \frac{V_s \times 1{,}000}{AT_fN_c \times l_s \times k} \quad . \quad . \quad . \quad . \quad (124)$$

The area and diameter of the required wire having a resistance of ω' ohms per 1,000 yards is easily obtained by reference to any table of the resistance of copper wires,[1] or by direct calculation. The international standard[2] for annealed copper wire, 1 metre in length and with a uniform cross-section of 1 sq. mm., being $\frac{1}{58}$th of an ohm

[1] Such as may be found, *e.g.* in Whittaker's *Electrical Engineer's Pocket Book*, p. 62 (4th edit.), or Munro & Jamieson's *Pocket Book of Electrical Rules and Tables*, p. 296 (18th edit.).

[2] In accordance with the Report adopted at the Berlin meeting (1913) of the International Electrotechnical Commission.

at 20° C. (68° F.), the resistance at that temperature of 1,000 yds. of annealed copper wire in ohms is 0·02445 ÷ area in square inches. Hence the area of the required wire is $0{\cdot}02445/\omega'$ sq. ins. ; or

$$\text{area of wire} = \frac{0{\cdot}02445\ AT_f N_c \times l_s \times k}{V_s \times 1{,}000} \text{ sq. ins.} \quad . \quad (125)$$

Thus for the same exciting volts and the same size of coil which gives the same l_s and practically the same heating coefficient k, the ampere-turns are directly proportional to the copper section of the wire. It will also be seen that the above equations are immediately applicable to one coil of AT_f ampere-turns on a multipolar machine or to all the $2p$ coils in series or in parallel or in any combination, if V_s be given the correct value appropriate to the given case.

If the wire is to be of rectangular section, the two dimensions which go to make up the area may be chosen to suit our own convenience in winding ; but if it be round, the necessary diameter in inches is

$$\sqrt{\frac{4 \times \text{area}}{\pi}} = \sqrt{\frac{4 \times \left(\frac{0{\cdot}02445}{\omega'}\right)}{\pi}}$$

whence at the above standard temperature

$$d = \frac{0{\cdot}176}{\sqrt{\omega'}} \quad . \quad . \quad . \quad . \quad (126)$$

From the above formulæ it is evident that if V_s and l_s be fixed, there is but one area or diameter of wire which will satisfy the equation and give the required number of ampere-turns ; and further, that this area or diameter is entirely independent of the actual number of turns in the coils, since neither T_f nor A appears separately in the final equations. This result may at first sight seem surprising, but is easily followed when it is remembered that if the number of turns be, for instance, doubled, the resistance of the coils is also doubled, which, with a given V_s, halves the current through them, and therefore leaves the total number of ampere-turns unaltered. If the number of turns be doubled by winding twice as many layers on the same length of bobbin, it is true that l_s is increased, since the depth of the winding is doubled ; but the effect of this upon the necessary diameter of wire is partially counterbalanced by the reduction which must be made in the value of k. When the number of turns is doubled, the current and the rate at which it generates heat are halved ; the cooling surface is also itself increased, owing to the perimeter of the coil being greater ; and therefore, for both reasons, the mean temperature attained by the coils will be less than before. The lesser value of k thus partly compensates for the increased value of l_s.

§ 17. The mean length of turn.—With a round pole-core of diameter D_1 when insulated or including allowance for insulating packing strips between coil and pole and wrapping on the inside of the coil, the length of an inner turn is $l_i = \pi D_1$, and the mean length of a turn in the central layer is

$$l_x = l_i + \pi t = \pi(D_1 + t) \quad . \quad . \quad . \quad . \quad (127)$$

where t is the total thickness of the copper winding.

In the case of a bobbin wound on a former which closely embraces a pole of rectangular section, if the length of a turn in the bottom layer is $l_i = 2\{(A + \frac{3}{8}'') + (B + \frac{3}{8}'')\}$, allowance for the insulated spool being made by adding $\frac{3}{8}''$ to each of the component dimensions, the length of a turn in any other layer is made up of this length *plus* a quadrant of a circle at each corner, the radius of this circle being equal to the depth of the winding at the layer in question. The mean length of a turn in the central layer is then again $l_i + \pi t$.

But if the pole of rectangular section is cast with a radius r at each corner, and its two dimensions are A and B, its periphery is $2\{(A - 2r) + (B - 2r)\} + 2\pi r$, and with an allowance of $\frac{3}{8}''$ for clearance and insulation

$l_i = 2\{(A - 2r) + (B - 2r)\} + 2\pi (r + \frac{3}{8})$.

The length of the mean turn is then

$$l_x = l_i + \pi t = 2\{(A - 2r) + (B - 2r)\} + 2\pi\left(r + \tfrac{3}{8} + \frac{t}{2}\right) \quad (127a)$$

and of an outer turn is

$l_o = l_i + 2\pi t = 2\{(A - 2r) + (B - 2r)\} + 2\pi (r + \frac{3}{8} + t)$.

It may be worth while to mention that if a continuous length of round wire is wound on to a bobbin in several layers, the layers are of alternate hand, and at one point each turn of an upper layer has to cross over the turn of the layer below it; hence, if $d_1 = d + \delta$ be the diameter of the wire with its insulating covering, and n be the number of layers, the depth of winding at this point is nd_1. Except, however, at the crossing point, the turns of the upper layers bed into the hollows between the turns of the layers underneath. The depth of winding is then

$$\begin{aligned} d_1 + (n - 1)d_1 \sin 60^\circ &= d_1 \{1 + (n - 1)0{\cdot}866\} \\ &= d_1 (n \times 0{\cdot}866 + 0{\cdot}134) \end{aligned}$$

While allowance must be made in the space allotted for the bobbins for the larger value, the mean depth of winding from which l_x is to be calculated may be taken as lying between the two values, or

$$t = 0{\cdot}9nd_1 \quad . \quad . \quad . \quad . \quad . \quad (128)$$

§ 18. Determination of weight of field-wire and dimensions of bobbin.—In order, therefore, to determine the actual number of turns which must be used, or the weight of wire, the necessary area or diameter of which has been determined, some other factor of

the problem must be known. This may be either the number of watts, W, to be lost in the coil or coils under consideration, or the current A; the latter case is, in reality, identical with the former, since $A = W/V_x$, and in many cases W and A form two of the data given at the outset, whence $V_x = W/A$ is at once derived. From our knowledge of W or of A, the total number of turns is at once fixed as

$$T_f N_e = \frac{AT_f N_e}{A} = \frac{AT_f N_e \times V_x}{W}$$

and the total length of wire required is $T_f N_e \times l_x$ yards. The weight of one yard of annealed high-conductivity commercial copper is $11{\cdot}55 \times$ area in square inches $= 11{\cdot}55 \times \frac{0{\cdot}02445}{\omega} = \frac{0{\cdot}282}{\omega}$ lb., therefore the weight of wire is

$$T_f N_e \times l_x \times \frac{0{\cdot}282}{\omega} \text{ lb.};$$

whence, by simple substitution, from equation (124)

$$\text{weight in lb.} = T_f N_e \times l_x \times 0{\cdot}282 \times \frac{AT_f N_e \times l_x \times k}{V_x \times 1{,}000}$$

$$= \frac{AT_f N_e}{A} \times l_x \times 0{\cdot}282 \times \frac{AT_f N_e \times l_x \times k}{V_x \times 1{,}000}$$

$$= \frac{(AT_f N_e)^2 \times l^2_x \times k \times 0{\cdot}282}{W \times 1{,}000}$$

While the above gives the net weight of copper, a small addition must be made for the weight of the cotton covering in order to obtain the gross weight of the insulated wire. Such addition ranges from 7 per cent. with a wire 0·040″ diameter to 1½ per cent. for ⅛″ diameter, its importance becoming less and less as the size is increased.

It should be observed that if our object is simply to form an estimate of the weight of wire required from the above data it is unnecessary first to determine ω, or the actual numbers of turns and layers of the wire. The final settlement of the winding will, however, require the latter to be determined, and when the number of layers has been decided it will be well to check the correctness of the assumed depth of winding underlying the first estimate of l_x, the mean length of a turn.

Round wire single or double cotton-covered, rectangular wire double-cotton-covered, and also wide copper strip, insulated with intervening layers of paper or calico, are used for the winding of field-magnet bobbins according to the circumstances of the case. If a = the section of the copper wire or strip, and a_1 = the space that must be allotted to each insulated wire with due allowance

for interstices between adjacent turns in the case of round wires, $\sigma = \frac{a}{a_1}$ is the ratio of the net volume of copper to the gross volume occupied by the insulated wire. The effect of bedding which increases σ from $0{\cdot}7854 \frac{d^2}{d_1^2}$ to approximately $\frac{0{\cdot}7854\, d^2}{0{\cdot}9\, d_1^2} = 0{\cdot}875 \frac{d^2}{d_1^2}$, improves the efficiency with which the space is utilized considerably. The total thickness of double cotton-covering in the case of round wire is usually from 12 to 15 mils, and in the case of rectangular wire 20 mils. Owing to the bedding, the curve of σ for a field-magnet coil with small round wires double-cotton-covered falls in between curves a and d of Fig. 147, but rises more steeply, and for wires above 0·050″ diameter more nearly coincides with curve b of Fig. 147. The ratio thus in practice ranges from 0·4 to 0·65, and the greater volume of coils in which a high voltage demands a great number of turns of very small wire as compared with low voltage coils with the same number of ampere-turns is worthy of special note. On this account for shunt coils it often becomes advisable to adopt single-cotton-covered wires, in which the total thickness of insulation averages about 8 mils; the voltage between adjacent turns of a field winding is not large even in 500-volt machines, and the thinner insulation suffices to withstand this voltage, while σ becomes as high as curve a of Fig. 147. Small round wire specially insulated with a thin layer of a tough, elastic, and heat-resisting enamel has also been tried for field coils with some success, but has not as yet come into ordinary use.[1] On larger machines, and wherever the area of wire which is to be wound is considerable, the use of a rectangular section is advantageous, even though the thickness of insulation has then to be increased to 20 mils. Indeed, a thin wide strip of which the one dimension is many times the other becomes quite permissible, and from its convenience in winding has much to recommend it.

In special cases, where the dynamo is subjected to very high temperatures, the field wires may be insulated with an asbestos covering; or if a wide flat section of copper can be employed the turns may be insulated from one another by a thin strip of asbestos paper of a few mils in thickness.

Since the cross-sectional area through one side of a coil must be equal to the number of turns in the coil multiplied by the area a_1 taken up by one wire, an important relation exists between the dimensions of a coil and the ampere-turns which it can furnish with due regard to its heating. The necessary dimensions of any coil can, in fact, be determined from equations (124), or (127) by

[1] Bare aluminium wire on which an insulating coating of oxide is formed has also been used, *Elektr. Kraftbetriebe u. Bahnen* (Hopfelt), Vol. 4, p. 401, and *E.T.Z.* (B. Duschnitz), Vol. 34, p. 1334.

consideration of the resistance or volume of copper as related to its external surface. Thus the resistance of the coil is $T_f \times \frac{l_x}{1000} \times \omega' \times k = T_f\, l_x\, k \,.\, \frac{0{\cdot}00002445}{a}$, where $a =$ the net copper section of the wire or strip in square inches. The watts of the coil are thence $W = A^2\, T_f\, l_x\, k \,.\, \frac{0{\cdot}00002445}{a}$. If, as is usually the case, the dimensions of the coil are fixed by the necessity of obtaining sufficient cooling surface to dissipate the heat loss without exceeding a certain fixed rise of the surface temperature, let $\xi t^\circ = \frac{W}{36(L\,.\,l_o + 2t\,.\,l_x)}$ be the permissible rate of loss in watts per square inch of external cooling surface to give a rise of t° in temperature, l_o being the external perimeter of the coil expressed in yards (so as to correspond with the perimeter of the central layer l_x which is in yards), L its axial length in inches, and t the thickness in inches of the winding in a radial direction. We then have

$$36\, \xi t^\circ\, (L\,.\,l_o + 2t\,.\,l_x) = A^2 T_f\, l_x\, k \,.\, \frac{0{\cdot}00002445}{a}$$

$$L\,.\,\frac{l_o}{l_x} + 2t = \frac{A^2\, T_f\, k \times 0{\cdot}68 \times 10^{-6}}{\xi t^\circ a}$$

Since $a_1 =$ the space in square inches taken up by each insulated wire, including allowance for any waste by interstices, $a = \sigma\,.\,a_1$, and $Lt/a_1 = T_f$. Multiplying both sides of the above equation by Lt, the necessary relation between the dimensions of the coil and its ampere-turns is obtained, namely,

$$L^2\,.\,t\,.\,\frac{l_o}{l_x} + 2L\,.\,t^2 = (AT_f)^2\,.\,\frac{k}{\xi t^\circ}\,.\,\frac{1}{\sigma} \times 0{\cdot}68 \times 10^{-6} \qquad (129)$$

$$\sqrt{L^2\,.\,t\,.\,\frac{l_o}{l_x} + 2L\,.\,t^2} = \frac{AT_f}{1{,}000}\,.\,\sqrt{\frac{k}{\xi t^\circ}}\,.\,\frac{1}{1{\cdot}21\,\sqrt{\sigma}}$$

or

$$\frac{AT_f}{1{,}000} = \sqrt{L^2 t\,.\,\frac{l_o}{l_x} + 2Lt^2} \times \sqrt{\frac{\xi t^\circ}{k}} \times \sqrt{\sigma} \times 1{\cdot}21$$

The value of $\sqrt{\sigma}$ $\left(\text{with round wires} = 0{\cdot}935\,\frac{d}{d_1}\right)$ rises from 0·75 for wire of 0·050″ diameter insulated with 12 mils of cotton covering or 0·81 with wire 0·100″ diameter and 15 mils of insulation to as high as 0·95 with copper strip, and is approximately known from the nature of the winding and the exciting voltage. Further, ξt° and k are mutually dependent upon one another, the ratio of watts to

cooling surface fixing the heating coefficient. The ratio l_o/l_x is in the case of circular bobbins equal to $\frac{\text{diam. of inside turn} + 2t}{\text{diam. of inside turn} + t}$, and in the case of a rectangular bobbin of which the two inside dimensions are A and $B = \frac{2(A+B)+2\pi t}{2(A+B)+\pi t}$. If the desired depth of winding is known, the necessary length L can be definitely determined. The influence of l_o/l_x is small, since it only varies from 1·33 in small to 1·1 in large machines, so that an approximate estimate of the necessary cross-section of coil can be made with close accuracy not only without calculating out the exact size of wire and number of turns, but also without calculating the exact depth of winding. Such values can thus be beforehand assigned to L and t as will enable the required number of ampere-turns to be obtained without overheating.

If the cooling influence of the end-flanges could be neglected, the ampere-turns of a coil as fixed by its heating would then be simply proportional to its length and to the square root of the depth of its winding, but even if this simple approximation is not adopted it is evident that, when a given number of ampere-turns is to be obtained with a fixed rise of temperature, an increase in the length is of considerably greater value than an increase in the depth.

The volume of copper in the coil is $Lt\sigma \times 36 l_x$ cubic inches, and since the weight of a cubic inch of copper is 0·32 lb., its weight is $Lt\sigma \cdot l_x \times 11{\cdot}5$ lb. Since by (129), $Lt\sigma = \dfrac{(AT_f)^2}{L \cdot \frac{l_o}{l_x} + 2t} \times \dfrac{k}{\xi t} \times 0{\cdot}666 \times 10^{-6}$

we have

$$\text{weight of coil in lb.} = \frac{(AT_f)^2}{L \cdot \frac{l_o}{l_x} + 2t} \cdot l_x \cdot \frac{k}{\xi t} \cdot 7{\cdot}66 \times 10^{-6} \quad . \quad (130)$$

and the total weight of copper on the machine is this quantity $\times\, 2p$. From this it follows that, for a given rise of temperature and given dimensions of pole, the requisite weight of copper is very much reduced by increasing the length of the coil. A limit is, however, set to the possible decrease in the weight of copper used by two considerations. As the length of the coil is increased, although the same temperature rise is attained, the loss in watts is continually increasing, and cannot be allowed to reach such a value as to impair the efficiency seriously. Further, the length of pole increases with the increase of the bobbin length; the magnetic circuit is in consequence longer, the necessary ampere-turns more, and finally, the size, weight, and cost of the iron magnet are greater. The best length of coil must therefore be determined by a compromise between

a number of conflicting considerations, the advantage of a reduction in the weight of wire being finally balanced by the increased cost of the longer iron or steel poles and yoke-ring, and by the lower efficiency of the machine as a whole.

§ 19. Winding of field-magnet bobbins. The field coils of small machines are usually wound in a lathe or small winding

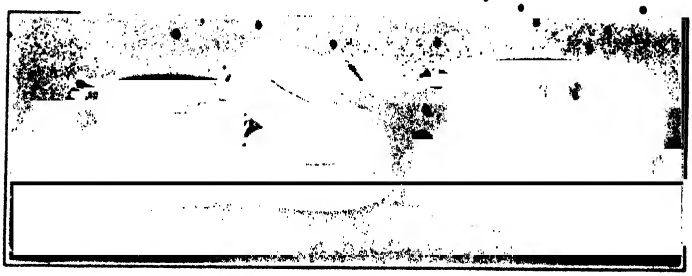


Fig. 276. Round shunt magnet coils.

machine on a wooden former between temporary cheeks; the numerous layers of wire are built up during the process of winding with interlaced strips of tape, so that the wooden cheeks and finally

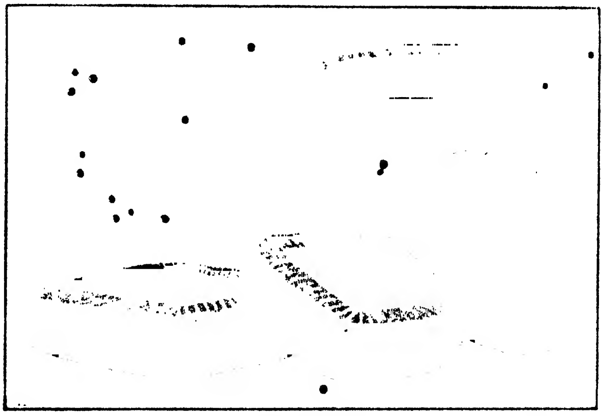


Fig. 277. Rectangular magnet coils.

the former itself may be withdrawn without the coil losing its shape. The coil is thus left self-supporting, and no spool is required. The ends and the whole of the inside are then covered with paper, press-spahn, or micanite of different degrees of thickness, according to the voltage which the coil has to withstand, especial care being

taken to secure a thoroughly sound covering on the inside edges where it will be in contact with the iron of the pole-core. Finally, the whole coil is wrapped round with an overlapping layer of strong tape to retain the true insulating covering in place, and is varnished inside and out. Such coils, both round and rectangular, are shown in Figs. 276 and 277. They are threaded over the poles, and are fixed in place so as not to become chafed by vibration, preferably being secured between two iron or brass retaining plates, the one resting against the projecting pole-shoe and the other fastened by pins driven into the iron pole. The thickness to be allowed for the insulation on the inside, after the scheme shown in diagrammatic section in *Fig.* 278, rises from 0·045″ for 100 volts to 0·070″ for 250, and 0·110″ for 500 volts. In machines subjected to especially trying conditions of dampness the wound coil is wrapped with a double

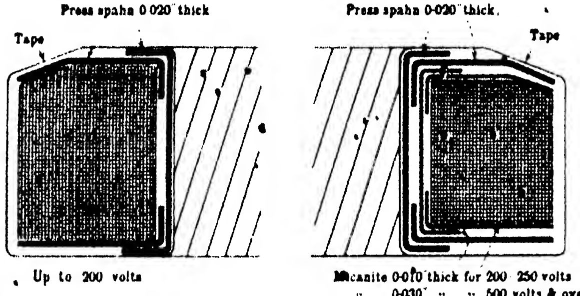


FIG. 278.—Insulation of magnet bobbins.

layer of webbing, and is then soaked in bitumen or other insulating varnish and thoroughly dried, until, to use an Americanism, it is "mummified." For larger machines spools are in many cases employed. These may be either of insulating material, such as vulcanasbest, specially moulded to fit over the poles, with grooves formed in their flanges to receive the wire leading to the bottom layer; such spools are convenient for winding, and provide a very high insulation from the iron, but they have the disadvantage of being heat-retaining. Or the spools may be of metal, usually with a sheet-iron cylindrical body riveted to malleable iron or brass end-flanges; the latter being pierced with holes or star-shaped to give as much ventilation as possible without impairing the mechanical support of the winding. Such spools conduct the heat better to the iron of the pole, but their winding surface must be insulated with several layers of varnished paper or micanite cloth. In order to obviate the danger of a break of the leading-in wire rendering the lower layer inaccessible, a third insulating partition may be

placed against the end-flange, and within this, as each layer is wound, a turn of the leading-in wire is taken in the reverse direction, until it is finally brought up to the top, flush with the bulk of the winding, and in a readily accessible position. Or a thin strip of copper is soldered to the first turn of the lowest layer, and is brought out under the winding and completely insulated therefrom by intervening mica. Fig. 279 shows the field-coils of an 8-pole machine in place on the magnet-cores, the coils being wound between metal flanges and connected in series round the circle of poles.

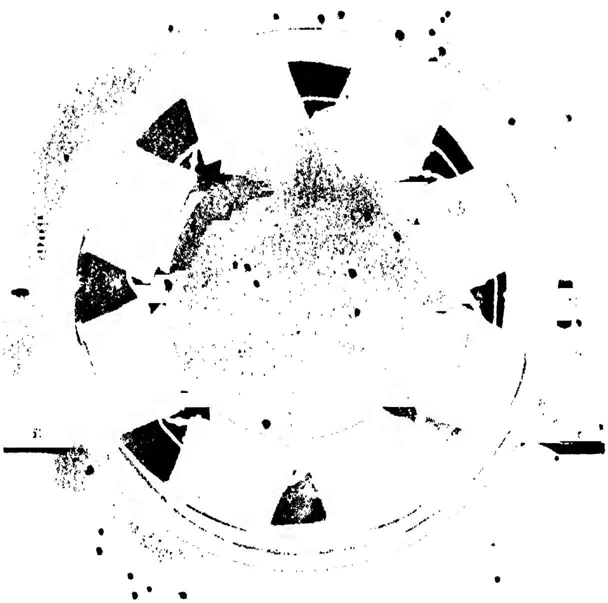


FIG. 279.—Eight-pole magnet-frame with coils in place.

For the better ventilation of large coils it is very common to find them divided into two or three separate sections with air-ducts between them. Such air-ducts may be either radial and communicating with a central air-space between the inside of the coil and the pole-core, or they may run longitudinally through the coil, the separate sections being kept apart by strips of wood or fibre which are interposed at intervals so as to form a circular air-gap. The end-flanges in either case are pierced or cut away to form corresponding openings through which the air may circulate. Figs. 280 and 281 illustrate the first method, with which, owing to the smaller size of the sections, we again return to separate coils wound

so as to be self-supporting and afterwards slipped over the pole without any internal spool; wood strips boiled in paraffin-wax are disposed round the pole and wedge the coil in place, so that no further internal insulation is required, and small black fibre distance-pieces dovetailed to the wood strips serve to keep the sections apart. The inner ends of the windings are then conveniently brought out through the radial openings between the sections.

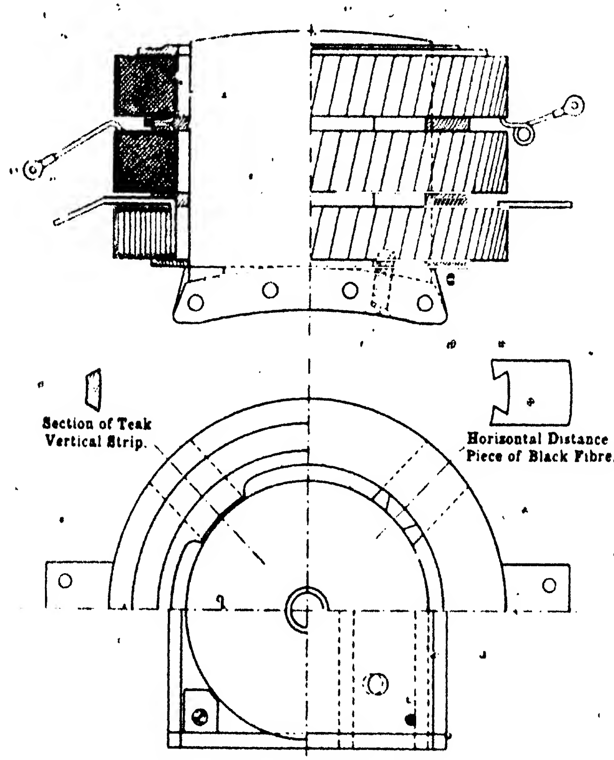


FIG. 280.—Magnet-coil in separate sections.

Series coils are advantageously made by winding flat copper strip on edge with insulation laid in between the turns; the heat is conducted directly to the outside edges of the copper, and the utilization of space is good. Such a construction is used commonly for the coils on commutating poles, which carry the full armature current, and if the copper strip is of sufficiently substantial dimension it may be left bare with an air-space only between adjacent turns. Fig. 282 shows a commutating field-coil of copper strip wound

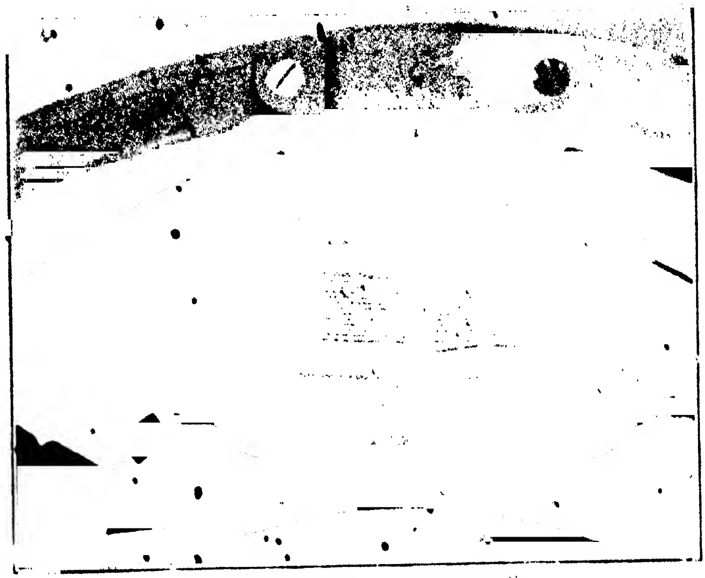


FIG. 281.—Magnet-coil in separate sections.

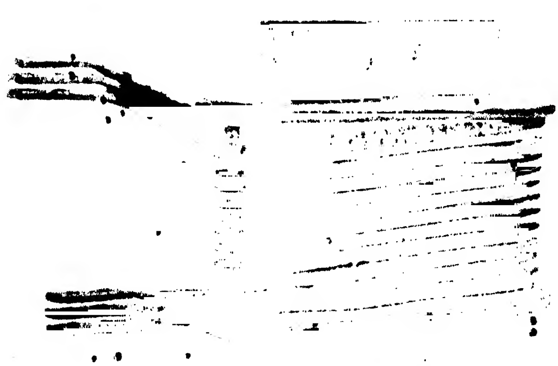


FIG. 282.—Commutating pole and coil (The British Thomson-Houston Co., Ltd.).

directly on the insulated pole-core, the connections between the coils for large currents being made by interleaved bolted joints,

FIG. 283. 500 kW 300 r.p.m. 3-wire generator for direct driving by engine. (The British Thomson-Houston Co., Ltd.)

and in Fig. 283 may be seen the complete assembly of main and commutating poles.

CHAPTER XVII

SHUNT, SERIES, AND COMPOUND WINDING

§ 1. Methods of excitation of the field-magnet. Separate excitation.—The excitation of the electro-magnet of a dynamo is, in general, effected either (1) by coils connected to a separate source of current, the machine being then said to be "separately excited;" or (2) by coils forming a shunt to the external circuit, the dynamo being then known as a "shunt" machine; or (3) by coils in series with the external circuit, when the dynamo is called a "series" machine; or (4) by both shunt and series coils in combination, a method known as "compound winding."

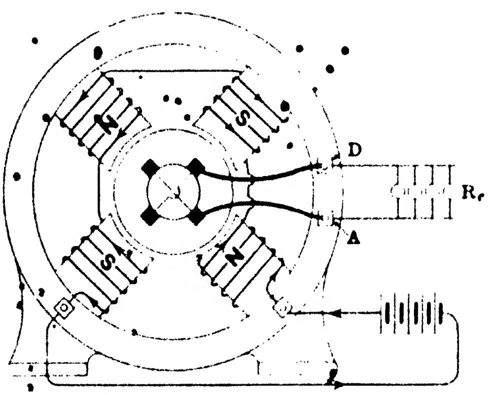


FIG. 284.—Separately excited dynamo.

The excitation of a machine from a separate and entirely external source of electrical energy is the most obvious method, and was the one first adopted in practice, a small dynamo with permanent magnets of steel being used to furnish current for exciting the field-magnet of a larger dynamo. Fig. 284 shows a *separately excited machine*, the electric circuit of the magnetizing coils being entirely distinct from the circuit of the main dynamo; in the diagram the main external circuit, R_e, is indicated by incandescent lamps strung in parallel across from the positive to the negative lead. The source of the magnetizing current is represented by a battery of cells, but, of course, is usually a separate continuous-current dynamo. The principle is still retained in connection with alternators, a separate "exciter" furnishing continuous current

for their excitation, and in special cases of continuous-current dynamos when their magnetization would otherwise tend to become unstable. The calculation of the field-winding for separate excitation is made at once from equations (124–6).

§ 2. Shunt excitation.—The second step in the order of development was made when it was suggested that a part of the electrical output delivered by the main dynamo might be used to maintain or increase the magnetism of its own field-magnet. This was, however, only possible if the machine were furnished with a commutator by which the current was commuted into a steady flow in one direction, and was thus rendered suitable for magnetizing purposes. Two distinct methods were then invented by which this suggestion was realized.

By the first, magnetizing coils were arranged as a *shunt* to the

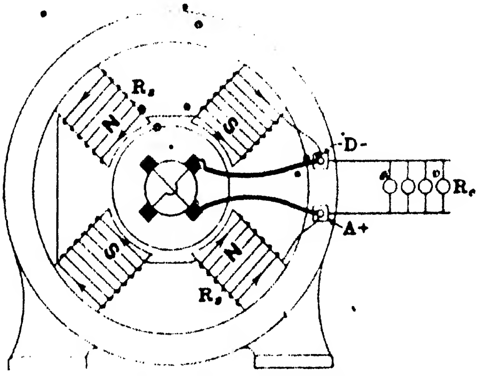


FIG. 285. Shunt wound dynamo.

external circuit proper; thus, in Fig. 285, from the brushes of the dynamo two paths, R_s and R_e, are presented, and the armature current divides into two portions, the relative magnitudes of which will vary inversely as the resistances of R_e and R_s; while the one portion of the total armature current flows through the external circuit R_e, wherein the useful electrical energy is manifested, the other is shunted through the magnetizing coils, and both reunite to flow through the armature. The voltage on the shunt is, of course, the same as that on the external circuit, since the same terminals, A and D, serve for both. If the resistance of the shunt, R_s, be relatively high as compared with R_e, only a small proportion of the total energy developed will be absorbed in exciting the field-magnet, and this is the case under load, owing to the resistance of the external circuit being then comparatively low. Hence the shunt coils consist of a large number of turns of small wire, and are represented in the diagram by fine lines.

§ 3. Series excitation.—By the second alternative method, magnetizing coils are arranged in *series* with the external circuit, and the whole of the armature current passes alike through the turns of the field-magnet coils, R_m, and the external circuit, R_e (Fig. 286). A portion of the total voltage developed at the brushes is expended in the magnetizing coils, and the remainder is available for useful work at the terminals, B, D, to which the external circuit is applied. If the resistance of the series coils, R_m, be low as compared with the resistance of the external circuit, the percentage of energy absorbed in the field will be small as compared with the useful output; on this account the series coils will usually be a few turns of thick copper strip, and are so represented in the diagram. The number of ampere-turns on the magnet of the dynamo may be the same whether it

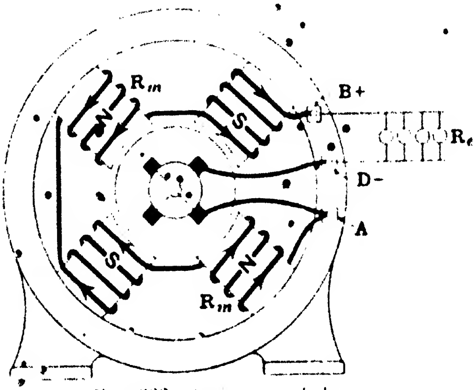


FIG. 286.—Series wound dynamo.

be shunt- or series-wound, since in the one case a small current flows through a large number of turns, and in the other case a large current flows through a few turns; and in both cases the amount of energy absorbed in securing any given number of ampere-turns is simply a question of the amount of copper in the field-coils.

§ 4. Self-excitation.—A further and most important step was next found to be practicable, namely, the *self-excitation* of the machine, whether shunt- or series-wound. The field-magnet of a dynamo is slightly magnetized, even when the machine has ceased running; it may then appear to be perfectly demagnetized by its failing to attract, say, a bunch of keys held near the poles, but a more delicate test will show that both forged and cast iron or steel retain a certain amount of magnetic flux (Chapter XIV, § 4). The presence of this feeble residual flux is sufficient to start the process of self-excitation, so that when the armature of either Fig. 285 or Fig. 286 is rotated, the active conductors cut the residual lines, and a small E.M.F. is thereby set up within the winding of the

armature. If the brushes are "down" on the commutator, and a closed circuit is thereby made, this small E.M.F. sends a feeble current through the magnetizing coils; the ampere-turns of the latter, although as yet they may be small, still serve to increase the number of lines passing through the armature, and these, as they are cut by the wires, produce an increased E.M.F.; the exciting current is thereby in turn increased, and its increase is again followed by a further increase in the flux and E.M.F. Thus the magnetism gradually grows, the increments becoming less and less as the iron of the field-magnet becomes more and more saturated. The time taken by the "building up" process will vary from a few seconds in a small machine to a minute or more in a large magnet, but after a short time a small increase in the exciting current produces very little increase in the flux; finally, for a given speed of rotation the voltage and exciting current reach a settled state, and the machine will run for any length of time, maintaining its own constant flux. Thus the presence of residual flux renders it unnecessary to impart to the field magnet any initial excitation, and the continuous-current machine becomes self-exciting by the mere rotation of the armature. One difference, however, between the shunt- and the series-machine will be apparent from their respective diagrams. In the latter case the external circuit must be closed before the process of excitation will begin, since the circuit of the field-coils is only completed through the external circuit. In the former case, if an external circuit of very low resistance be closed on the brushes, so small a portion of the feeble initial current will be shunted through the magnet-coils that the machine may fail to excite. Hence, in the shunt machine, the excitation is most quickly and surely obtained if the external circuit be left open until the magnet is thoroughly excited.

It only remains to remark that even when a machine has just come from the workshops, and is run for the first time, there is usually sufficient residual flux in the magnet, due to the effect of the earth's magnetic field or of other magnetic bodies upon it during the process of its manufacture, to enable it to excite itself. The first excitation may, however, require a higher speed than will be subsequently necessary, and in cases of large shunt machines, running at slow speeds, separate excitation may have to be resorted to in the first instance.

§ 5. Energy stored in the excited field.—When once the machine is normally excited, and is running under settled conditions, the energy spent in the magnetizing coils is entirely converted into heat; but during the process of creating the magnetic field a certain amount of energy is absorbed, which is given back as the field demagnetizes. If the circuit of a series-wound dynamo be opened while it is running, the stored-up energy reappears in the form of

a spark at the switch; the rapid collapse of the lines of flux which circle through the turns of the field-coils generates a high E.M.F., which tends to keep up the strength of the current; and if the field be powerful and the turns encircling it be numerous, the intensity of the spark is great. In the case of a shunt-wound dynamo the opening of the external circuit does not interrupt the flow of the shunt current round the coils; the stored energy discharged at the switch is then chiefly that of the external circuit. Owing, however, to the large number of turns in the shunt, the effect of suddenly breaking the magnetizing circuit is even more marked than in a series machine. Thus, if a person accidentally lifts the brushes of a large shunt machine running on open circuit, the self-induced E.M.F. may rise to thousands of volts, may damage the insulation of the machine, and, further, may give perhaps a fatal shock. If the machine be running on a closed circuit, say, of incandescent lamps, the person would be less liable to receive such a severe shock, since the so-called "extra-current" is discharged through the external circuit, and causes the lamps to flash up momentarily. In order, therefore, to open the shunt circuit of a machine when running, a shunt-breaking switch is necessary, in which at the moment of opening a non-inductive resistance is closed upon the terminals of the shunt.

§ 6. Determination of shunt-winding.—The application of the formulæ of Chapter XVI to the design of the field-winding of the shunt machine is easy. Let V_t be the terminal or external voltage which the machine is required to give when supplying its full external current; then the difference of potential on the ends of the shunt[1] is likewise V_t, and it is only necessary to substitute its value for V_s in equations (124-5) in order to determine the necessary area or diameter of wire. In order further to determine the number of turns and weight of wire, it is necessary to know, at least approximately, the rate in watts at which energy may be expended in the field-coils.

While, however, the necessary size of wire is rigidly determined, there is, in fact, no hard-and-fast rule for deciding the weight that is to be used, and this must be left to the designer's judgment. If a large quantity be used, the weight and cost of the machine are increased, while if a smaller quantity be used the heating of the field-coils is greater, and the efficiency of the machine is decreased. A further disadvantage which arises when the weight of copper is reduced is that there is a greater difference between the resistances of the shunt when cold and when hot, owing to its rise of temperature being greater; this difference causes a difference in the shunt current for the same terminal voltage, and therefore in the excitation and total flux produced, so that when

[1] Unless there is a rheostat in circuit, as will be described in § 8.

the dynamo runs at a constant speed the E.M.F. induced is greater at starting than it is after it has run for several hours continuously and has attained its final temperature. If this difference be great, it will necessitate an alteration in the speed of rotation in order to maintain the correct voltage, or it must be corrected by means of a variable rheostat in series with the shunt. Difference of temperature is minimized by the winding of the shunt coils in separate sections with intervening air-ducts through which the air is brought into contact with the more heated central parts, as in Figs. 280, 281; but, on the other hand, it has to be borne in mind that, as compared with a single coil filling the whole of the bobbin-length, the lesser number of turns and weight of wire are always obtained at the expense of a slightly lower efficiency. The settlement of the rate in watts at which energy may be expended in the shunt coils depends, therefore, upon their allowable heating, and upon the efficiency which the dynamo is to have; from both considerations combined, experience enables us to fix upon a preliminary estimate from which the completed design need differ but little. In practice, the loss of energy in the field of a shunt machine varies from about 0·75 per cent. of the output in a 1,000-kilowatt machine, 1·5 per cent. in a 100-kilowatt machine, to 7 or 8 per cent. in a 4-kilowatt machine. The heating question will be more fully discussed in Chapter XXI. If it be settled that W watts may be lost in the field-winding, then $W = V_e \times I_s$ watts, and $I_s = V_e/R_s$, whence the resistance of the shunt and its composition can be at once determined. The preliminary result thus arrived at should only require such slight revision as will lead to the winding forming a complete number of layers.

§ 7. Determination of a shunt-winding with two sizes of wire.—As stated in Chapter XVI, § 16, there is only one diameter of wire which will give a specified number of ampere-turns for a given applied voltage V_e at its ends, if the whole of the coil is wound with wire of the same gauge. It is, however, often desired to make up a composite shunt winding out of two sizes of wires. Any size of shunt wire can be specially drawn, but for manufacturing reasons it is advisable to adhere to certain standard sizes of wire which are held in stock. Given two sizes, one of which is larger and the other smaller than the correct diameter, they can be wound in such proportions as to produce the same magnetizing result as the correct intermediate size. The use of two gauges of wire in series implies a different rate of heat-generation in them owing to their unequal current-density, but in practice, if the two sizes do not differ greatly, this leads to no disadvantageous result.

To determine the necessary proportions by trial and error is usually a somewhat lengthy process, and the calculation may be much shortened by attention to the following points:

From equation (125) the AT of any coil wound with wire of area a sq. inches is

$$\frac{V_x \times 1{,}000 \times a}{0{\cdot}02445 \times l_x \times k}$$

When two sizes of wire are to be employed, let the suffixes l, s, c indicate respectively the large, small and correct sizes of wire. The current through the combination of wires is

$$\frac{V_x \times 1{,}000}{0{\cdot}02445\left(\dfrac{T_l \times l_{xl} \times k_l}{a_l} + \dfrac{T_s \times l_{xs} \times k_s}{a_s}\right)}$$

and this when multiplied by $(T_s + T_l)$ must give

$$\frac{V_x \times 1{,}000 \times a_c}{0{\cdot}02445 \times l_x \times k}$$

The turns T_s and T_l must now be expressed in terms of the number of turns of either the larger or the smaller wire which will fill the entire winding space of sectional area A. Let the larger wire be taken and let T_l' be the number of turns of it which will fill the area A.

The true number of turns of the larger wire when a part A_s of the winding space is filled with the smaller wire instead of the larger wire is then

$$T_l = T_l' \cdot \frac{A - A_s}{A}$$

and the true number of turns of the smaller wire is

$$T_s = T_l' \cdot \frac{A_s}{A} \cdot \frac{a_{1l}}{a_{1s}}$$

where a_{1l} and a_{1s} are the areas which one turn of each of the two sizes of wire occupies when insulated and after taking into account the interstices between the wires.

The total turns are therefore

$$T_s + T_l = T_l'\left\{1 + \frac{A_s}{A}\left(\frac{a_{1l}}{a_{1s}} - 1\right)\right\}$$

or since $A_s = T_s \cdot a_{1s}$ and $A = T_l' \cdot a_{1l}$

$$= T_l' + T_s\left(1 - \frac{a_{1s}}{a_{1l}}\right)$$

$$= T_l' + T_s\left\{1 - \left(\frac{d_{1s}}{d_{1l}}\right)^2\right\}$$

since whatever the conditions of bedding, a_1 is $\propto d_1^2$, where d_1 is the insulated diameter of the wire.

Identifying k_1, k_2, and k_3, it is found by substitution that

$$T_s = T_l' \cdot \frac{\left(\dfrac{d_l}{d_e}\right)^2 - \dfrac{l_{xl}}{l_x}}{\left(\dfrac{d_l}{d_s}\right)^2 \cdot \dfrac{l_{xs}}{l_x} - \left(\dfrac{d_l}{d_e}\right)^2 + \left(\dfrac{d_{1s}}{d_{1l}}\right)^2 \left\{\left(\dfrac{d_l}{d_e}\right)^2 - \dfrac{l_{xl}}{l_x}\right\}}$$

$$T_l = T_l' \cdot \frac{\left(\dfrac{d_l}{d_s}\right)^2 \cdot \dfrac{l_{xs}}{l_x} - \left(\dfrac{d_l}{d_e}\right)^2}{\left(\dfrac{d_l}{d_s}\right)^2 \cdot \dfrac{l_{xs}}{l_x} - \left(\dfrac{d_l}{d_e}\right)^2 + \left(\dfrac{d_{1s}}{d_{1l}}\right)^2 \left\{\left(\dfrac{d_l}{d_e}\right)^2 - \dfrac{l_{xl}}{l_x}\right\}}$$

When the coil is divided into sections so that the mean length of a turn in each section is the same, and $l_{xl} = l_x = l_{xs}$, the equations are immediately soluble. As a general rule, it suffices in this case to calculate the relative widths of the sections of the two wires. If y_s = the fraction of the net winding length (after deducting any radial air-spaces) which the small wire must occupy,

$$y_s = \frac{1}{1 + \left(\dfrac{d_{1l}}{d_{1s}}\right)^2 \cdot \left(\dfrac{d_l}{d_s}\right)^2 \cdot \dfrac{d_e^2 - d_s^2}{d_l^2 - d_e^2}} \qquad (131)$$

In this expression the differences in the squares require to be worked out with accuracy.

But if one size of wire is wound above the other, $\dfrac{l_{xl}}{l_x}$ and $\dfrac{l_{xs}}{l_x}$ in the previous equations must be approximately estimated, as roughly indicated below. Under average conditions of depth, etc.,

	$\dfrac{l_{xl}}{l_x}$	$\dfrac{l_{xs}}{l_x}$
(a) *With the small size wound on the inside,*		
and nearly right in size	1·17	0·98
half and half in depth	1·08	0·91
large size nearly right	1·02	0·82
(b) *With the small size wound on the outside,*		
large size nearly right	0·98	1·17
half and half in depth	0·91	1·08
small size nearly right	0·82	1·02

Since with the same winding length $T \propto n/d_1$, where n is the number of layers, the ratio between the numbers of layers of the two sizes is

$$\frac{n_s}{n_l} = \frac{d_{1s}}{d_{1l}} \cdot \frac{\left(\dfrac{d_l}{d_e}\right)^2 - \dfrac{l_{xl}}{l_x}}{\left(\dfrac{d_l}{d_s}\right)^2 \cdot \dfrac{l_{xs}}{l_x} - \left(\dfrac{d_l}{d_e}\right)^2} \qquad (132)$$

The ratio $\left(\dfrac{d_l}{d_e}\right)^2$ need not necessarily exceed $\dfrac{l_{xl}}{l_x}$, since if the smaller wire is wound inside and will at a certain thickness give the required

number or more than the required number of AT, the function of the larger wire at its larger mean turn is to *reduce* the AT. But if this is the case and the numerator of the above equations becomes negative, $\left(\frac{d_s}{d_c}\right)^2$ must exceed $\frac{l_{xs}}{l_x}$, which makes the denominator also negative. When the sizes are thus so nearly commensurate with their respective mean turns, the equations cannot be solved with sufficient correctness on the slide rule, the differences being too small to determine accurately. It is therefore best to wind the smaller wire outside, for purposes of calculation, since it is practically a matter of indifference which course is pursued. Under these circumstances $\left(\frac{d_l}{d_c}\right)^2$ always exceeds $\frac{l_{xl}}{l_x}$ and $\frac{l_{xs}}{l_x}$ exceeds $\left(\frac{d_s}{d_c}\right)^2$ which gives a clue to the right values to be inserted for the ratios $\frac{l_{xl}}{l_x}$ and $\frac{l_{xs}}{l_x}$.

The following simple approximate formulæ have been developed by Mr. R. G. Jakeman[1] to determine directly the winding depths of the two sizes of wire when one is wound above the other.

1. If the large size is wound first

$$\frac{x}{x_l} = 1 + 0{\cdot}45\,\frac{a_l - a_c}{a_c - a_s}$$

2. If the small size is wound first

$$\frac{x}{x_l} = 1 + 1{\cdot}0\,\frac{a_l - a_c}{a_c - a_s}$$

where x and x_l are the total winding depth and the depth of the large size respectively. These formulæ give good results in practice for any size of coil, whether round or rectangular.

When the wires as usual are not far removed from one another in size, the differences are so small that the numbers of turns for a given depth of winding can be varied through a wide range with practically little effect on the total AT which the combination yields.

§ 8. Example of shunt-winding calculation.—To illustrate the above, the winding for the 21″ × 11″ dynamo of Chapter XVI, § 9, as a shunt machine will now be worked out in detail. A shunt-wound dynamo is almost invariably furnished with a rheostat or resistance in series with the magnet winding, for the purpose of regulating its voltage; the coils of such a rheostat can be successively thrown into or out of the shunt circuit, and since for a given voltage at the brushes the exciting current can be thereby reduced or increased, the voltage of the machine can be lowered or raised by successive small steps. It will be assumed in the present case

[1] "A Direct Method of Calculating Shunt Field Coils having Two Gauges of Wire," *Electr.*, 21st Nov., 1919.

that a shunt rheostat is to be used, so that the electrical connections of the machine will be as diagrammatically shown in Fig. 287, and the current in the shunt is $I_s = \frac{V_b}{R_s + R_r}$, where R_r is the resistance corresponding to the particular contact on which the arm of the rheostat rests. Under these circumstances at full load, and when the machine is heated after a prolonged run, it will be advisable to retain a small margin of resistance in the rheostat; in other words, the full voltage at the brushes will not be used to excite the field-magnet. It will be remembered from Chapter XVI, § 16, that, when once the gauge of shunt wire has been selected and the coils wound, the ampere-turns are not increased but rather reduced by winding on more turns, while a limit to the removal of turns is quickly reached through the overheating of the coil which results. But if one or two steps of resistance in a rheostat have been held in reserve, should it become necessary to make up any small deficiency in the calculated ampere-turns or to cover any variation from strictly normal conditions, the last steps can be cut out and the voltage thereby raised slightly.

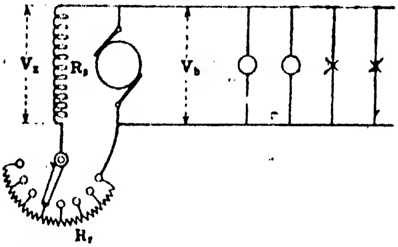


FIG. 287.—Adjustable rheostat in shunt circuit.

But on the other hand, the difference between the terminal voltage at the brushes V_b, and that applied to the terminals of the field-coils, V_s, *i.e.* the loss of volts over the rheostat at full load, or $V_b - V_s$, must be kept small, so as not to impair appreciably the efficiency of the machine. In the present case, therefore, V_s will be taken as about 2½ per cent. less than V_b, or, say, 225 out of the 230 volts at the brushes or terminals of the external circuit.

With the same loss of volts over the armature resistance and brushes as in Chapter XVI, § 9, the induced E.M.F. must now be 238 volts, or slightly less than before, since no part of the field-winding is in series with the armature; the necessary flux is therefore $\Phi_a = \frac{238 \times 60 \times 10^8}{450 \times 500} = 6{\cdot}35 \times 10^6$, and the total ampere-turns on the four poles to give this flux is from Fig. 274, $AT_f \times 2p = 4 \times 9{,}600$.

The diameter of the pole being 9½″, the inside diameter of the bobbin over the wood strips running axially down the magnet-core, after the fashion of Fig. 280, will be 9⅞″. A depth of winding of 2½″ may be assumed, so that the mean diameter of a central turn will be 12⅜″, and its length $l_s = \pi \times 12{\cdot}375$ inches = 1·08 yard.

With the assumed thickness of winding, $\frac{l_0}{l_s} = \frac{9.875 + 5}{9.875 + 2.5} = 1.2$, and $\sqrt{\sigma} = 0.935 \frac{d}{d_1}$ will be about 0.8. As will be explained in Chapter XXI, if the coil were closely wound without any ventilating channels, a ratio of about 0.5 watt per square inch would give a surface rise of temperature of about 45° F., and the ratio of the mean rise to the surface rise would be, say, 1.72, so that from Chapter XVI, § 16, assuming the temperature of the engine-room to be 70° F. while the standard temperature of the wire table is 68° F., $k = 1 + 0.0022 \times 79.5 = 1.175$, and $\sqrt{\frac{k}{\xi}} = \sqrt{\frac{1.175}{0.5}} = 1.53$. Since each coil has to give 9,500 ampere-turns, by equation (129),

$$\sqrt{1.2L^2t + 2Lt^2} = \frac{9.5}{1.21 \times 0.8} \times 1.53 = 15$$

and with $2\frac{1}{2}''$ depth of winding $L = 6.82''$.

The above gives a first idea of the necessary length of coil, and will hold good equally in the present case when the coils are to be divided up into sections with air-canals between them. The effect of this construction will be to reduce the value of k and at the same time to increase the watts per square inch that may be allowed for the same surface rise; thus, as in Chapter XVI, § 16, k may be taken as 1.16, and 0.64 watt per square inch may be allowed, the gross length of coil being reckoned in the cooling surface. It results that, even when two air-canals each $\frac{1}{4}''$ wide are added to divide the coil into three sections (Fig. 288), its gross length may remain 6.8″. It will be assumed, therefore, that the above calculations or the known design of the magnetic circuit, as in Fig. 266, have led to an axial length of pole of 7.3″, a surplus of $\frac{1}{4}''$ being allowed at each end of the coil.

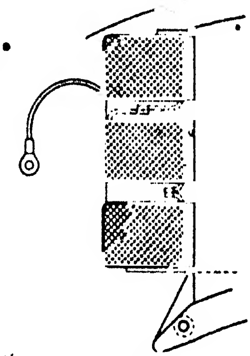


FIG. 288.

The several shunt coils, one on each pole, are almost invariably connected in series, so that by equation (124)

$$\omega' = \frac{225 \times 1{,}000}{38{,}000 \times 1.08 \times 1.16} = 4.72 \text{ ohms per 1,000 yards,}$$

and the necessary diameter of the shunt wire is by equation (126)

$$d = \frac{0.176}{\sqrt{4.72}} = 0.0813''$$

The net winding length is $7 \cdot 3 - 1 \cdot 25 = 6 \cdot 05''$, and each of the three sections will be, say, $2 \cdot 02''$ long. In order to illustrate the procedure that would be followed in working out the details of the coils, a winding with the above single size of wire will first be completed as follows. The number of turns in a layer of one section will be $\frac{2 \cdot 02}{0 \cdot 0963} = 20 \cdot 9$. Strictly speaking, one turn must be deducted from this quotient, which gives 19·9, or 59·7 per layer of the three sections. With a depth of about $2 \cdot 5''$ there will be 28 layers, since with allowance for bedding we have by equation (128)

$$0 \cdot 9 \times 28 \times 0 \cdot 0963'' = 2 \cdot 43''$$

The total number of turns on a complete bobbin will be, therefore, $28 \times 59 \cdot 7 = 1{,}670$, or 6,680 in all on the four bobbins. The mean length of a turn being $l_s = (9 \cdot 875 + 2 \cdot 5)\pi = 38 \cdot 8$ inches $= 1 \cdot 08$ yard, the total length of wire is 7,215 yards, and its resistance at 68° F. $= 7 \cdot 215 \times 4 \cdot 72 = 34$ ohms, or when heated 34 ohms $\times$ $1 \cdot 16 = 39 \cdot 45$ ohms. In order to give a total of 38,000 ampere-turns the current must be 5·7 amperes, corresponding to the assumed $V_s = 225$ volts, and to a loss of energy in the shunt winding at the rate of $5 \cdot 7 \times 225 = 1{,}280$ watts. The perimeter of the outer layer will be $(9 \cdot 875 + 5)\pi = 46 \cdot 6''$, and the over-all length of the coil being $6 \cdot 81''$, the total cooling surface of the four coils (measured as in Chapter XV, § 5) will be

$$\begin{aligned} 46 \cdot 6 \times 6 \cdot 81 &= 318 \\ 2 \times 2 \cdot 5 \times 38 \cdot 8 &= 194 \\ \hline 512 \times 4 &= \text{say, } 2{,}000 \text{ sq. inches.} \end{aligned}$$

As will be explained in Chapter XXI, the ratio $\frac{1{,}280}{2{,}000} = 0 \cdot 64$ watt per square inch will, in the present type of construction with well ventilated coils, give a surface rise of temperature of about 48° F., or sufficiently near to our assumed amount. The design may therefore be proceeded with, since the assumed depth of winding tallies with the requirements in regard to heating.

It will next be assumed that for manufacturing reasons we are limited to No. 14 S.W.G. $= 0 \cdot 080''$ diameter, and a size intermediate between Nos. 14 and 13, say $\cdot 085''$ diameter. The diameter of the double-cotton-covered wire with 15 mils insulation will be in the two cases $0 \cdot 095''$ and $0 \cdot 100''$. Substituting these values in equation (131), y_s is 0·7, and in view of the small difference between the wires, there will be but little effect if its proportion of the total length of the winding is reduced to two-thirds. Three sections of equal axial length have a good appearance, and it will then be feasible to avoid having to change the wire during the winding of a section, the whole of the central section alone requiring to be wound with

the larger wire. In the two kinds of section there will then be respectively $\frac{2\cdot02}{0\cdot095} = 21\cdot2$ and $\frac{2\cdot02}{0\cdot100} = 20\cdot2$ turns per layer, or more accurately with one turn deducted 20·2 and 19·2 turns. In order to bring the ends of the sections together for soldering and the ends of the bobbin as a whole away from the end flanges, it is with three sections advantageous to have an even number of layers in the upper and lower sections and an uneven number of layers in the central section. The final winding will therefore be arranged with the two end sections each of 28 layers of 0·080″, and a depth in each of 2·4″. The central section will be given 27 layers of 0·085″, with a depth of 2·43″. The mean length of a turn will be slightly less than in the preliminary calculation, or, say, $l_t = 1\cdot072$ yard. In tabular form we thus have—

	No. of Turns.	Yards.	
28 layers of 0·080 . .	566	607	—
27 ,, 0·085 . .	518	—	556
28 ,, 0·080 . .	566	607	—
Per bobbin . . .	1650	1214	556
In 4 bobbins . .	6600	4856	2224
ρ' per 1,000 yards . .	...	4·86	4·32
Resistance . . .	—	23·7	9·7
		33·4 ohms.	

The corrected cooling surface shows so slight a reduction that the temperature will remain practically the same as first calculated, and the difference in the rate at which energy is lost in the central as compared with the outer sections will not be sufficient to demand any further adjustment.

The total resistance of the four bobbins when hot will therefore be $33\cdot4 \times 1\cdot16 = 38\cdot8$ ohms, and the required ampere-turns will be obtained with an exciting current of 5·75 amperes. The exciting voltage will then be 223·1, which falls within our assumed limit, and the loss of energy in the shunt coils at the rate of 1,280 watts will be 1·6 per cent. of the output, with a total loss in the shunt and rheostat combined at the rate of 1,320 watts, or 1·65 per cent. of the output of 80 kilowatts. The weight of wire will be 283 lb. of 0·080″ and 145 lb. of 0·085″, or 428 lb. on the whole machine.

§ 9. External characteristic curve of shunt machine.—If a dynamo be run at a constant speed, and the resistance of its external circuit be varied so as to alter the value of the external current, the curve connecting simultaneous values of the terminal voltage, V_t, and the external current, I_t, for a given speed of rotation is known

as the *external characteristic* of the machine for that speed; since from it the behaviour of the machine under varying conditions of load can be graphically studied. If a rheostat is employed in conjunction with the field circuit, its resistance is assumed to be maintained constant, as well as the speed. The amperes of such a curve are usually plotted horizontally as abscissae with the corresponding volts as vertical ordinates. The two may either be obtained by direct measurement, voltmeter and ammeter being read off simultaneously, or the curve may be derived from the flux curves of the machine. In the case of a shunt-wound machine

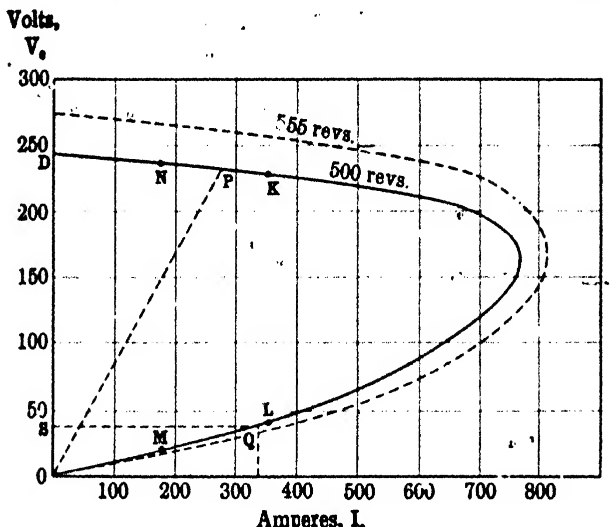


FIG. 289.—Characteristic curve of shunt-wound dynamo.

the curve of external voltage and current will be found to take the characteristic form shown in Fig. 289, *OLKD*. The manner in which it may be derived from the flux-curves has now to be explained, and first it will be necessary to consider the process by which the three curves of Fig. 274 may be determined by direct experiment on a shunt-wound dynamo.

From equation (44*a*) it follows that E_c for a constant speed is always proportional to Φ_a, the numerical relation between the two depending upon the nature of the winding and number of active conductors. In a shunt dynamo $I_a = I_s + I_e$; and $E_a = V_b + I_a (R_a + R_b)$, where R_b is the resistance of the brushes; whence $\Phi_a \propto V_b + I_a(R_a + R_b)$. If, therefore, V_b be observed for different excitations, the flux Φ_a can be thence deduced. The ampere-turns

on the field-magnet are $AT_s = I_s \times T_s$. $I_s = \frac{V_b}{R_s + R_r}$ may be obtained by measuring the voltage at the brushes and dividing it by the resistance of the shunt and rheostat; but for greater accuracy it is preferable to measure it directly by an ammeter, since the resistance of the shunt is continually altering if the machine is in process of warming up during the tests.

When the external circuit is open, the only current flowing is that through the shunt. As a rule, this is so small that $I_a R_a$ becomes negligible, and V_b and E_a are almost identical. If, therefore, the armature be run at different speeds, and a series of simultaneous readings of the speed, the voltage at the brushes, and the shunt current be taken, a curve can be plotted from the readings which will connect Φ_a and AT_s; the armature ampere-turns due to the shunt current alone being very small, their influence on the field may practically be neglected, and the curve thus obtained is the highest or "no current" curve of flux of Fig. 274. Owing to the small amount of residual magnetism which persists in the machine even when not excited, the flux-curve as thus practically determined does not descend strictly to zero; but towards the origin curves round slightly, and ends a little above zero (cp. Fig. 292); there is further a small difference in the curves according as the readings are taken with the excitation progressively increasing or progressively decreasing. To obtain the lower curves of flux for either "half-current" or "full-current," the same readings must be taken as for the "no-current" curve; but between the readings at each different speed the resistance of the external circuit must be altered, so that when the measurements are made for either curve, the current through the armature ($I_a = I_s + I_e$) has the constant value required. The loss of volts over the resistance of the armature and brushes must now be added to the terminal voltage to obtain E_a.

Since $AT_s = \frac{V_x}{R_s} \times T_s$, the potential difference applied to the terminals of the whole shunt-circuit, or $V_x \frac{R_s + R_r}{R_s}$ is $\propto AT_s$ when R_s and R_r are constant, as is the case when a machine has been running for some time and has attained a steady temperature, and further, when the position of the contact-arm of the shunt rheostat is not altered. If, therefore, we possess the flux curves of Fig. 274, these can be converted into curves connecting armature E.M.F. and exciting voltage (or more strictly $V_x \times \frac{R_s + R_r}{R_s}$, if a rheostat is present in the shunt circuit), when some fixed speed is assumed. The full-line curves of Fig. 290 show the curves of Fig. 274, thus converted on

the assumption that in our dynamo the number of armature conductors is 450, the number of shunt-turns 1,650 per bobbin, the total resistance of the 4 bobbins 38·8 ohms, and that the constant speed of rotation is to be 500 revs. per minute. From the design of § 8, the rheostat must be given a resistance of 1·2 ohms at full load, so that the combined resistance of shunt and rheostat is 40 ohms, at which it is to be maintained throughout. The armature

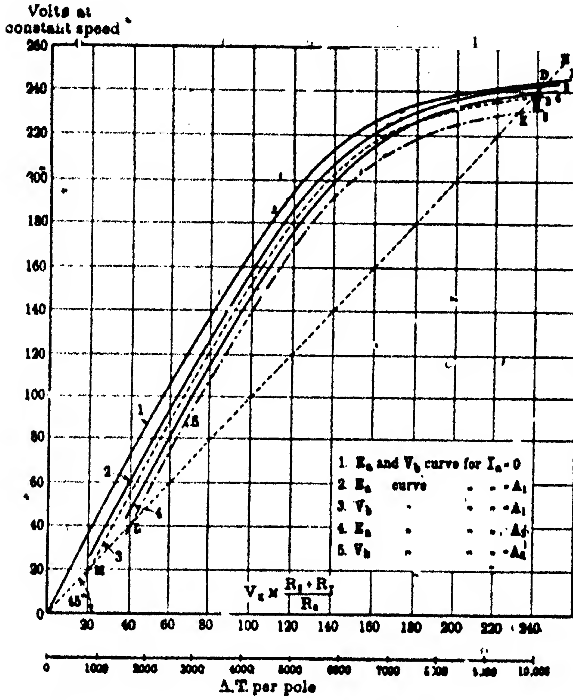


FIG. 290.—Construction for deriving external characteristic of shunt-wound dynamo.

E.M.F. is thus $E_a = \Phi_a \cdot 450 \cdot \frac{500}{60} \times 10^{-8} = \Phi_a \times 37{\cdot}5 \times 10^{-8}$, and when the exciting voltage on the terminals of the shunt alone is V_s, the corresponding voltage on the terminals of the shunt and rheostat is $V_s \times \frac{R_s + R_r}{R_s} = V_s \cdot \frac{40}{38{\cdot}8}$. The loss of volts over the armature when warm at half-load is 3, and at full load 6 volts; and to each of these has to be added the loss over the brushes which with hard carbon as their material may be taken as a constant amount = 2 volts. Curve 1 connects E_a or V_b, which has nearly

the same value, with $V_x \cdot \frac{R_s + R_r}{R_s}$ when $I_a = 0$. Curve 2 connects E_a and $V_x \cdot \frac{R_s + R_r}{R_s}$ when I_a has the value A_1 ($= 181$ amperes), and curve 4 connects E_a and $V_x \cdot \frac{R_s + R_r}{R_s}$ when $I_a = A_2$ ($= 356$ amperes). From curve 2 is obtained the dotted curve 3, connecting V_b and $V_x \cdot \frac{R_s + R_r}{R_s}$, when $I_a = A_1$ amperes, by deducting from the ordinates of curve 2 the loss of volts over the brushes and armature resistance; the vertical distance between curves 2 and 3 is thus throughout equal to $A_1(R_a + R_b) = 5$ volts. Similarly from curve 4 is derived the chain-dotted curve 5, connecting V_b and $V_x \cdot \frac{R_s + R_r}{R_s}$ when $I_a = A_2$, the vertical distance between the two curves being then 8 volts.

Now, since in a shunt-wound machine V_b is the same as $V_x \cdot \frac{R_s + R_r}{R_s}$, those points on curve 5 (such as L or K) for which the ordinate $V_b =$ the abscissa $V_x \cdot \frac{R_s + R_r}{R_s}$ are the only values of voltage which the machine would give when run at the fixed speed with A_2 amperes flowing through the armature; and in general those points on any curve connecting V_b and $V_x \cdot \frac{R_s + R_r}{R_s}$, for which ordinate and abscissa are equal, give the only possible values of the terminal voltage for the particular armature current. Deducting the shunt current from the total armature current, we obtain the external current; thus at point K, $I_s = \frac{230 \text{ volts}}{40 \text{ ohms}} = 5{\cdot}75$, $I_a =$ nearly 355·75 amperes, and therefore $I_e = 350$ and $V_e = 230$ are simultaneous values of external current and terminal voltage. All such points are necessarily passed through by the straight line OH, which is drawn at an angle of 45° from the axis of abscissae, as e.g. K and L, N and M, D and Q; and after deduction for the shunt current these may be plotted as part of the external characteristic, as shown by the corresponding letters in the full-line curve of Fig. 289. As the current is increased, the two points at which the V_b curve intersects the diagonal draw together, and the maximum current is that given by the V_b curve, which is just touched by line OH as a tangent.

§ 10. Deductions from characteristic of shunt dynamo.—It will be seen from Fig. 289 that for the shunt-wound dynamo running at any one fixed speed there is a maximum value for the external

current which it can give. Owing to the heat which would be generated in the armature winding by a long-continued passage of this maximum current it may be, and usually is, impossible to work the machine at the point of maximum current; the maximum current, in fact, is usually more than double the working current. Apart, however, from the question of the heat damaging the winding, it is for another reason inadvisable to work close to this point. After rounding the point of maximum current the characteristic curve descends very rapidly, and almost in a straight line to the origin. Thus in the diagram (Fig. 289) the part OQ is practically a straight line, and this implies that for an external resistance $\dfrac{QR \text{ volts}}{OR \text{ amperes}}$ the terminal voltage may have any value between zero and OS volts, the external current correspondingly varying between zero and OR amperes; apparently, therefore, the machine might give widely different voltages although running at exactly the same speed on the same external resistance. But in reality the flux of the field is then unstable owing to the voltage on the shunt not being sufficient to magnetize the iron properly; hence if, when working near the point of maximum current, the resistance of the external current becomes lowered to a value below that which corresponds to the maximum current, the machine loses its magnetism altogether, and the voltage runs down to zero. The practical importance of this is that if a shunt-wound machine be accidentally short-circuited when at work, or, in other words, if R_e is reduced almost to = 0, the armature winding is not burnt up by the continued passage of an abnormally large current; the machine, on the other hand, becomes demagnetized and gives no current, although the driving engine may be running at a speed higher than the normal.[1] Neither will a shunt-wound machine excite if an external circuit of abnormally low resistance be closed on it, so that it is unharmed if, when running alone, its terminals are short-circuited through an accidental misconnection of its leads. For every value of the external current except the maximum there are two values of the external voltage, and which of the two voltages is obtained depends entirely upon the resistance of the external circuit; this latter is equal to V_e/I_e, or the slope of the line to any point, P, on the curve, *i.e.* to the tangent of the angle made by OP with the horizontal axis multiplied by the ratio of the scales to which volts and amperes are plotted, or simply to tan α when V_e and I_e are plotted to the same scale. The slope of the full-line curve between L and O marks the *critical resistance* of the external circuit for a speed of 500 revolutions per minute; at this value the magnetism

[1] Occasionally a complete short-circuit of a shunt dynamo will, through certain secondary reactions, end in magnetizing its field slightly in the reverse direction, so that, when again self-excited, its polarity is reversed.

is unstable, and at any lower value the machine will fail to excite or maintain its excitation.

For each constant speed the same shunt-wound dynamo gives a different external characteristic. Thus in Fig. 289 the dotted curve shows the external characteristic of the same dynamo at a constant speed of 555 revolutions per minute instead of 500, and it will be seen that at the new speed the maximum current is increased. The higher the speed the less is the slope of the descending branch of the curve; or, in other words, the smaller is the critical resistance which the external circuit may have, without the dynamo losing its magnetism.

§ 11. Instability of magnetism.—Instability of the magnetism requires to be carefully guarded against in the design of self-exciting dynamos. Figs. 289 and 290 show that the unstable portion of the curve between *L* and *O* occurs when the flux falls on the initial straight portions of the curves in Fig. 274, where the ampere-turns over the air-gaps form almost the whole of the excitation; hence if the machine is designed so that the working number of lines falls on the lower part of the flux curve where it descends rapidly towards the origin, the magnetism of the machine as a self-exciting dynamo will be *unstable*. Even if the field hold its magnetism, a very slight variation of the speed will cause a large variation in the voltage—a result which is always undesirable, and especially so if the machine be feeding lamps directly. Thus in the case of a shunt-wound machine, suppose that its speed is raised slightly, the increased E.M.F. causes an increase in the current through the shunt, which will again increase the flux and the E.M.F., and the lower down the curve of magnetization that the machine is worked, the greater is the effect of any variation in the shunt current upon the E.M.F., owing to the magnet being less saturated.

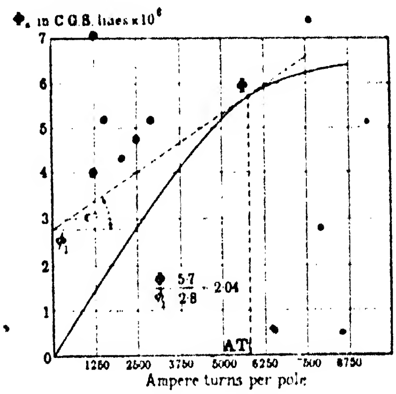


FIG. 291.

A measure of the percentage variation in the voltage which follows from a certain percentage variation in the speed may be obtained by the following construction.[1] Let $O\Phi$ (Fig. 291) be a portion of the flux curve containing the working point

[1] Cp. Poynder and Wimperis, *Engineering*, 3rd May, 1901.

corresponding to Φ lines at which it is desired to investigate the effect of change of speed. The same curve with a different scale of ordinates will also represent the voltage, or $E_a = kN\Phi$, where k is a constant converting lines of flux into volts at the speed of N revs. per minute. At the point Φ draw a tangent to the curve; let it cut the vertical axis at ϕ_1, making an angle a with the horizontal. The equation to the tangent line is $\Phi = \phi_1 + X \tan a$; and in the shunt machine the excitation is practically proportional to the armature voltage, or $X = cE_a$, where c is some constant depending upon the field winding. For a very small alteration of the voltage the working point may be taken as moving up or down the tangent, and for a very small alteration of the speed k may be assumed constant. Hence $E_a = kN(\phi_1 + cE_a \tan a)$. Differentiating with respect to the speed, the rate of change of the voltage when the speed begins to alter is

$$\frac{dE_a}{dN} = k\left\{\phi_1 + c\left(N\frac{dE_a}{dN} + E_a\right)\tan a\right\}$$

whence $\quad \dfrac{dE_a}{dN}\left(1 - kcN \tan a\right) = k(\phi_1 + cE_a \tan a) = k\Phi$

Since $\Phi = \phi_1 + kcN\Phi \tan a$, we have $\Phi\,(1 - kcN \tan a) = \phi_1$

or $$(1 - kcN \tan a) = \frac{\phi_1}{\Phi}$$

therefore $$\frac{dE_a}{dN} = k\frac{\Phi^2}{\phi_1}$$

Multiplying both sides by $\dfrac{N}{E_a}$, and substituting $kN\Phi$ for E_a on the right-hand side,

$$\frac{dE_a}{dN} \times \frac{N}{E_a} = \frac{1}{\Phi} \cdot \frac{\Phi^2}{\phi_1}$$

or $$\frac{\dfrac{dE_a}{E_a}}{\dfrac{dN}{N}} = \frac{\text{fractional change of voltage}}{\text{fractional change of speed}} = \frac{\Phi}{\phi_1} \qquad (138)$$

The percentage change of the voltage corresponding to a given percentage change of the speed is thus directly given by the ratio Φ/ϕ_1, and this must not be allowed to exceed a certain limiting value if instability of the voltage is to be avoided. It can only be unity if the machine is separately excited. Fig. 291 illustrates the case of the 21″ × 11″ dynamo which has been above considered when giving 220 volts on open circuit at a no-load speed of 515 revolutions per minute; the flux required is then $\Phi_a = \dfrac{220 \times 60 \times 10^8}{450 \times 515} =$

$5{\cdot}7 \times 10^6$ C.G.S. lines, and it has been designed to work on a fairly stable part of the curve so as to be satisfactory under these conditions which would have to be met in practice. The ratio $\Phi/\phi_1 = 2{\cdot}04$, and a 1 per cent. variation of the speed implies a 2·04 per cent. variation of the voltage. As the working point approaches the origin, the ratio Φ/ϕ_1 rises, and in practice, in order to avoid instability, should not exceed 3, so that the change of the voltage becomes 3 per cent. for a 1 per cent. variation of speed.[1] As will be explained

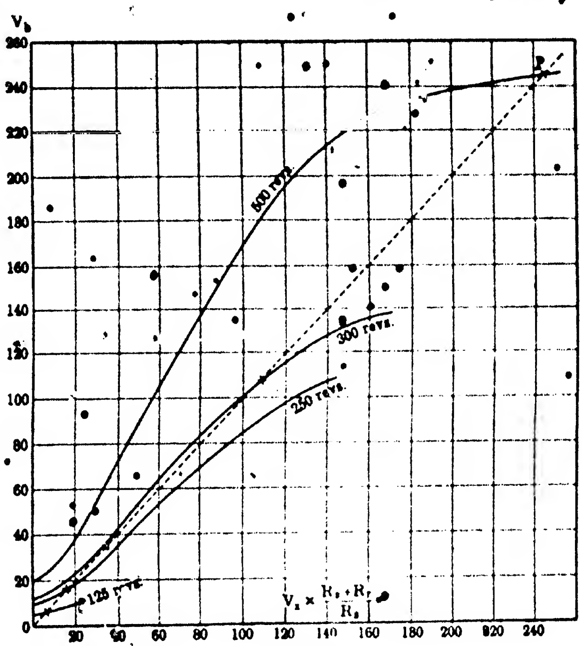


FIG. 292.

later, it is especially in over-compounded dynamos that the safe limit is likely to be approached in regard to the open-circuit voltage given by the shunt winding. As a matter of fact, the change of voltage for any change of speed is to a certain extent held in check by the hysteresis, which assists in maintaining the stability of the magnetism; but the determination of the ratio Φ/ϕ_1 affords a very useful warning to the designer when in doubt as to the lowest point on the flux-curve at which it is safe to work.

A further disastrous consequence of designing a shunt-wound

[1] Cp. Boucherot (*Trans. Intern. Elect. Congress, St. Louis* 1904, Vol. 1, p. 669), who calls the ratio $\frac{dE}{E}\Big/\frac{dN}{N}$ the kinetic variation of voltage, and recommends that its value should be limited to 2.

machine to work on the initial straight portion of the flux-curve that, even when the external circuit is open and the machine is ru very nearly at the normal speed for which it is designed, it may b difficult to reach the proper excitation. When the flux curve i converted into a voltage curve, the diagonal must not fall too nea its initial part. As already stated, the no-current flux-curve owing to the presence of residual magnetism, really ends at som point on the vertical axis slightly above the origin. If now a number of voltage curves similar to those of Fig. 290 are plotted for several speeds, the points at which they cut the diagonal (Fig. 292) give the voltages for the different speeds, and these when plotted as in Fig. 293 show that there is a *crucial speed* at which the voltage rapidly rises. Below this the effect is almost entirely due to the residual magnetism, and as in practice this is very small it has been exaggerated in the diagram in order to render the matter clearer. If the dynamo is run up to its full speed before the circuit of the shunt is closed, then on closing the latter (and neglecting the secondary effects from self-induction) the rising voltage follows the upper curve of Fig. 292; the voltage at the brushes is initially higher than that corresponding to the required excitation, and up to a certain point the surplus ampere-turns continue to increase, after which they decrease until a condition of stable equilibrium is reached at P. If the dynamo is run up to full speed with the shunt closed, the voltage follows the diagonal line, and a rapid increase is obtained when the speed reaches the crucial value which in our $21'' \times 11''$ dynamo as a shunt-wound machine is about 270 revolutions per minute. This crucial speed is determined by the air-line, and is easily calculated from the flux-curve as follows.

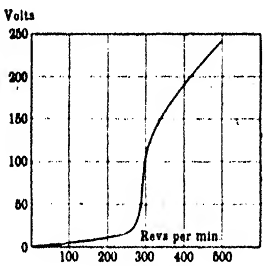


FIG. 293.

In the shunt machine the ampere-turns are themselves dependent upon the voltage at the brushes which is nearly proportional to the flux at any given speed. Hence $AT_s \propto$ flux $\times$ speed, or speed $\propto \frac{AT_s}{\text{flux}}$. In Fig. 294, if Φ be the working point at no-load with the designed speed N_1, $N_1 \propto \frac{AT_s}{\Phi}$. At the crucial speed N_0 the ampere-turns are practically proportional to the air-gap reluctance, and the ratio of the two is given by any point on the air-line, and therefore by point ϕ_2. Hence $N_0 \propto \frac{AT_s}{\phi_2}$ and $\frac{N_0}{N} = \frac{\Phi}{\phi_2}$. In our case from

the characteristic curve of Fig. 287 the open-circuit E.M.F. at 500 revolutions per minute is 245 volts, corresponding to $\Phi_a = 6{\cdot}54 \times 10^6$, so that, as shown in Fig. 294, $\dfrac{N_0}{N_1} = \dfrac{6{\cdot}54}{12{\cdot}1} = 0{\cdot}54$. In other words, the crucial speed $N_0 = N_1 \cdot \dfrac{\Phi}{\phi_1}$ is approximately $500 \times 0{\cdot}54 = 270$ revolutions per minute.

The crucial speed is, in fact, immediately derivable from the constructional data of the machine as follows. On open external

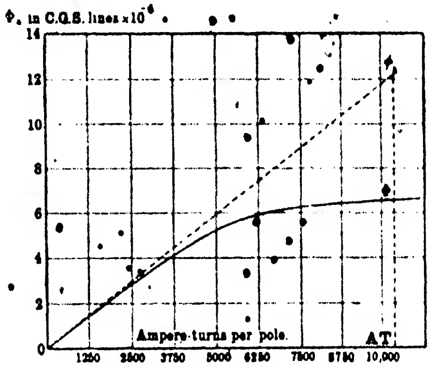


FIG. 294.

circuit at the normal speed the armature induced E.M.F. is $E_a = I_s(R_a + R_s + R_r)$, and is also $\dfrac{p}{a} \times \Phi_a \times Z \times \dfrac{N}{60} \times 10^{-8}$, whence $\Phi_a N_1 = \dfrac{I_s(R_a + R_s + R_r) \times 60 \times 10^8}{Z} \times \dfrac{p}{a}$. The M.M.F. required over the single air-gap of one magnetic circuit is equal to the product of flux $\times$ air-gap reluctance, the latter being $\dfrac{Kl_g}{a_g}$, so that the ampere-turns of air-gap excitation are equal to the flux $\times \dfrac{K \cdot l_g}{1{\cdot}257 a_g}$. Hence for the particular flux ϕ_2 they are $0{\cdot}8\, Kl_g \cdot \dfrac{\phi_2}{a_g}$, and this is also equal to $I_s T_s$, where T_s is the number of shunt-turns in a bobbin corresponding to one pole. The crucial speed is thus

$$N_0 = \frac{0{\cdot}8\, K \cdot l_g (R_a + R_s + R_r) \times 60 \times 10^8}{a_g \cdot T_s \cdot Z} \times \frac{p}{a}$$

In our case $p = a$, and $R_a + R_s + R_r = 0{\cdot}02 + 38{\cdot}8 + 1{\cdot}2 = 40{\cdot}02$ ohms, and $T_s = 1{,}650$; as in Chapter XVI, § 9, the constant by

which the flux must be multiplied in order to give the ampere turns of air-gap excitation is $\frac{0.8K.l_g}{a_g} = \frac{0.8 \times 1.09 \times 0.795}{838}$ $= 0.00083$, and

$$N_0 = \frac{83,000 \times 40.02 \times 60}{450 \times 1,650} = 270 \text{ revolutions per minute.}$$

It is evident from Fig. 292 that the crucial speed is that for which the curve of V_b in relation to $V_s . \frac{R_s + R_r}{R_s}$ (assuming that there is no residual magnetism and that the no-current flux curve descended strictly to the origin) would be a tangent to the diagonal at the origin. For self-excitation to commence there must be some residual magnetism, but in order to reach the true self-excitation that is required the condition $N_1 > N_0$ must hold. Failure to excite in this sense is seldom to be feared in practice with any but very small machines, except in the case of electroplating and similar dynamos of very low voltage. Yet even if self-excitation takes place it is always important, as has been already emphasized, to avoid instability of the magnetism and to design the machine so that it attains a definite and high degree of magnetism when self-exciting on the shunt winding. It must therefore be worked fairly high up on the curves of flux, and in Fig. 274 the best portions whereon to work fall within the horizontal lines marking 5½ and 6½ million lines respectively. These portions may be said to lie on the bend or knee of the curve, but it must be remembered that the apparent position of this bend depends largely on the relative scale to which lines and ampere-turns are plotted. Another advantage of working high up the curve is that the difference of the ampere-turns of the shunt when hot and when cold then produces but small effect upon the flux and E.M.F. of the machine, since the magnet is well saturated. Also, the vertical distance between the no-current and full-current curves decreases as the total flux is increased, so that the variation in the voltage of the machine between full-load and no-load is small when the magnet is strongly excited. If, however, the working flux-density be taken too high, a large amount of copper will be required on the field, and if for any reason the actual curve comes below the predetermined curve, it may be practically impossible to rewind the magnet so as to obtain a greater flux, so that care must also be exercised not to exceed the due limit in this direction.[1]

§ 12. Fall of volts in shunt-wound dynamo, and its regulation.—It will already be apparent that much information can be obtained from the shape of the characteristic curve of the dynamo; in the

[1] Cp. Miles Walker, *The Diagnosing of Troubles in Electrical Machines*, pp. 246 and 293.

case of a shunt-wound dynamo run at a constant speed, we see that its E.M.F. is highest when the external circuit is open, and the only current flowing is that through the shunt. When the external circuit is closed, and the armature current is swelled by the addition of an external current, not only does the loss of volts over the resistance of the armature progressively increase as the armature current increases, but the demagnetizing effect of any back armature ampere-turns increases. Hence not only does the exciting voltage at the brushes of the dynamo fall, but the flux due to a given number of shunt ampere-turns is decreased. Owing, therefore, to the combined effect of the two causes, namely, the increase of any back ampere-turns, AT_b, and the loss of volts over $R_a + R_b$, the characteristic gradually falls throughout the entire working range of the dynamo. If, however, the armature be of low resistance, the drop in volts for any current within the working range will be but small, and consequently the shunt-wound dynamo may in such cases be practically regarded as giving a nearly constant voltage when run at a constant speed.

The shunt-wound dynamo of Fig. 289 has been above designed to work high up on the flux-curve of a well saturated field-magnet, so that it affords a good example of the regulation which may be obtained under favourable conditions with simple shunt winding.[1] By "regulation" in the present connection is understood the degree in which under different loads the machine approximates to a constant voltage at the terminals when run at its normal speed and without alteration of the rheostat inserted in the shunt-circuit; *i.e.* it is the rise of volts when the full load is switched off expressed as a percentage of the full-load volts. From Fig. 289 it is seen that if the rheostat resistance is adjusted to give 230 terminal volts at full-load and at the normal speed of 500 revolutions per minute, the switching off of the full-load with the speed maintained constant will cause the volts to rise to 245—an increase of 6½ per cent. In practice this rise of volts is magnified by the difference between the no-load and full-load speeds as fixed by the governor of the prime mover. When a dynamo is driven by an engine and forms the whole or a considerable portion of the load on the engine, the greater the output the slower will be the speed of the engine, even though it is governed for an approximately constant speed. This difference of speed between full- and no-load in an ordinary engine varies, but usually ranges with a good governor from 3 to 4 per cent. of the no-load speed. Its effect is cumulative, since not only does removal of the full load cause the E.M.F. to rise for the same flux, but the excitation is thereby increased, which again causes a further rise of volts. Since any such rise of volts is detrimental to the good working of a central station or other installation

[1] Cp. Miles Walker, *loc. cit.*, pp. 248–268.

of motors and lamps, the shunt machine is almost invariably supplied with a rheostat of sufficient range to enable the terminal volts not only to be kept constant but even to be reduced at light loads. The least favourable conditions requiring the highest resistance in the rheostat will occur when the machine is working at no-load and before it has become in any way heated by prolonged running. Thus in the case of the 21″ × 11″ dynamo above considered the necessary resistance which the rheostat must provide is to be calculated as follows. On the assumption that it is to give the reduced voltage of 220 at no-load, at a speed of 515 revolutions per minute, or 3 per cent. above the full-load speed to allow for the governor range, a flux of $5{\cdot}7 \times 10^6$ lines per magnetic circuit will suffice, and from Fig. 274 this is obtained with 5,800 ampere-turns per pole, or with a shunt current of 3·42 amperes. The resistance of the shunt when cold being 32·76 ohms, $I_s(R_s + R_r) = 3{\cdot}42 (32{\cdot}76 + R_r) = 220$, whence $R_r = 31{\cdot}74$, say 32 ohms, which will be divided up between a number of intermediate steps proportionate to the degree of fineness which is required in the regulation, and with a current-carrying capacity tapering approximately from 6 to 3·4 amperes.

While the rise of volts on switching the full load off varies in a shunt-wound machine from 7 to 20 per cent., it is evident that the fall of volts on switching the full-load on is with the same initial volts in the two cases considerably greater. Indeed, it may result in the machine becoming almost demagnetized, owing to the resistance of the shunt circuit with its rheostat fully inserted bearing a high ratio to the resistance of the external circuit, and owing to the loss of volts over the armature resistance and brushes with the full current.

§ 13. Uses of shunt-wound dynamos.—Large continuous-current dynamos for central-station work, which supply current at 220 or 550 volts to a network of mains, are often shunt-wound. The voltage at the station end of the feeders to the network is in such cases conveniently regulated by means of rheostats. For the direct lighting of incandescent lamps in smaller installations, shunt-wound dynamos are not so suitable, owing to the necessity for constant attendance in order to regulate their E.M.F. according to the load, if a constant voltage is to be maintained on the lamps. They are, however, frequently used in conjunction with accumulators, and in such cases, by means of a supplementary resistance in the shunt, they are arranged to charge accumulators during the day up to about 130 volts, and at night to work in parallel with the battery, lighting incandescent lamps at 100 volts, the speed under the two conditions being maintained at nearly the same value.

For the electro-deposition of metals, shunt-wound dynamos are employed; for a single electrolytic bath with its leads only some

four or five volts are required, but several baths are commonly worked in series at a correspondingly higher voltage. For charging accumulators, and for electrolytic or electrometallurgical work, shunt-winding is alone suitable; if the voltage of the dynamo becomes less than the E.M.F. of the cells, or the current through the electrolytic baths becomes reversed through polarization at their electrodes, the direction of the current round the shunt remains unchanged, although the current through the armature will be reversed. The result is that the machine runs as a motor in the same direction as it did when acting as a generator, and therefore opposes a back E.M.F. to the discharge from the cells; this back E.M.F. prevents the flow of a large current which might burn up the armature and also injure the cells.

At high voltages the necessary diameter of wire for a shunt-winding on a small machine becomes very small, and fine wire is not only expensive in itself, but the time and trouble of winding the great length of it which is required become very considerable. Hence the maximum voltage for a small machine which has all or most of its excitation provided by a shunt-circuit may be set at about 500 volts, but in large generators of, say, 400 kilowatts output this may be raised to 2,250 volts.

§ 14. Determination of series winding.—Series-winding, although common in motors, is but seldom met with in dynamos; yet for purposes of comparison with shunt-winding it will be instructive to work out the details of a series winding, its necessary size of wire and its disposition, especially as many of the practical considerations thereby brought to light will be found equally applicable to the compound-wound dynamo that has later to be discussed.

Assuming, then, that it is desired to convert our 21" × 11" dynamo into a series-wound machine without alteration of the armature, and that the terminal voltage is still to remain 230 at full load, an addition must be made to the armature induced E.M.F. in order to cover the loss of volts over the series winding. With the same dimensions of pole as in § 8, we know that a suitable depth of coil and satisfactory heating conditions will be attained if we allow a loss of about 1,280 watts over the field-winding. The full armature current now becomes available externally, but as this differs so slightly from 350 amperes the latter figure will be again taken, so that the external volts and amperes forming the output will remain unaltered. The voltage expended over the series-winding must therefore not exceed $\frac{1{,}280 \text{ watts}}{350 \text{ amperes}}$ $= 3{\cdot}65$ volts, or 1·6 per cent. of the terminal volts. When this is added to the loss of 5·9 volts over the armature resistance and the constant loss of about 2 volts over the brushes, E_a must now be 241·55, and at the same full load speed of 500 revolutions per minute the required flux will be $\Phi_a = \frac{241{\cdot}55 \times 60 \times 10^8}{460 \times 500} = 6{\cdot}43 \times 10^6$, which will from Fig. 274 necessitate 10,200 ampere-turns per pole.

Assuming all the 4 bobbins to be connected in series, and the same surface rise of temperature, the required area of wire is therefore by equation (125)

$$\frac{0{\cdot}02445 \times 10{,}200 \times 4 \times 1{\cdot}08 \times 1{\cdot}16}{3{\cdot}65 \times 1{,}000} = 0{\cdot}344 \text{ square inch,}$$

and the number of turns per coil is $T_m = \frac{AT_f}{I_e} = \frac{10{,}200}{350} = 29\frac{1}{4}$. A single

round wire of the necessary diameter to give this section is out of the question owing to the difficulty of winding it. A number of alternatives are, however, possible. Thus in a 4-pole machine two of the 4 bobbins may be placed in parallel and the pair in series with the remaining pair; the current in any turn will then be halved, and the number of turns doubled, but the section of each wire need be only half as large. Or all the bobbins could be placed in parallel, when the current in each, being again halved, becomes $\frac{350}{4} = 87 \cdot 5$ amperes. There are, however, two objections to any parallel connection of the bobbins, and these become of increasing importance as the number of parallel paths is increased. In the first place, as the size of each wire is reduced, the ratio of the copper area to the space occupied or the value of σ diminishes, so that it may finally become impossible to find room for the necessary volume of copper. In the second place, owing to the different location of the several bobbins on the machine they may be differently affected as regards the effectiveness of their cooling surface, so that even if their resistances are exactly equal when cold, they differ when hot; the total current then becomes unequally divided between them, and one or more magnetic circuits may have a greater number of ampere-turns than the remainder—a result which is especially disadvantageous with lap-wound armatures. For both reasons, therefore, it is advisable to adhere to the arrangement of all the bobbins in series.

Returning, then, to the series connection, a rectangular strip may be tried, double-cotton-covered and wound on flat-wise. Six layers of 4·9 turns per layer will give the required number of turns per pole; the thickness of the insulated strip must not exceed $\frac{2 \cdot 5''}{6}$, say $0 \cdot 4''$, and the thickness of the bare copper will be $0 \cdot 4 - 0 \cdot 025 = 0 \cdot 375''$. The width must therefore be $0 \cdot 915''$, or when insulated $0 \cdot 94''$. In calculating the required winding length, especially with a wide strip, one turn must always be added to the actual number which each layer is to give, since the space lost from the spiral way in which the strip must be wound amounts to the width of one turn; hence in our case (number of turns + 1) (width of insulated wire) $= (4 \cdot 9 + 1) \times 0 \cdot 94 = 5 \cdot 55''$. Or *vice versâ*, in calculating the number of turns per layer that can be wound within a certain length one turn must be deducted from the quotient $\frac{\text{winding length}}{\text{width of insulated wire}}$. If the strip be wide and the length of coil be short, this loss of space amounts to an appreciable percentage of the total length. For this reason the proposed use of rectangular strip wound flatwise, although it has in itself a very high value for σ, does not lend itself conveniently to a subdivision of the bobbin into the three sections with air-gaps between them which is required in order to secure the best results on the score of ventilation. Further, the copper even when divided into two strips each $0 \cdot 1875''$ thick, wound in parallel, will be troublesome to wind. An equivalent section of copper wound on edge, say $2 \cdot 45'' \times 0 \cdot 14''$, reduces the above-mentioned loss of space very considerably; the winding length, in order to give $29\frac{1}{4}$ turns per pole, must now correspond to 30 turns, or if the neighbouring turns are insulated with strips of calico to a thickness of $0 \cdot 160''$, the length must be $30 \times 0 \cdot 16 = 4 \cdot 8''$. Whether a quarter turn can be secured depends upon where the wire is led on to or off the bobbins, and a half-turn is usually more convenient as bringing the ends to the opposite sides of each bobbin and in line all round the circle of the poles in a convenient position for connecting together. In spite of the lower value of σ the edgewise arrangement requires less space and is the best. But the winding of a wide strip on edge calls for special tools and skill in manipulation, so that as an alternative a thin wide strip wound on the flat may be tried. The thickness of the strip will be $\frac{2 \cdot 5''}{30} = 0 \cdot 083''$ when insulated with calico strips pasted to one side, or say $0 \cdot 060''$ bare. The necessary width is then $5 \cdot 75''$, and this may be conveniently divided into three sections each $1 \cdot 9''$ wide. The sections of each bobbin are coupled together into parallel where the connections are made to the neighbouring coils by which the several bobbins are put in series.

Finally, therefore, each pole will be wound with three sections, each containing $29\frac{1}{2}$ turns of a copper strip 1·9" wide × 0·060" thick, insulated to 0·080". The joint area of the three strips is 1·9 × 3 × 0·06 = 0·342 square inch, and their joint resistance per 1,000 yards is $\frac{0{\cdot}02445}{0{\cdot}342} = 0{\cdot}0716$ ohm. The resistance of the 4 bobbins in series is $\frac{29{\cdot}5 \times 1{\cdot}08 \times 4}{1{,}000} \times 0{\cdot}0716 = 0{\cdot}00915$ when cold, or 0·00915 × 1·16 = 0·0106 ohm when hot. The loss of volts is 0·0106 × 350 = 3·71, and of watts is 1,298. Our original allowance for the armature E.M.F., etc., may therefore be permitted to stand. The weight of wire is 11·55 × 0·342 × 128 = 506 lb. It will thus be seen that in spite of the greater number of ampere turns which are required on the field for the series machine to give the same terminal volts as the shunt-wound machine, the better utilization of the space which results from the use of strip as compared with round wire enables the same size of bobbins to be retained, the weight of copper contained therein being greater.

Owing to the higher value of σ with rectangular strip copper as compared with round wire, and in general owing to the fewer turns of comparatively large area, the current-density in series winding is usually lower than in shunt

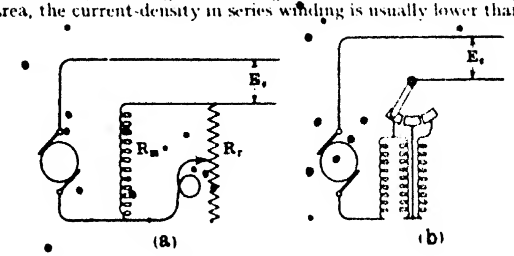


FIG. 295.—Regulation of series-wound dynamo.

coils, since the watts per cubic inch of total space should be the same for coils of similar dimensions. If, as usual in continuous-current dynamos, the field-magnet is stationary, the current-density in the series winding seldom exceeds 1,000 amperes per square inch, and on an average ranges from 750 to 800 amperes per square inch.

Regulation of the series machine is effected either (*a*) by a *diverter*, or less frequently (*b*) by alteration of the number of turns through which the external current passes (Fig. 295). In the latter case the series winding is divided into a number of sections from the ends of which leads are brought out to the contacts of a regulating switch; by moving the lever of the latter the groups of sections are brought into action or cut out one after the other. This method is, however, seldom employed, since it can only be adapted to give a few large steps with consequent sparking at the contacts.

A more gradual change can be obtained with a diverter (Fig. 295*a*); this latter consists of an adjustable resistance connected in parallel with the series winding. By moving the arm of the regulating switch over the contacts connected to the resistance, the amount which is in parallel with the series winding is gradually altered; thus by decrease of the ohms of the resistance a gradually increasing proportion of the total current is diverted away from the series coils, and the field is correspondingly weakened.

§ 15. External characteristic of series-wound dynamo.—The external characteristic of the dynamo when series-wound as described in § 14 for a speed of 500 revolutions per minute, is shown in Fig. 296. It may be derived from the flux-curves of Fig. 274 in the following manner. Taking any current, I_1, multiply it by T_{se}, the number of turns per pole in the series-winding, and then find on the flux-curve for the particular current I_1 the Φ_a corresponding to the exciting power, $I_1 T_{se}$. Thence $E_a = \frac{p}{a} \frac{\Phi_a \times Z \times N}{60 \times 10^8}$ can be determined; the loss of volts over the resistance of the armature,

brushes, and series-winding $= I_1 \times (R_a + R_b + R_m)$, so that V_e corresponding to a value of $I_e = I_1$ is $E_a - I_1 (R_a + R_b + R_m)$, and the corresponding point on the external characteristic can be plotted. Next, take another value for I_e, and in the same way find the corresponding value of V_e, and so on until sufficient points have been obtained to draw in the curve. If only three or four flux-curves are to hand it will be necessary to interpolate or add other curves in order to determine the characteristic throughout

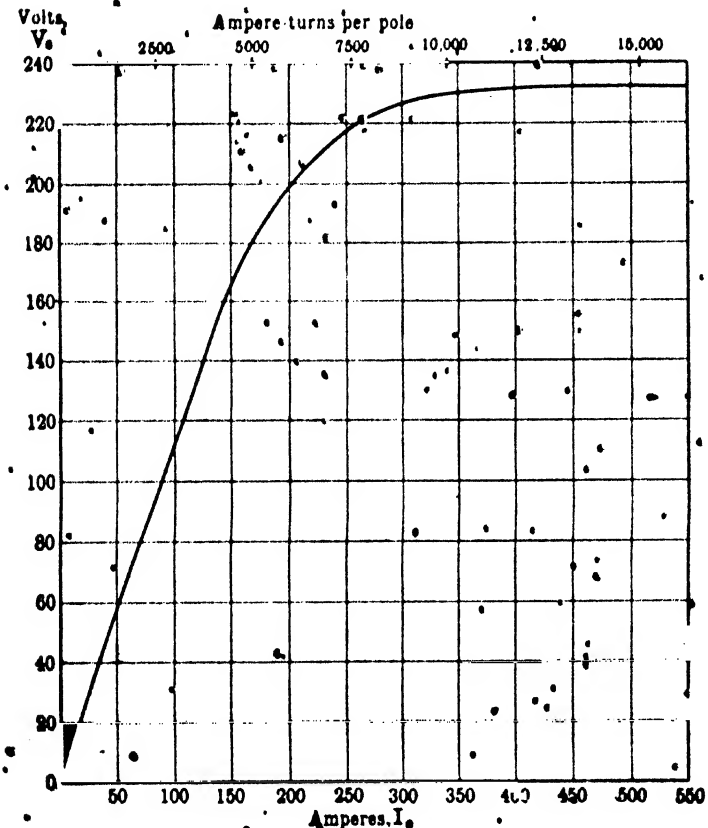


FIG. 298.—Characteristic curve of series-wound dynamo.

a large range of current. Thus the curve for $\frac{3}{4}$ths of full current will fall approximately midway between the curves for half and full current, but slightly nearer to the former; the back ampere-turns of the armature are less at three-quarters than at full-load, owing to the decreased current, and the proportion of leakage is thereby decreased.

The shape of the external characteristic for a series-wound dynamo is widely different from that for a shunt machine. Since the external current is also the magnetizing current, it resembles in the main a curve of flux. But if the external current can be progressively raised by lowering the resistance, a point is always finally reached at which the terminal voltage decreases

as the external current is increased, although this point is usually well beyond the working range of current. The fall of the external characteristic is due to the loss of volts over the resistance of the armature and series winding, and to the demagnetizing effect of any back ampere-turns on the armature. The former loss is directly proportional to the current, while the effect of the back ampere-turns increases faster than the current. On the other hand, when the magnet is approaching saturation, if the external current is increased,

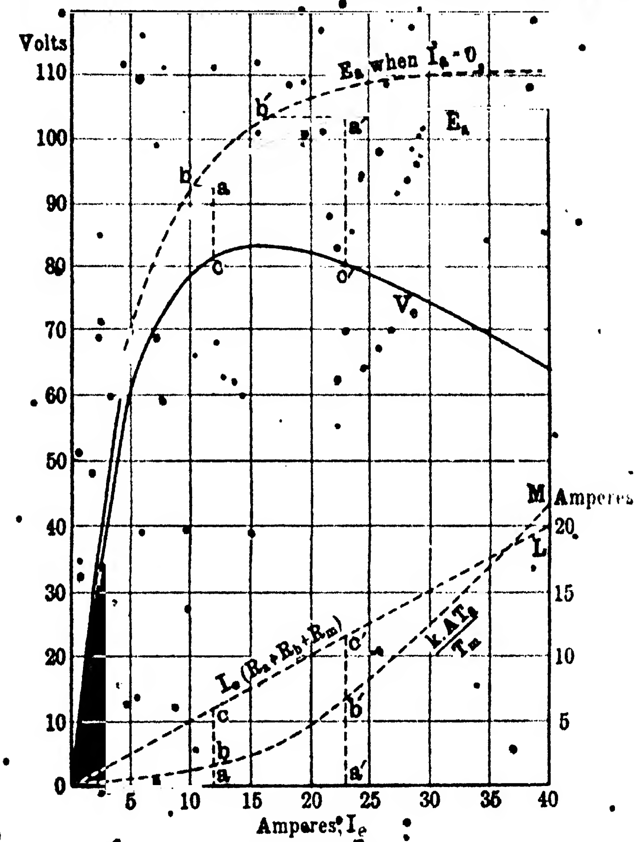


FIG. 297.—Method of deriving external characteristic of series-wound dynamo.

the increase in the total flux is by no means proportional to the increase in the exciting power of the field ampere-turns. Hence the induced E.M.F. rises less and less rapidly as the current is increased, and after a certain point the increase in the induced E.M.F. does not so much as compensate for the increased loss of volts over the resistances of the machine; the external characteristic curve therefore attains a maximum height, and then bends gradually downwards.

If the brushes have a fixed position the descent is very gradual, as in the case of the small series-wound dynamo illustrated in Fig. 297, which also

shows how the external characteristic for a series-wound dynamo may be graphically constructed. The open-circuit flux-curve of the dynamo is first converted into an E.M.F. curve for the given constant speed, as shown by the upper dotted curve in Fig. 297. For any particular value of the current the ampere-turns required to balance all the effects of armature reaction, including the increased leakage, may be expressed as $k.AT_a$ where k increases as saturation is approached; this again may be expressed by the equivalent number of amperes which with the given value of the field turns T_m per pair of poles yield $k.AT_a$, i.e. $\frac{k.AT_a}{T_m}$. The dotted curve marked $\frac{k.AT_a}{T_m}$ is thus drawn, of which the scale in amperes to the right of the diagram is equal to the horizontal scale of abscissae. Inclined to the horizontal is then drawn the line OL, which to the volt scale at the left hand of the diagram measures the loss of volts over the resistances of the armature, brushes, and series coils for any particular current. The induced E.M.F. of the armature for any value of the current is then derived by finding a series of points such that their horizontal distance from the upper dotted curve is equal to the corresponding ordinate ab of the curve OM, and at the same time such that they fall on the vertical from the particular value of the current in question. From the curve of the armature E.M.F. E_a so derived has then to be deducted the loss of volts AC corresponding to each value of the current. If the brushes are given a greater angle of lead as the current is increased, the back ampere-turns of the armature then rise faster than the field ampere turns, and gradually overpower the latter, so that the actual flux may diminish. The curve of the armature E.M.F. then begins itself to bend over, and the descent of the curve of the external voltage is rendered much steeper.

The external characteristic at any other speed is obtained by simply altering the height of the ordinates; the slide rule is set up for the ratio of the speeds, and the calculated values of E_a are altered in proportion to the alteration of the constant speed.

§ 16. Uses of series-wound dynamos.—Series-wound dynamos have been chiefly used for the electric transmission of energy over considerable distances, when one or more generators are employed to supply current to similar series-wound motors through a single series circuit (Chap. XXIII, § 2); and also for high-potential arc lighting.

For running a number of arcs in series since a nearly constant current is required, the drooping external characteristic is a distinct advantage, and on this account the series-wound machine is worked on the descending portion of the curve. If the resistance of any one or more lamps be decreased owing to the carbons being fed together, or if a lamp be entirely cut out of circuit, the momentary increase in the external current is then accompanied by a decrease in the terminal volts; this helps to bring the current back to its normal strength, and although it cannot be sufficient to make the machine entirely self-regulating for constant current at varying potentials, it tends in the right direction, and leaves less for the automatic constant-current regulator to do. Thus in the constant-current dynamos as formerly used for arc lighting the drooping characteristic was expressly exaggerated.

§ 17. Compound winding.—In the case of a shunt-wound dynamo with armature of low resistance its terminal voltage, although approximately constant over a considerable range of current, necessarily falls. A constant voltage over the full range of a dynamo's capacity or even a voltage rising with the load is, however, essential for most purposes, and this requirement can be closely fulfilled by the *compound-wound* dynamo (Fig. 298). If a dynamo be shunt-wound to give a certain voltage at no-load, and the magnet be in addition wound with a certain number of turns connected

in series with the external circuit, then, as the load is increased, the current flowing through the series, or as they are also sometimes called "main," turns, will progressively increase the excitation. Now this increase in the excitation may be made not only to balance any increase of the back ampere-turns, as more load is thrown on, but also to increase the total flux through the armature. As a

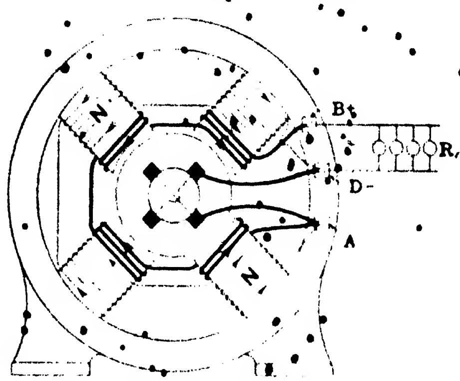
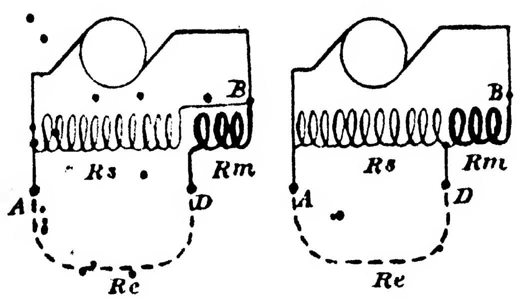


FIG. 298.—Compound wound dynamo.



Short Shunt. Long Shunt

FIG. 299.

consequence the induced E.M.F. is increased, and when properly proportioned this increase may be made to exceed the loss of voltage over the series-turns sufficiently to compensate for the increased loss of volts over the armature resistance and brushes. Thus while the armature E.M.F. rises with the load, the terminal voltage remains practically constant, however the current be varied over a very considerable range. The external characteristic is therefore practically a straight line throughout the entire working range of current. In the compound-wound dynamo the shunt may, in fact, be regarded as providing a certain initial flux and voltage;

these latter are maintained and reinforced against the combined causes which tend to reduce them by means of the series-turns the exciting power of which varies directly with the external current.

The connections of a compound-wound dynamo may be arranged in two slightly different ways, shown diagrammatically in Fig. 299. By the first, which is also shown in Fig. 298, and is known as the "short-shunt" method, the shunt is placed across the brushes of the machine, and the voltage acting on it is $V_b = V_e + I_e R_m$, the current flowing through the series winding being simply the external current I_e. By the second, or "long-shunt" method, the shunt is placed across the terminals of the external circuit; the voltage acting on it is then V_e, while the whole armature current $I_a = I_e + I_s$ flows through the series winding. Of the two methods the first is the more common, but their practical difference is not very important.

As regards the winding of the two sets of magnetizing coils on the magnet bobbins, the series may be wound over the shunt or *vice versâ*, but it is more convenient that they should form separate coils. In the latter case, however, care must be taken that the equality of the turns on each pole is such as to suit the necessities of the armature winding. Thus in the multipolar machine the series-winding may be confined to alternate poles if the armature be wave-connected, but if it be lap-wound all the poles are preferably similarly wound, each with its proportion of shunt and series-turns.

§ 18. Over-compounding.—A constant speed and a required constant voltage have been tacitly assumed above for the compound-wound dynamo; but such are not in strictness the ordinary conditions of working, owing to the drop in speed between no-load and full-load which results from the governor range, and which averages from 3 to 4 per cent. of the no-load speed. Apart from this mechanical lowering of the speed, there is also the electrical loss of volts over the main leads between the dynamo and the load, which increases in direct proportion to the current passing. If, therefore, a constant potential is required at the far end of the line (as, for example, in a factory with 220-volt incandescent lamps or motors run directly from a dynamo some distance away) the dynamo must give a terminal voltage of, say, 230 volts when supplying full current in order to allow for the loss of potential over the leads. When, however, the load is very light, the dynamo must give little more than 220 volts, and for any external current between zero and the maximum the external voltage should rise in proportion from 220 to 230. Hence in such a case the compound-wound dynamo must give a higher voltage at full-load than at light loads, and that at a lower speed. The desired external characteristic is thus a straight line, not horizontal, but slightly inclined upwards, or, as it is termed, the dynamo is to be *over-compounded*.

§ 19. Example of calculation of compound winding.—Let us suppose that the 21" × 11" dynamo of Chapter XVI, § 9, has to be compounded to fulfil the following conditions: at 500 revolutions per minute, $V_e = 230$ volts when $I_e = 350$ amperes, and at 515 revolutions $V_e = 220$ when $I_e = 0$, *i.e.* the voltage is to rise by 10 volts while the speed drops about 3 per cent. as the load is increased from no-current to full-current.

To give the required voltage at no-load, Φ_a must, as in § 11, be $5{\cdot}7 \times 10^6$, and from the no-current curve of Fig. 274 we see that the shunt-winding must then give $AT_{s1} = 5{,}800$ ampere-turns. At full load, allowing a loss of 8 volts over the armature and brush resistances (as before), and of 1 volt over the series-winding, the induced E.M.F. must be 239 volts; the total number of useful lines to give this E.M.F. is $6{\cdot}38 \times 10^6$, and, as calculated in Chapter XVI, § 9, the excitation required per pole to give this flux is $AT_f = 9{,}750$ ampere-turns. At full load the exciting voltage on the shunt is 231, if the machine has a "short-shunt" winding, and the ampere turns due to the shunt are then

$$AT_{s2} = AT_{s1} \times \frac{V_{b2}}{V_{b1}} = 5{,}800 \times \frac{231}{220} = 6{,}090$$

The ampere-turns which the series-winding must produce at full-load are therefore $AT_f - AT_{s2} = 9{,}750 - 6{,}090 = 3{,}660$, and the number of series turns is $T_m = \frac{AT_m}{I_e} = \frac{3{,}660}{350} = 10\frac{1}{2}$ per pole.

Assuming a total loss of about 1,280 watts in the field, and a depth of winding of 2·5 inches as in the previous case of the shunt machine, we have still to settle the question of how to divide the total winding space between the shunt and series coils so as to obtain the best results.

The shunt and series coils will be arranged as shown in Fig. 281, with the series winding in a section below the shunt, the depth of winding being in both cases as nearly as possible the same so as to give a neat appearance to the complete bobbin. Under these conditions it is evident that, in order to obtain an approximately uniform temperature rise over the length of the bobbin, the loss per unit length of winding space should be approximately uniform; *i.e.* if L_s and L_m are the axial lengths of the shunt and series coils respectively, and W_s and W_m are the watts lost in them, $\frac{W_s}{L_s} = \frac{W_m}{L_m}$, or $\frac{L_s}{L_m} = \frac{W_s}{W_m}$. In Chapter XVI, § 18, it was shown that the watts in any coil are $W = A^2T \times l_s \times k \times \frac{0{\cdot}00002445}{a}$, and since $a = \sigma \cdot a_1 = \sigma \cdot \frac{Lt}{T}$, we have in general $W = \frac{(AT)^2 \times l_s \times k \times 0{\cdot}00002445}{Lt\sigma}$,

of in the present case, since the depth and the length of the mean turn are to be the same, $W_s \propto \frac{(AT)^2}{L\sigma}$. Hence $\frac{W_s}{W_m} = \frac{\frac{AT_s^2}{L_s \cdot \sigma_s}}{\frac{AT_m^2}{L_m \sigma_m}}$ and finally $\frac{L_s}{L_m} = \frac{AT_s}{AT_m}\sqrt{\frac{\sigma_m}{\sigma_s}}$. The total winding length of the coil being L, the correct length for the shunt coil is thus given by the relation $\frac{L_s}{L - L_s} = \frac{AT_s}{AT_m}\sqrt{\frac{\sigma_m}{\sigma_s}}$, or $L_s = L \times \frac{1}{1 + \frac{AT_m}{AT_s}\sqrt{\frac{\sigma_s}{\sigma_m}}}$

With copper strip for the series, and round wire for the shunt-winding, σ_m will be higher than σ_s, and $\sqrt{\frac{\sigma_m}{\sigma_s}}$ will be greater than unity, say, $\sqrt{\frac{0{\cdot}85}{0{\cdot}6}} = 1{\cdot}19$. Since $\frac{AT_s}{AT_m} = \frac{6{,}090}{3{,}660} = 1{\cdot}66$, the fractions of the total winding length which should be assigned to the shunt and series respectively are 0·665 and 0·335. If any great departure from this ratio be contemplated, the tendency should be in small machines to assign more space to the series-winding, so as to reduce its loss of watts, since this loss has the indirect effect of increasing the necessary armature E.M.F. and the ampere-turns to give the corresponding flux;[1] in large machines the opposite

[1] The sections at the top and bottom of the coil will actually be cooler than the central section, but in the above this effect, due to the cooling influence of the end surfaces, has been neglected. On this latter approximate assumption, since in equation (129) the second term on the left-hand side, namely, $2Lt^2$ is not taken into account, it follows at once that, in order to satisfy the heating conditions, the axial length of any coil or part of a coil expressed in terms of the ampere-turns which it contains is

$$L\sqrt{t} = \frac{AT \times 10^{-3}}{1{\cdot}225\sqrt{\sigma}}\sqrt{\frac{k}{\xi i}}\sqrt{\frac{l_x}{l_o}}$$

and since in our present case, with the shunt and series coils wound abreast, the only difference between them lies in σ, we have

$$\frac{L_s}{L_m} = \frac{AT_s}{AT_m}\sqrt{\frac{\sigma_m}{\sigma_s}}$$

Further, the current-density in the two kinds of winding is then inversely proportional to the square root of σ in the two cases.

If the series coils are wound over the shunt, or *vice versâ*, there is a further difference in that the mean length of a turn and the mean temperature of each division of the coil must depend upon its position. Let t_i, AT_i, k_i, σ_i refer to the inner coil, whether shunt or series, t_o, AT_o, k_o, σ_o to the outer coil, and let p_i, p_o be respectively the inside and outside perimeters of the complete coil; then the minimum watts with a given space to be filled are reached when the ratio of the depths is

$$\frac{t_i}{t_o} = \frac{AT_i}{AT_o}\sqrt{\frac{\sigma_o}{\sigma_i} \cdot \frac{k_i}{k_o} \cdot \frac{p_i}{p_o}}$$

With circular bobbins of which the inside diameter is d and the total radial

course is better. When the total loss of about 1,280 watts is divided in the above proportion, the respective amounts are $W_s = 850$ and $W_m = 430$; the latter corresponds to a loss over the series winding of 1·23 volts, which is thus higher than the allowance that has been made above. The difference is, however, but a very small percentage of the whole, and since a division of the winding into three equal sections conduces to the neat appearance of the coil as a whole, the total winding length of 6″ will be divided between the shunt and series in the proportion of $\frac{2}{3} : \frac{1}{3}$.

The resistance which the shunt wire must have is from equation (124)

$$\omega' = \frac{220 \times 100}{5{,}800 \times 4 \times 1{\cdot}08 \times 1{\cdot}16} = 7{\cdot}55 \text{ ohms per 1,000 yards}$$

and the necessary gauge is therefore to be obtained by the use of wire 0·064″ diameter with a small amount of 0·072″ wire wound on one of the two sections. When the two sizes are insulated with 12 mils of double cotton covering to 0·076″ and 0·084″, in each section of 2″ axial length there will be 25·2 and 22·8 turns per layer respectively, while the required depth of 2¼ inches will be given by 36 layers of the smaller size and by a division of the second section

		Turns per layer.	Turns per section.	l_s in yards.	Yards.	
1st section	36 layers of 0·064	25·2	906	1·08	980	. .
2nd „	25 „ 0·064	25·2	627	1·01	635	. . .
	10 „ 0·072	22·8	228	1·225	—	280
Per bobbin			1761	–	1615	280
In 4 bobbins			7044	.	6460	1120
ω' per 1,000 yards				..	7·6	6·01
Resistance					49·1	6·74
					55·84 ohms	

into 25 layers of the smaller and 10 layers of the larger wire. The radial depth of the 25 layers will be 1·71″ and the mean length of a turn $\pi(9{\cdot}875 + 1{\cdot}71)$ inches = 1·01 yard. Similarly, the depth

depth is t, the ratio $\frac{p_i}{p_o} = \frac{d}{d + 2t}$ and with rectangular bobbins of which the two inside dimensions are A and B, $\frac{A + B}{A + B + \pi t}$. Since the ratios $\frac{k_i}{k_o}$ and $\frac{p_i}{p_o}$ vary to some extent in opposite directions, the ratio of the depths reduces approximately to $\frac{t_s}{t_m} = \frac{AT_s}{AT_m}\sqrt{\frac{\sigma_m}{\sigma_s}}$ as before. In actual practice a large percentage difference in the value assigned to the ratio of the depths produces but a small percentage difference in the total watts, so that even an approximate division will not cause the watts to exceed the minimum appreciably.

of the outer layers of the larger wire being 0·79″, the mean length of a turn is $\pi(9{\cdot}875 + 3{\cdot}42 + 0{\cdot}76)$ inches = 1·225 yard. When hot, $R_s = 55{\cdot}84 \times 1{\cdot}16 = 65$ ohms; thence $I_{s1} = \frac{220}{65} = 3{\cdot}38$ amperes, giving $AT_{s1} = 5{,}950$ and 745 watts, while $I_{s2} = \frac{231}{65} = 3{\cdot}55$ amperes, giving $AT_{s2} = 6{,}250$ and 820 watts under the full-load condition. There is therefore a small margin in hand. The total weight of the smaller wire is 240 lb. and of the larger 53 lb., making 293 lb. in all.

The series coils are to consist of wide copper strip wound on flatwise, and insulated with intervening strips of thin calico. The width may be nearly equal to that of the finished section, say, 1·9″, since the whole is to be bound over with cotton tape and the insulation beween the layers need not extend round the edges. Since the several sections of series-winding are to be connected in series from pole to pole, in order that the start and finish of each coil may be on opposite sides of the pole, there will be 10½ turns per section, and the depth of the winding will correspond to 11 turns. The thickness of each turn when insulated must therefore be $\frac{2{\cdot}5}{11} = 0{\cdot}227''$; and allowing 27 mils for the insulation between neighbouring turns, the copper strip is, for ease in winding, best divided into two thicknesses of 0·100″ wound together in parallel. Its area is therefore $1{\cdot}9 \times 0{\cdot}100 \times 2 = 0{\cdot}38$ square inch, and its resistance per 1,000 yards at 68° F. $\omega' = \frac{0{\cdot}02445}{0{\cdot}38} = 0{\cdot}0645$. The total length that is required is 42 turns $\times$ 1·08 = 45·4 yards, and its resistance $0{\cdot}0454 \times 0{\cdot}0645 = 0{\cdot}00293$, or when hot, $R_m = 0{\cdot}00293 \times 1{\cdot}16 = 0{\cdot}0034$ ohm. When carrying 350 amperes, the loss of volts over the series winding is thus 1·19, and of watts is 417, while the weight of the series winding is 200 lb. The total loss in the field-winding at full load is 1,237 watts or 1·55 per cent. of the output; it is rather less than in the corresponding shunt-wound machine—a result which is due to the better utilization of the space with the rectangular strip of the series winding, while the σ of the shunt winding has been kept up to the same figure, even though the size of wire is smaller, by the employment of a thinner insulation. The total weight of wire is correspondingly greater, namely, 493 lb.

In order to compensate for differences of temperature of the compound-wound dynamo it is convenient to insert in series with the shunt a rheostat, just as in the case of the simple shunt machine, but of much smaller range. The difference between the resistance of the shunt itself when hot and when cold would be compensated by the addition of an external resistance of 10 ohms; but since,

when the whole machine is cold, the armature voltage which is required at full-load to give 230 volts at the terminals will be reduced to about 238, and only 9,500 ampere-turns per pole are needed, while the series winding still gives 3,660, the shunt current must be reduced to $\frac{5,840}{1,761} = 3\cdot3$ amperes; whence $R_s + R_r = \frac{231}{3\cdot3} = 70$ ohms. The maximum resistance of the rheostat must then be at least $70 - 55 = 15$ ohms.

§ 20. Necessary imperfections of compound winding.—Although a dynamo may compound perfectly at full and no-load, it will not

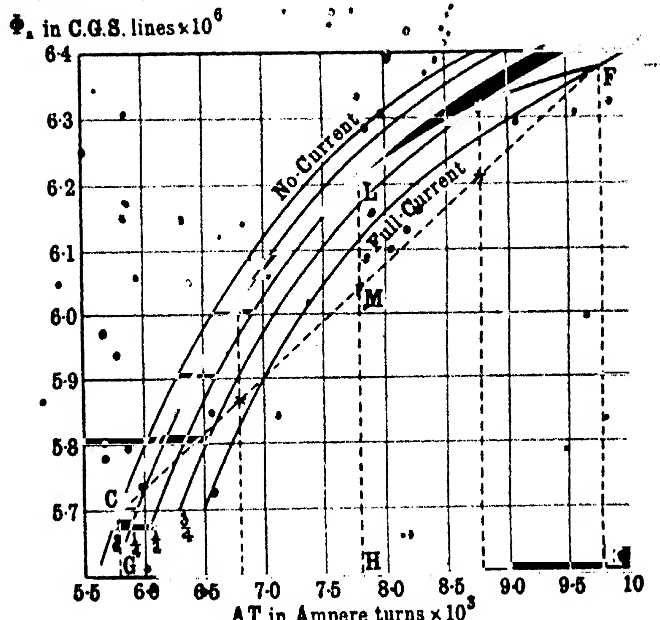


Fig. 300.

do so with equal accuracy at intermediate points. This is due partly to the flux curves being rounded between the working points of no and full load, as, *e.g.* between the limits of $5\cdot7 \times 10^6$ and $6\cdot38 \times 10^6$ lines in Fig. 274, and partly to the fact that when the brushes have to be given forward lead in the absence of commutating poles, the back ampere-turns at half-current are somewhat less than half the back ampere-turns at full-current, and those at quarter-current less than half of those at half-current, and so on.[1] The

[1] Even if the brushes are retained in a fixed position the effect of the armature reaction in increasing the leakage helps to prevent the rise of the flux in the compound-wound machine from following the rise of the current in strict proportion.

effect of these two causes is seen in Fig. 300, which shows a portion of the flux-curves of Fig. 274 on an enlarged scale. At no-load (or practically no-load) the working point of our compound-wound dynamo is C, and at full load F; since the length GK gives the increase in the ampere-turns, when the current is increased up to full load, the increase in the ampere-turns at half-load will be $GH = \frac{GK}{2}$, and the corresponding Φ_a on the half-current curve is HL. But for perfect compounding only HM lines are then required, and the result is that the voltage is rather higher than it should be; this, of course, increases the shunt-current, and the excess voltage is thereby rendered even greater. If therefore the terminal voltage is correct at no-load and at full load, it is higher than is required, not only at half-load, but at all loads between zero and the maximum. The excess voltage is roughly proportional at any point to the vertical distance between the full line CLF, and the dotted line CMF, from which it will be seen that it is greatest between quarter and half-load. In order to minimize this imperfection, a compound machine should be worked as far as possible on the upper part of the flux-curves; the higher the working limits, the flatter and more horizontal do the curves become; and the effects of the unequal spacing of the intermediate curves, and of the increase of the shunt-excitation, are lessened. But in slow-speed machines, e.g. in an 18-kilowatt dynamo running at 250 revolutions per minute, the percentage loss of volts over its electrical resistance may amount to about 8 per cent. of the terminal voltage (6 per cent. over the armature and brushes, and 2 per cent. in the series coils); allowing a further loss of 3 per cent. in the leads between the dynamo and the lamps, and also a decrease of 5 per cent. in the speed of the engine at full-load, it will be found that Φ_a at full-load is $1{\cdot}11 \times 1{\cdot}05 = 1{\cdot}17$ times the no-load Φ_a, although the amount of over-compounding is not great. In such a case it is practically necessary to work the machine on the rounded knee of the magnetization-curve, since to work higher up would require an excessive amount of series wire. It results that slow-speed machines and those which have to work over a large range of voltage compound more or less imperfectly, and the terminal volts for any current between ¼ and ¾ of full load may be from 3 to 4 per cent. too high.[1] In machines running at comparatively high speeds the voltage should not exceed the required amount by more than 3 per cent., even at the point of greatest difference.[1]

Machines with small magnets compound less perfectly than those with large magnets, and multipolars with their greater number of smaller magnetic circuits show in general a greater percentage rise

[1] Cp. Miles Walker, *The Diagnosing of Troubles in Electrical Machines*, p. 273.

at intermediate loads than would large 2-pole dynamos. Differences of temperature also considerably affect the perfection of the compounding of a dynamo; if worked on the rounded portion of the curve at no-load, and if the temperature of the field-coils rises during working 50° or 60° F., a machine designed to compound when hot will give too high a voltage at no-load, and, in general, as the temperature of the dynamo changes, the relative value of the shunt- and series-turns is altered. Such effects can, as mentioned above, be compensated by inserting a small variable resistance in the shunt circuit, which can be gradually short-circuited as the dynamo warms up during working. In other cases, where it may be necessary to alter the amount of compounding to meet different conditions, a diverter can be employed just as with a series-wound dynamo, or the series-turns are connected at intervals to contact-blocks, which enable some or all of them to be short-circuited by the insertion of a plug. Since the compound-wound machine is usually required to excite itself on open circuit, the same precautions must be taken against instability of the magnetism as in shunt machines.

Traction generators are very frequently wound to give 500 volts at no-load, and are then over-compounded to give 550 volts at full-load, so as to balance the loss of volts in the transmission line. Compound winding is also largely employed in the dynamos of isolated installations, as for driving motors in a factory or for the lighting of steamships, in order to maintain an approximately constant voltage on the motors or lamps under all conditions of load.

CHAPTER XVIII

THE FLUX-DENSITY CURVE ON NO-LOAD

I. THE SALIENT-POLE MACHINE

§ 1. The determination of the flux-density curve over a pole-pitch.—Given a salient-pole machine under design with a total number of ampere-turns AT_f of excitation on each pole, it is not possible to pre-determine the spacial distribution of the flux, *i.e.* the flux-density over a complete pole-pitch, without in the first instance assuming one or other of two closely related sets of quantities; and the accuracy of the assumption must subsequently be checked when it can be compared with the final result obtained by its means. If found to be inaccurate, it must then be corrected and the process repeated, until agreement is finally reached. The necessity for the initial judicious guess-work even for the no-load condition follows for the reason now to be given, and the same necessity also holds when the armature carries current.

In Fig. 301 the M.M.F. of a field coil, AT_f in ampere-turns, is marked in a circle on each pole. If 2 AT_c be the ampere-turns required to pass the flux from one bifurcation plane CD in the armature core below the teeth to the next bifurcation plane EF, the magnetic potential of the core in ampere-turns will be $+AT_c$ at the bifurcation point under a N. pole, and $-AT_c$ at the bifurcation point under a S. pole. Either value will decrease numerically as we proceed from the bifurcation plane towards the interpolar plane of zero potential, and at each point x the potential will have some lesser value $\pm AT_{cx}$. The positive and negative values of the magnetic potential at the surface of the armature will under each pole be numerically higher by the ampere-turns expended over the reluctance of the teeth AT_{tx}. The magnetic potential of the N. and S. pole-faces will then be respectively $+AT_p$ and $-AT_p$, where

$$AT_p = AT_{cx} + AT_{tx} + AT_{gx}$$
$$\text{or } AT_p - AT_{cx} = AT_{tx} + AT_{gx}.$$

If 2 AT_y be the ampere-turns required to carry the flux from one pole through the yoke to the next pole, the potentials at the roots of the N. and S. poles are respectively $-AT_y$ and $+AT_y$; as the flux passes down through the N. pole, its potential is raised by the M.M.F. of the exciting coil AT_f less the loss AT_m over the pole from $-AT_y$ to $+AT_p$, while as it passes up through the S. pole, the potential is correspondingly raised from $-AT_p$ to $+AT_y$.

Hence a second expression for the potential of the pole-face as obtained from the yoke side is

$$AT_p = AT_f - AT_y - AT_m.$$

All the above symbols have reference to one half of a magnetic circuit, corresponding to one pole, while the complete circuit calls for $X_f = 2\,AT_f$ ampere-turns of excitation.

Now AT_y, AT_m and AT_{cx} are at first unknown, since the total flux has not been exactly determined. Either then the total flux must be assumed from which to calculate AT_y and AT_m and thence to determine AT_p, or AT_p must itself be assumed as an initial datum. There is then only left the necessity either to guess the value of AT_c and thence by gradual reduction of it step by step as

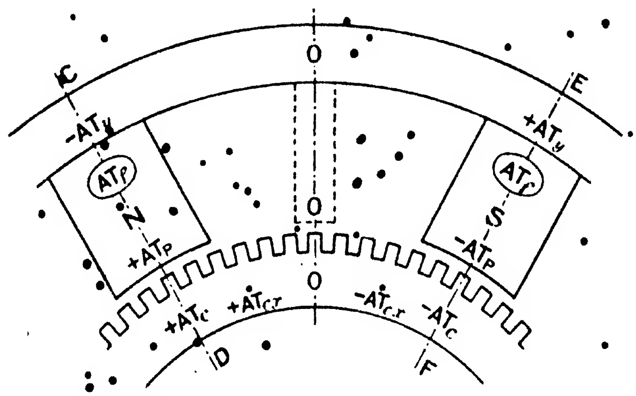


FIG. 301.—Magnetic potentials on no load in salient-pole machine.

flux is removed from the core to obtain AT_{cx}, or to assume initially a certain distribution of the flux and by calculation from the gradually increasing core density to estimate provisionally the value of AT_{cx} at each point (cp. Figs. 327 and 329). In both cases the correctness of the assumptions can only be checked at the end of the process, but it is surprising how quickly such "trial and error" solutions converge on the true result.

§ 2. The core-density curve on no load.—On no load from considerations of symmetry the plane of zero magnetic potential passes radially down through the centre of the interpolar gap between two main poles, or down the centre of a commutating pole if such be present (as shown dotted in Fig. 301). Out of the yoke total ampere-turns $2AT_y$, the expenditure will be similar for equal distances on either side of the zero plane, and so also in the armature core for symmetrical points the numerical value of $\pm AT_{cx}$ will be the same. Starting from the bifurcation plane which must on

no-load fall in line with the centre of the pole-face, at or the specific AT per cm. length of path increase at first but slowly, but as the flux-density in the armature core grows, it begins to rise very rapidly towards the pole-edge and then remains nearly constant over the part of the interpolar gap where the fringe is weak (Fig. 302). Conversely the integral of the curve or AT_{cx} as we proceed in the opposite direction from the interpolar line of symmetry to the pole-face centre rises at first rapidly and then becomes nearly flat under the pole.

As an example, in an 8-pole machine with toothed armature of diameter 45 ins. = 114·2 cm. with a pole-pitch of 45 cm. the polar arc was 33 cm., leaving an interpolar gap of 12 cm. so that $c = 6$. The equivalent air-gap length $Kl_g = 1{\cdot}11 \times 0{\cdot}795 = 0{\cdot}882$ and

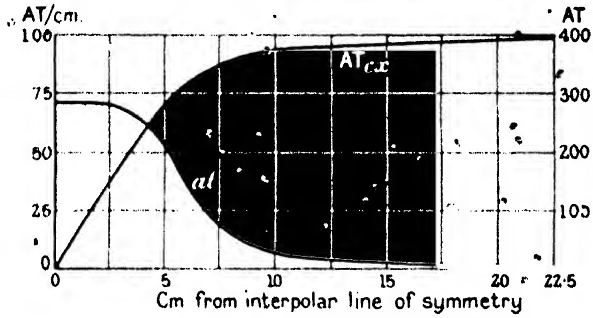


FIG. 302. AT_c over half magnetic circuit in armature core.

with a normal $B_g = 8400$, $AT_g = 5900$, while from Fig. 303 for this density in the air-gap $AT_t = \dfrac{0{\cdot}087 \times 8400}{1{\cdot}257} = 582$. The total flux over the polar arc would then be $8400 \times 33 = 277{,}000$ per cm. of armature core-length, and with allowance for the inter-polar fringe, this was increased to 294,000 as the actual total flux of a pole-pitch per cm. of core. With deduction for insulation and air-ducts, and a depth of core 4·5 ins. = 11·4 cm. below the teeth, the single core-area per cm. axial length of core will be taken as $11{\cdot}4 \times 0{\cdot}9 \times 0{\cdot}83 = 8{\cdot}5$ cm^2, giving a maximum density in the core of 17,300, for which the at per cm. = 72. Thence AT_c under a pole was estimated at 390. The magnetic potential of a pole-face was thus $AT_r = AT_g + AT_t + AT_c = 5900 + 580 + 390 = 6870$.

If now the density in the air-gap at intervals of $2\frac{1}{2}$ cm. over half the pole-pitch, obtained from equations (134) and (135) of the following section, be entered as shown in column 2 of Table X, the flux collected in each interval is given in the intermediate rows; the actual flux in the core at the points x, and the density B_c are then

as shown in the 4th and 5th column respectively, opposite to the values of x. The average at over a section is given in the intermediate rows. The teeth of the armature being 4 cm. long, the mean diameter of the core below the teeth is $114{\cdot}2 - (8 + 11{\cdot}4) = 94{\cdot}8$. A cm. on the surface of the armature therefore corresponds to $1 \times \frac{94{\cdot}8}{114{\cdot}2} = 0{\cdot}83$ cm. length in the centre of the core, and $2\frac{1}{2}$ cm. on the surface correspond to 2·07 in the core. Multiplying the average at's by 2·07, and summing up the products, the value of AT_{cx} at each point x is reached in the last column.

TABLE X.

AT_c OVER ARMATURE CORE.

x cm. from interpolar centre.	B_{g}	Flux per cm. length of core. Added in section.	Total in arm. core.	Density B_c	Specific at.	Average at for section.	× 2.07.	AT_{cx}
22·5	8370		0	0	0			395
		20,900				0·4	0·9	
20	8374		20,900	2,465	1			394·1
		20,910				1·1	2·3	
17·5	8382		41,810	4,930	1·2			391·8
		20,940				1·5	3·1	
15	8,390		62,750	7,390	1·8			388·7
		20,970				2	4·1	
12·5	8,400		83,720	9,850	3·1			384·6
		21,000				4	8·3	
10	8,410		104,720	12,360	6·4			376·3
		21,080				11	22·8	
7·5	8,420		125,800	14,800	18			353·5
		16,100				33	68·5	
5	9,500		141,900	16,700	54			285
		4,450				66	137	
2·5	1,000		146,350	17,250	71			148
		650				71·5	148	
0	0		147,000	17,350	72			0

§ 3. The air-gap flux-density curve on no load.—From the bifurcation plane the flux-density in the air up to a distance of l_p from the pole-edge is

$$1{\cdot}257\frac{AT_p - AT_{tx} - AT_{cx}}{Kl_g} = 1{\cdot}257\frac{AT_p - AT_{cx}}{Kl_g + \mathcal{R}_{tx}} \quad . \quad (134)$$

where $\mathcal{R}_{tx}$ is the reluctance of the teeth corresponding to a sq. cm. of cross-section of the path in the air-gap, *i.e.* for a peripheral width of 1 cm. on the armature surface and an axial length of 1 cm. along the armature. No attempt is here made to determine the actual undulations of the flux curve due to the varying density over teeth and slots at the armature surface or at any other level in the air-gap. But the smoothed-out curve of density is obtained,

from the area of which the total flux is correctly obtained. So important is the varying reluctance of the iron teeth that in the toothed armature it is best to consider the effect of $+TA_r$ and $-AT_r$ in relation, not to the armature surface, but to the cylindrical surface of the core at the level of the bottom of the slots. This may conveniently be done by calculating the equivalent reluctance of tooth per sq. cm. of path in the air, which is in series with the single gap. Thus if B_{gx} is the average density over a tooth-pitch

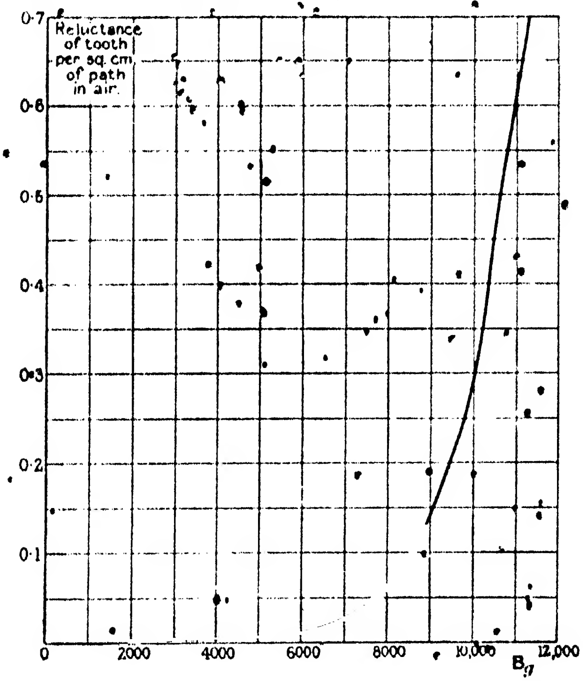


FIG. 603.—Tooth reluctance per sq. cm. of path in air-gap.

in the air-gap, the centre of the tooth-pitch being at point x, the equivalent tooth reluctance in series with the air-gap is—

$$\mathcal{R}_{tx} = \frac{\text{mean } H \text{ over tooth} \times l_t}{B_{gx}} = \frac{1{\cdot}257\ AT_{tx}}{B_{gx}}$$

From the data of the machine dimensions a curve such as Fig. 603 (which refers to the machine of § 2) can be plotted for the tooth reluctance per sq. cm. of air-gap for varying gap densities, and by reference to this values of B_{gx} and $\mathcal{R}_{tx}$ which are in agreement can quickly be found for any value of $1{\cdot}257\ (AT_x - AT_{cx})$:

Within the interpolar region, it has been stated in Chapter XVI., § 6 (a) that the counter effect of the adjacent pole dies away from its value on the interpolar line of symmetry very nearly to zero at the opposite pole-tip, and the rate of this decay may approximately be assumed as uniform over the half interpolar zone. The resultant flux-density on the armature surface in the interpolar gap for any distance x from the pole-corner up to $x = c$ would then, if the armature surface were throughout at zero potential, be closely represented by

$$1{\cdot}257\ AT_p \left\{\frac{m}{\xi x + Kl_g} - \frac{m'}{\xi c + Kl_g} \cdot \frac{x}{c}\right\}$$

where $\xi = 0{\cdot}9\gamma$ and takes into account the inclination of the pole-edges to the armature surface, and m' is the correction factor for c/l_g.

But the above approximation, as also the accurate solution of the ideal case upon which it is based, departs from the actual facts in assuming that the armature core is throughout at zero potential. In reality on the one side of the interpolar line of symmetry the armature core is at a varying positive magnetic potential AT_{cx}, being the ampere-turns expended over the armature core from any point x up to the interpolar line of symmetry, and on the opposite side it is at a varying negative potential $-AT_{cx}$. The effective AT over the air-path for flux from one pole is thereby decreased to $(AT_p - AT_{cx})$, and we are justified in assuming that the counter effect from the opposite pole is proportionately increased, the effective difference of potential becoming $-AT_p - (+AT_{cx}) = -(AT_p + AT_{cx})$. Next, the resultant AT_{tx} required by the tooth reluctance is that corresponding to the difference between the forward component flux from one pole and the counter component from the adjacent pole. The necessity for employing two different values AT'_{tx} and AT''_{tx} in the numerator for each component separately is at once avoided by inserting in the denominator of each item a term for the true tooth-reluctance. The final expression for the interpolar fringe from $x = 0$ to $x = c$ thus becomes—

$$1{\cdot}257\left\{\frac{m\,(AT_p - AT_{cx})}{\xi x + Kl_g + \mathcal{R}_{tx}} - \frac{m'\,(AT_p + AT_{cx})}{\xi c + Kl_g + \mathcal{R}_{tx}} \cdot \frac{x}{c}\right\}. \quad (135)$$

An approximate allowance for $\mathcal{R}_{tx}$ can alone be made in the first instance, but as soon as the pole-corner is left behind the density falls so quickly that the tooth reluctance becomes practically almost a negligible quantity. On the interpolar line of symmetry $m = m'$, $\mathcal{R}_{tx}$ vanishes, and also AT_{cx}, so that the resultant density is zero. No lines enter or leave through the armature surface, but the direction of the real flux just grazes the tooth-tip tangentially (cp. Fig. 249).

In this manner a close approximation to the flux-density curve over a pole-pitch on no-load can be made when the varying undulations due to the movement of slots and teeth are not immediately required.

II. The Non-salient-Pole Machine

§ 4. The distributed field-winding of the non-salient-pole machine.—In the cylindrical rotor of a non-salient-pole machine the exciting coils are embedded in slots which are distributed over a certain portion of its periphery. The number of slots and the fraction σ of the pole-pitch over which they are spread vary in different makes, but the practical range of variation is not great. Concentration of the winding in a single slot or in a very few per pole would be favourable to the production of a large magnetic flux for a given value of the ampere-turns of excitation. But the possible width and depth of a single slot are limited; if open and wide, it will cause exaggerated tooth ripples, and the accumulated mass of copper will be difficult to secure mechanically and to cool effectively. With an increase in the number of slots the intervening teeth greatly assist in conducting the heat to the surface where it may be carried away by air driven through the air-gap, and further, when the slots are partially distributed, a greater number of ampere-turns can be accommodated per pole. On the other hand, if the winding is distributed over the whole of the pole-pitch, the inner turns are comparatively ineffective, so that nothing is gained by making it extend over more than 80 per cent. of the pole-pitch.[1] A compromise must therefore be struck, and usually the wound fraction of the pole-pitch falls in practice between the limits of $\sigma = 0{\cdot}6$ to $\sigma = 0{\cdot}8$ with from 6 to 10 or 12 wound slots per pole.

In these circumstances the unwound pole-centre may be uniformly slotted in order to maintain the same magnetic relation to the air-gap as holds over the wound portion, but more usually it is left unslotted (Figs. 234 and 235) as giving the greatest permeance, or it has smaller grooves closed by magnetic wedges to give a smooth surface, such grooves being used for purposes of axial ventilation. On the average then it may be said that two-thirds only of the pole-pitch are wound, one-third of the pole-pitch forming a pole-centre which in a minor degree acts like a salient pole, and, so to speak, holds the flux more or less bound to itself.

§ 5. The trapezium of M.M.F.—When the exciting winding is located in two or more slots per pole, the sides of the spacial curve of the M.M.F. plotted in relation to one pole-pitch developed on the flat are stepped, and when the fraction σ of the pole-pitch Y only is wound, the curve has a flat top extending over $1-\sigma$ of the

[1] Cp. Dr. S. P. Smith, "The Non-salient Pole Turbo-alternator," *Journ. I.E.E.*, Vol. 47, p. 563.

pole-pitch. In the extreme cases of a winding perfectly uniformly distributed over the pole-pitch or $\sigma = 1$, the M.M.F. curve per pole is a triangle, of height $IT_m/2$ in ampere-turns, where I = the exciting current and T_m is the number of conductors per pole, and if the winding is closely concentrated in one slot per pole, it is a rectangle of the same height. Between these two limits lie all the intermediate cases of practice. Since the M.M.F. rises steeply at the same rate within each slot (when uniformly filled) as we pass from one wall to the other, a continuously sloping line may without much error be

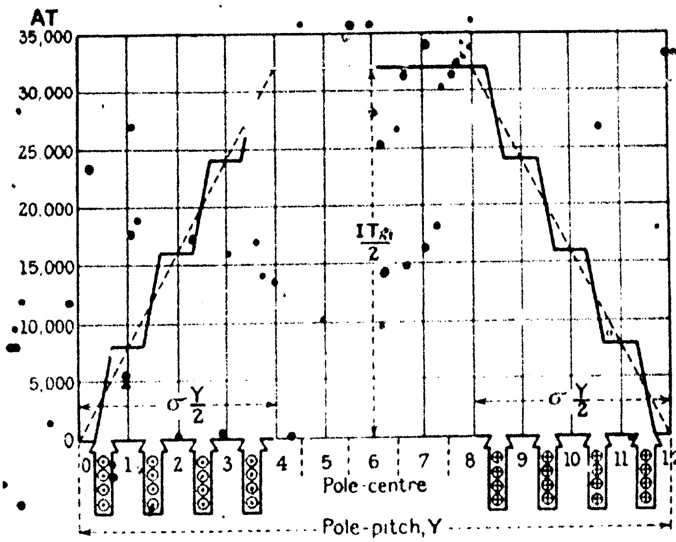


FIG. 304.—The M.M.F. trapezium.

substituted for the stepped sides, as shown dotted in Fig. 304, and a regular trapezium of M.M.F. as acting on the half magnetic circuit corresponding to a pole is obtained.

§ 6. Allocation of the M.M.F. to component parts of the magnetic circuit.—This M.M.F. has then to be allotted in proper proportions to the several parts of the magnetic circuit—stator core, stator teeth, air-gap, rotor teeth and rotor core. As in the case of the salient-pole machine (§ 1), in order to determine the shape of the flux-curve either the AT expended over rotor and stator cores, or the maximum flux through the rotor core, must provisionally be guessed at the outset.

Of the two, it is best to assume the former, thence to determine approximately the total flux, and lastly to check the first assumption, before proceeding with more detailed calculations. From

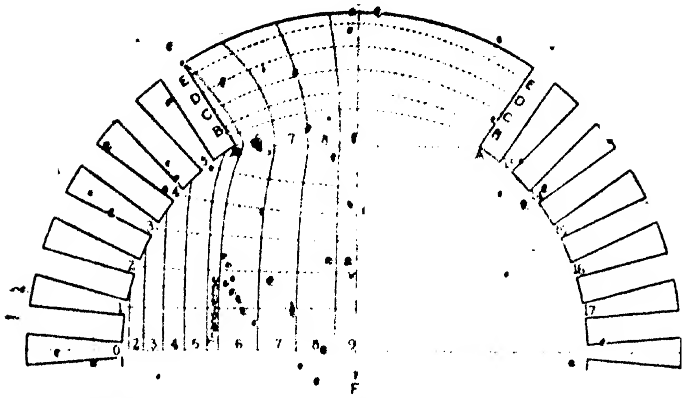


FIG. 305.—Division of rotor core by equipotential surfaces on no load.

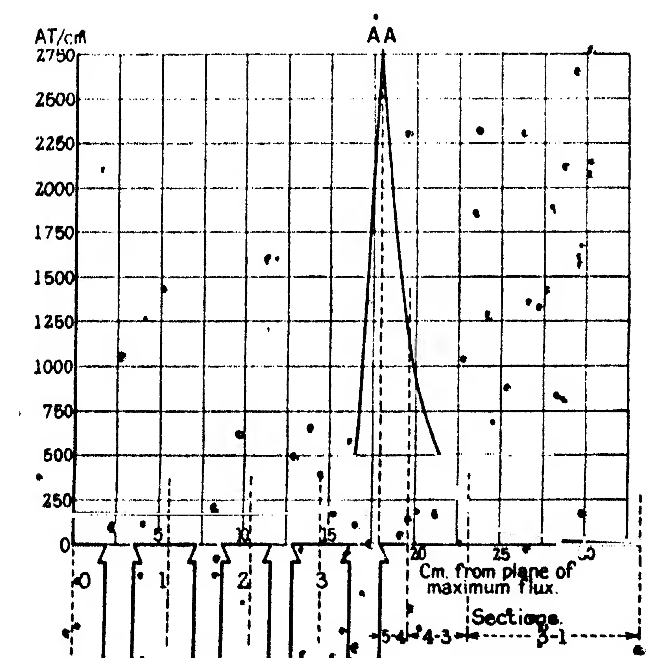


FIG. 306.—Specific AT/cm over rotor core.

considerations of symmetry, on no-load only one half of a pole-pitch need be considered.

Starting from a point opposite to the pole-centre, the curves for the specific AT/cm for the stator core, and for its integral in the

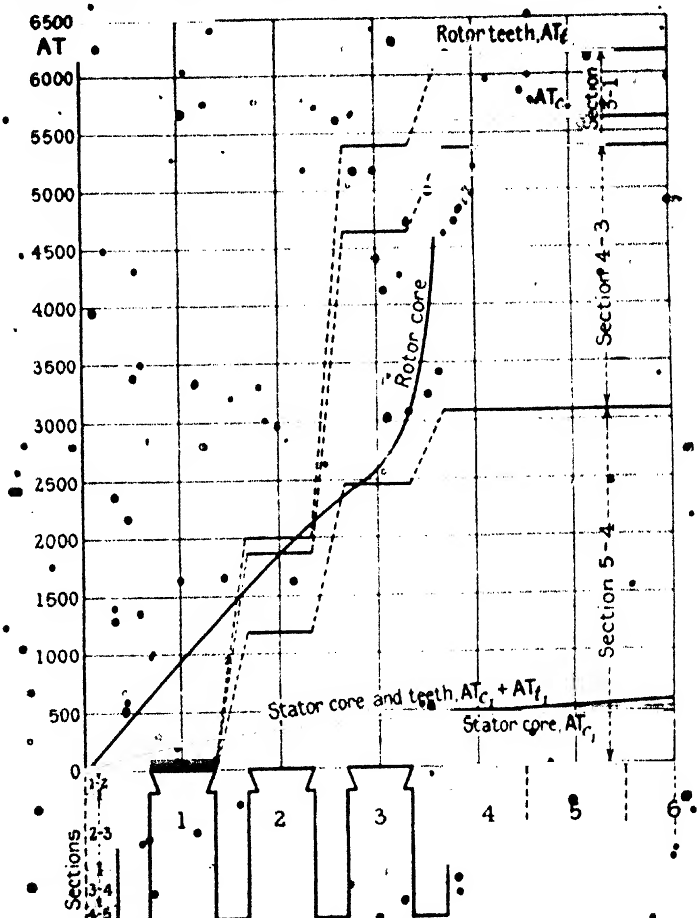


FIG. 307.—AT expended over component parts of the magnetic circuit.

reverse direction resemble those of Fig. 302. The experiments of Mr. Carl J. Fechheimer already mentioned in Chapter XIV, § 12, show that in a deep stator core the flux distribution is almost sinusoidal; if therefore the stator pole-pitch is divided up into small sections, each, say, 5 electrical degrees, and the maximum

density averaged over the half core-section is taken as the starting-point, the value of the density at each step of 5° therefrom is found in proportion to the cosine of the angle in electrical degrees; thence the specific AT/cm is obtained from a B-H curve for the sheet-steel, and when summed up, AT_{c3} for any distance from the plane of maximum flux or zero potential. The ampere-turns AT_{t1} for the stator teeth opposite the centre of each rotor tooth are obtained from a curve similar to Fig. 303, but are comparatively unimportant, as appears from Fig. 307.

The rotor core is treated by division into sections corresponding to rotor tooth-pitches and bounded by equipotential surfaces after

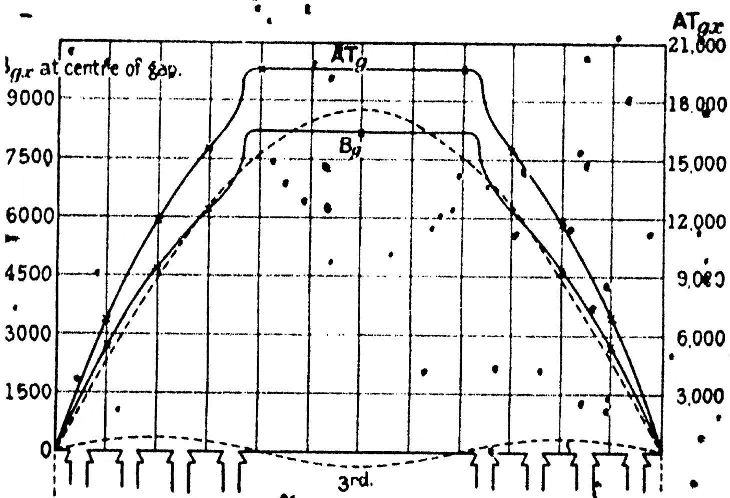


FIG. 308.—AT_g and B_g of non-salient-pole rotor on no load.

the manner of Fig. 305. The specific AT/cm or *at* for each section will depend on the density of the flux in the rotor core as affected by the presence or otherwise of axial ventilating holes or ducts through the core; but usually it is found to remain fairly constant from the plane of maximum flux up to the last tooth-pitch before the equipotential plane AA across the neck of the pole-centre. When this is closely approached, the density and the specific AT/cm rise very rapidly, as shown in Fig. 306. The integral again yields a continuous curve marked as AT_{c1} in Fig. 307 and rising very steeply towards AA, over which it remains constant; for the further passage of the flux, the unslotted pole-centre is reckoned as a single large rotor tooth or is divided up mentally into rotor tooth-pitches, as 4, 5, 6 in Fig. 307.

The AT expended over each rotor tooth increase greatly as the

pole-centre is approached, and are marked as AT_{t1} in Fig. 307, which also includes the pole-centre regarded as a tooth.

Adding up, then, the component items, $AT_{c1} + AT_{t1} + AT_{t2}$, AT_{c2} of Fig. 307 for the centre of each rotor tooth,[1] and deducting their sum from the corresponding values of the exciting ampere-turns of Fig. 304, the differences give AT_g acting on the air-gap, opposite each rotor tooth (Fig. 308). The effective length of this gap has two values according as it exists between the surfaces of rotor and stator where both are slotted or between the slotted stator surface and the unslotted surface of the pole-centre, to which may be added a third intermediate value for the tooth-pitch at the edge of the pole-centre which is only slotted on the one side. The values of B_{gx} thence deduced for each rotor tooth are marked by the crosses on the lower full-line curve of Fig. 308, the whole of the values employed in Figs. 304, 306–8 being derived from the 2-pole 3000 kW. 3-phase turbo-alternator analysed by Mr. S. Neville in his paper (Part III) supplementing the present writer's Part I on the Flux-Wave of the Turbo-alternator in *Papers on the Design of Alternating Current Machinery*.[2]

§ 7. The rotor transverse slot flux.—The difference from the analogous case of the salient-pole machine lies mainly in the additional complexity introduced by the transverse component of the flux crossing the rotor slots. As explained in detail in the above-quoted paper, the difference of magnetic potential in ampere-turns acting at any height x between the walls of a rotor slot of depth h_w below the wedge may be expressed as

$$AT_r \frac{x}{h_w} - AT_c \frac{w_s}{t_r} - (AT_x' - AT_x) \quad . \quad . \quad (136)$$

where AT_r are the ampere-conductors of one rotor slot, and AT_c are the ampere-turns expended over the rotor core between the centres of two adjacent teeth for which AT_x' and AT_x are the ampere-turns expended over their reluctances up to the height x from the bottom of the slots. Here AT_x' relates to the tooth farther from, and AT_x to the tooth nearer to, the plane of maximum rotor flux. The difference of potential thus varies greatly with the varying values of AT_x' and AT_x. Near the plane of maximum flux AT_x' always exceeds AT_x owing to the increase of the flux-density as we proceed up the side of the flux-curve, but for the unslotted pole-centre AT_x' will not so greatly exceed AT_x or may even be less. In the latter case, the bracketed term is negative, and the whole term with its sign becomes positive, increasing the transverse component due to AT_r. In consequence the transverse flux may

[1] The full line values of AT_{t1} are only joined up by dotted lines for convenience in following, and Fig. 307 must only be read on the centre lines of the teeth.

[2] Pp. 240–252 (Pitman & Sons

be greater within the slot at the edge of the pole-centre than at the centre of the exciting coil, and may decline to a minimum in an intermediate slot. Such a case is indicated in Fig. 309. The difference between the amount of the flux entering one side and leaving on the other side of a tooth within any section of its height is added to or subtracted from the radial component flux proceeding up or down the tooth, and is thus an addition to or a deduction from the useful flux passing through the air-gap into the stator surface.

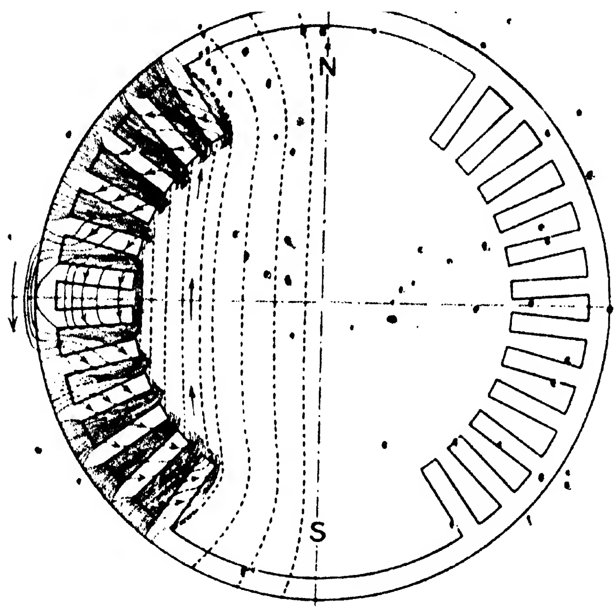


FIG. 309.—Rotor slot flux on no load.

The several items of the transverse M.M.F. can be isolated and considered separately, each as causing its own component in the transverse slot flux. But of these items the first two can be grouped together as being independent of the tooth reluctance and as being a nearly constant quantity until the tooth-pitch next to the edge of the pole-centre is reached. Here the rotor tooth-pitch λ_r must be interpreted as the distance only from the equipotential plane AA to the next lower equipotential surface, and the high value of the AT/cm over the part next to AA forces a larger amount of the total rotor flux out of the core into the slot.

To deal with the varying item of the tooth reluctance and the varying difference of potential that thence arises, each tooth is

best divided into at least three sections of height decreasing towards the root of the tooth where the saturation is greatest, as shown at the foot of Fig. 307. It is then simplest to calculate, first, the transverse flux that would be caused by AT_s' for any section if the opposite wall of the slot were at zero potential, and secondly the transverse flux that would be caused by AT_s for the similar section if the wall of the slot formed by the face of the tooth first considered be imagined in turn to be reduced to zero potential;[1] the algebraic difference of the two fluxes thus found will then be the actual component due to $AT_s' - AT_s$ for the considered section. Lastly, deducting this flux from the similar quantity for the next slot nearer to the plane of maximum flux, the difference is the flux added to or subtracted from that at the end of the section nearest to the rotor core, and, if positive, it is added thereto within the section. The process is fully illustrated by a worked example in Part III of the second *Paper on the Design of Alternating Current Machinery* and need not here be repeated.

§ 8. End-bell leakage.—The end-connections of the exciting coils are retained in place by end-bells, for which nickel steel is usually employed. Since this material is magnetic, and the edge of the bell abuts against the rotor body or is separated therefrom only by a small intervening air-gap, lines of flux escape from the tips of the rotor teeth and from the curved tip of the pole-centre into the end-bell. They are thus lost so far as any useful action is concerned, since they do not pass through the air-gap into the stator, but complete their path through the substance of the end-bell and re-enter the tips of the rotor teeth and pole-centre under a pole of opposite sign. The amount of this true *leakage* is very considerable, and thereby a further difference is introduced between the non-salient-pole and salient-pole cases.

The course of the leakage flux over a half pole-pitch is indicated with a few equipotential lines in Fig. 310, which shows the portion of the bell at one end developed on the flat. A first estimate of its total amount can be made by taking a somewhat smaller value than $\frac{1}{2}$ AT_r as acting over the half tooth-pitch between the lines AB' and OQ. From the AT/cm thus obtained the density for the steel employed is found, and this, when multiplied by the area of cross-section, gives a figure for the total leakage. Assuming certain values for the potentials of each rotor tooth at its tip, the process is continued, and the difference in the values as each tooth-pitch is passed is regarded as the leakage of the tooth-pitch. When the AT acting on each tooth-pitch has reached its minimum, trial must then be made as to how nearly the remaining flux can be carried round the curved paths by the remaining differences of potential, after the manner of Fig. 310. Adjustment of the equipotential

[1] As proposed by Mr. Neville in the above quoted paper.

lines by raising their outer ends through the distances D to D', F to F', will then enable a reasonable estimate to be made of the

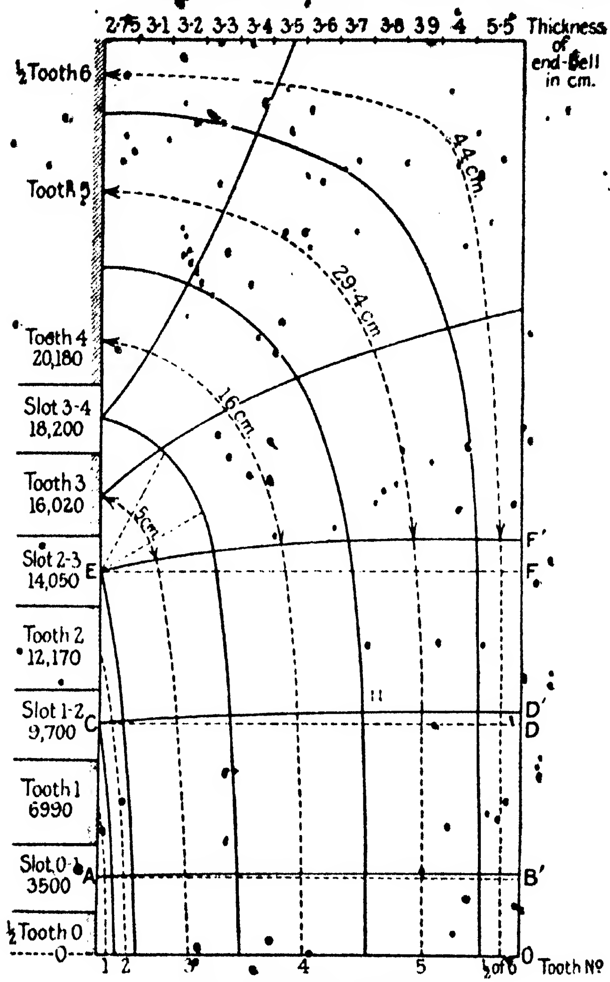


FIG. 310.—End-bell leakage.

maximum flux that the potentials of the tooth-tips and pole-centre can cause.[1] Generally the section of the end-bell on the plane of maximum rotor flux becomes saturated to a density of some 18,000

[1] See *Papers on the Design of Alternating Current Machinery*, pp. 199–201, for further details.

to 20,000 C.G.S. lines per sq. cm. The longer the rotor core, the less the percentage loss of flux by leakage into the end-bells.

§ 9. The fundamentals of the flux-density curve.—The importance of the fundamental B_{g1} in relation to the E.M.F. has been emphasized in Chapter IX, § 14, and it is evident that it is mainly dependent on the height of the flux-density curve over the flat portion corresponding to the pole-centre, and that when this has been determined, the value of the fundamental follows with but little range of variation for a given ratio σ.

If the AT expended over the rotor and stator cores and teeth increased in strict proportion to the distance or number of teeth

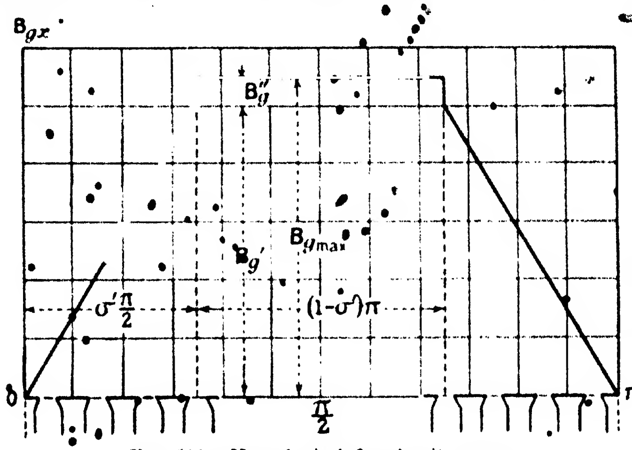


FIG. 311.—Hypothetical flux-density curve.

passed as we proceed from the plane of maximum flux, AT_{gx} and the density in the air-gap would rise as inclined straight lines reproducing the sloping sides of the trapezium of M.M.F. On reaching the unslotted pole-centre, the effective air-gap length changes from $K'l_g$ to some smaller value $K''l_g$, so that with some further constant loss of AT over the stator core and teeth and rotor pole-centre, deducted from $IT_m/2$, the remainder AT_{gx} will add a rectangle of flux over some fraction of the pole-pitch. The point of rapid rise of the density may be fixed at the centre of the slot nearest to the unslotted pole-centre. If γ_s = the pitch of the rotor slots in electrical radians, the flat portion of the flux-density curve extends over an angular width in radians of $(1-\sigma')\pi = (1-\sigma)\pi + \gamma_s$, and each sloping side over a width $\sigma'\frac{\pi}{2} = \sigma\frac{\pi}{2} - \frac{\gamma_s}{2}$. That is, $\sigma' = \sigma - \frac{\gamma_s}{\pi}$, or, if the slots are pitched as for a number s per pole,

$q' = \sigma - \frac{1}{s}$. The flux-density curve then has the geometrical shape of Fig. 311, which may be split up into a trapezium of height B_g' and a rectangle of height B_g'', the two together making up the height $B_{g\,max}$.[1]

The general expression for a trapezium in which the height B_g' extends over a fraction σ' of the pole-pitch (Fig. 312) is

$$B_x = \frac{8}{\pi^2} \times \frac{B_g'}{\sigma'}\left(\sin\sigma'\frac{\pi}{2}\sin\theta_x + \frac{1}{9}\sin 3\sigma'\frac{\pi}{2}\sin 3\theta_x + \ldots\right) \quad . \quad (137)$$

and the general expression for a rectangle of height B_g'' extending over the same fraction of the pole-pitch when the origin is at a point

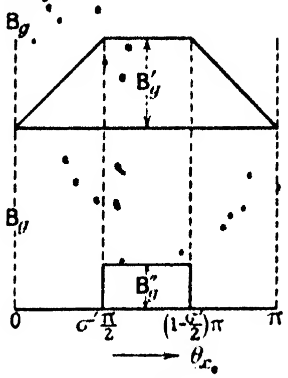


FIG. 312.

distant $\sigma'/2$ of the pole-pitch away from the side of a rectangle, *i.e.* midway between two rectangles, is

$$B_x = \frac{4}{\pi}B_g''\left(\cos\sigma'\frac{\pi}{2}\sin\theta_x + \frac{1}{3}\cos 3\sigma'\frac{\pi}{2}\sin 3\theta_x + \ldots\right) \quad . \quad (138)$$

The value of B_x for the combination is given by the sum of the two expressions, and the amplitude of the joint fundamental is

$$B_{g1} = \frac{4}{\pi}\left\{B_g'\frac{\sin\sigma'\pi/2}{\sigma'\pi/2} + B_g''\cos\sigma'\pi/2\right\} \quad . . \quad . \quad (139)$$

If *e.g.* we assume $\sigma = \frac{2}{3}$rds. and the slots are pitched as for 12 per pole, as in Fig. 308, $\sigma' = \frac{8}{12} - \frac{1}{12} = \frac{7}{12}$, and

$$B_{g1} = \frac{4}{\pi}\left\{0{\cdot}866\ B_g' + 0{\cdot}6088\ B_g''\right\}$$

[1] See Dr. S. P. Smith and R. S. H. Boulding, "The Shape of the Pressure Wave in Electrical Machinery," *Journ. I.E.E.*, Vol. 53, pp. 211–14.

Although the corners of the rectangle and the junctures of the sloping sides with it are in reality rounded off, the double value of the flux-density on the line of the juncture indicates that at this point, the same value of AT_g divided by $K''l_g$ and by $K'l_g$ yields the two values B_g' and $B_{g\ max}$. If the ratio of the two values of the effective air-gap length be, as usual, about 0·955, $B_g' = 0{\cdot}956\ B_{g\ max}$ and $B_g'' = 0{\cdot}0448\ B_{g\ max}$. Thence

$$B_{g1} = \frac{4}{\pi}\left\{0{\cdot}828 + 0{\cdot}027\right\} B_{g\ max}$$
$$= 1{\cdot}09\ B_{g\ max}$$

But the above case would only be reproduced at low values of $B_{g\ max}$. When the teeth become highly saturated, the sides of the curve become strongly bowed or concave to the horizontal axis, owing to the increasing *proportion* of the total AT expended over their reluctance, and the height of the rectangle becomes proportionately greater. Both effects are illustrated in Fig. 308, for which the amplitude of the fundamental (shown dotted)[1] is closely 8800 as compared with $B_{g\ max} = 8200$, *i.e.* $B_{g1} = 1{\cdot}07\ B_{g\ max}$. Hence for purposes of design, B_{g1} may be taken as, say, 1·09 to 1·05 $B_{g\ max}$ over the working range. At no load under full load excitation—a condition which may require to be tested in order to calculate the inherent regulation of the alternator—B_{g1} may sink to equality with $B_{g\ max}$.

§ 10. The practical process of design.—The necessity at the outset for a provisional estimate of the total flux has been stated in § 6, and we are now in a position to show how it can be quickly made and immediately checked, and the value of $B_{g\ max}$ be thus determined for a given excitation or *vice versâ.*

Assuming $AT_{c1} + AT_{t1}$ ampere-turns as expended over the stator core and teeth, and $AT_{c2} + AT_{t2}$ for the corresponding items in the rotor core up to the equipotential plane AA and onwards up to the surface of the unslotted pole-centre, the difference $T_m/2 - (AT_{c1} + AT_{t1} + AT_{c2} + AT_{t2}) = AT_g$ acting between pole-centre and stator face. Dividing by $0{\cdot}8\ K''l_g$, the density $B_{g\ max}$ is found, which practically will hold over the pole-centre or $1 - \sigma'$ of the pole-pitch, save for a small falling off towards its edges.

Multiplying $B_{g\ max}$ by the effective air-gap area over half of the pole-centre, the flux in the single section of the stator core opposite to the pole-centre edge is obtained. The same value of AT_g divided by $0{\cdot}8\ K'l_g$ gives a value for B_g' at the juncture of rectangle and trapezium which cannot be exceeded and when the rotor teeth are highly saturated will not be reached. Plotting therefore a slightly

[1] The amplitude of the 3rd harmonic shown at the foot of the diagram is 366 = 4·3 per cent. of the fundamental.

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